Type Synthesis and Static Balancing of a Class of Deployable Mechanisms

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ABSTRACT

This thesis addresses the type synthesis and static balancing of a class of deployable mechanisms, which can be applied in applications in many areas including aerospace and daily life.

Novel construction methods are proposed to obtain the deployable mechanisms. First, the type synthesis of the foldable 8-revolute joint (R) linkages with multiple modes is presented. Two types of linkages are constructed by connecting planar 4R linkages and spherical 4R linkages. The obtained linkages can be folded into two layers or four layers, and have multiple motion modes. A spatial triad is also adopted to build single-loop linkages, then the single-loop linkages are connected using spherical (S) joints or RRR chains to obtain deployable polyhedral mechanisms (DPMs). The DPMs have only 1-degree-of-freedom (DOF) when deployed, and several mechanisms with 8R linkages and 10R linkages have multiple motion modes and can switch modes through transition positions. In addition, when connecting single-loop linkages using half the number of the RRR chains, the prism mechanisms obtain an additional 1-DOF rotation mode.

Furthermore, the DPMs are developed into statically balanced mechanisms. The geometric static balancing approaches for the planar 4R parallelogram linkages, planar manipulators, spherical manipulators and spatial manipulators are developed so that the mechanisms can counter gravity while maintaining the positions of the mechanisms. Only springs are used to design the statically balanced system readily, with almost no calculation. A novel numerical optimization approach is also introduced which adopts the sum of squared differences of the potential energies as the objective function. Using the proposed static balancing approaches, the 8R linkages and the DPMs presented in this thesis can be statically balanced.
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LIST OF ABBREVIATIONS

Revolute joint: R joint
Universal joint: U joint
Spherical joint: S joint
Centre of mass: CM
Degree-of-freedom: DOF
Devavit-Hartenbergh: D-H
Parallel mechanism: PM
Deployable polyhedral mechanism: DPM
Remote centre of motion: RCM
LIST OF PUBLICATIONS


CHAPTER 1 – INTRODUCTION

Robotics is one of the key technologies being adopted worldwide. There is an increasing need for robots from food industries to the manufacturing sector. Mechanism design is an old subject and is still one of the most important aspects of robotics. With the development of advanced technologies, people pay less attention to it. However, mechanism design is indispensable to the industry. The developments of a number of technologies depend on the development of mechanical design which provides the essential machines.

Manipulators belong to mechanisms. Manipulators are composed of moving links and joints, and can be classified into serial manipulators and parallel mechanisms (PMs). Serial manipulators are composed of a series of links connected by joints and have simple structures and large workspaces, while PMs are consisted of two platforms and several limbs and have better stiffness, dynamic performances and can carry heavy loads. Two examples of serial manipulator and PM are respectively presented in Fig. 1.1. Serial manipulators are widely used in industry, for example, for painting, welding and assembly. PMs can be applied to flight simulators [1].

![Fig. 1.1 Two types of manipulators: (a) serial manipulator; (b) PM.](image)

The single-loop linkage is a special case of PM. In this thesis, only the loops with even numbers of links (which have symmetric structures) comprised of revolute (R) joints are discussed. 4R linkages include planar 4R linkage [Fig. 1.2(a)], spherical 4R linkage [Fig. 1.2(b)] and Bennett 4R linkage [Fig. 1.2(c)]. The axes of the R joints of the planar 4R linkage are parallel, and those of the spherical linkage intersect at a point. In the Bennett 4R linkage, the axes of the R joints are neither parallel nor intersected at a point. The
joints are designed in a particular way that makes this over-constrained linkage movable [2-3]. The Bennett 4R linkage is an important compositional mechanism, many linkages are based on Bennett 4R linkages, such as the Myard 5R linkage [4], Goldberg 5R linkage [5] and 7R linkages [6].

![4R linkages](image)

**Fig. 1.2** 4R linkages: (a) planar 4R linkage; (b) spherical 4R linkage; (c) Bennett 4R linkage.

![6R linkages](image)

**Fig. 1.3** 6R linkages: (a) Sarrus 6R linkage [7]; (b) Bricard 6R linkage [8]; (c) Bennett 6R hybrid linkage [10]; (d) Bennett plano-spherical hybrid linkage.

The first over-constrained 6R linkage is the Sarrus 6R linkage proposed by Sarrus [7], as shown in Fig. 1.3(a). The two platforms are always parallel and the three joint axes of
each limb are also parallel. Then the Bricard 6R linkage [Fig. 1.3(b)] was designed [8]. There are six types of Bricard linkages, categorized by the directions of the R joints, such as the line-symmetric Bricard linkage and the plane-symmetric Bricard linkage [9]. The Bennett 6R hybrid linkage in Fig. 1.3(c) was proposed by Bennett [10]. It is comprised of two spherical 4R linkages. The joints axes 1, 2 and 3 intersect at a point and the joint axes 4, 5 and 6 intersect at another point. The Bennett plano-spherical hybrid linkage, as shown in [Fig. 1.3(d)], has three joints with parallel axes and another three joints with intersecting axes. The other 6R linkages can be found in [11]. The 4R linkages and 6R linkages mentioned above all have single degree-of-freedom (DOF).

1.1 Foldable or Deployable Mechanisms

The foldable or deployable mechanisms refer to the mechanisms that can change their sizes or shapes to facilitate transportation and storage. Due to their characteristics, they are widely used in aerospace and daily life. Foldable mechanisms and deployable mechanisms sometimes intersect.

Foldable mechanisms are evolved from origami, by regarding the ceases as revolute joints and the panels as links. There are various methods that have been proposed for the structure of which the thickness of the panels cannot be neglected. Generally, we can offset the panels to facilitate folding. The foldable mechanisms are widely used on solar panels and roofs.

Chen et al. [12-13] developed a 1-DOF foldable mechanism base on the Bricard linkage. The mechanism can be spread onto a plane and be folded into a bundle (Fig. 1.4). In [14], another foldable 6R linkage was presented. It is a two-fold symmetric mechanism, and the axes of the joints intersect at two points, as shown in Fig. 1.5. Like the mechanism in [13], it can be spread onto a plane and be folded into a bundle.

![Fig. 1.4 Threefold-symmetric Bricard linkage [13]](image-url)
Fig. 1.5 Twofold-symmetric Bricard linkage [14]

Chen and You [15] also investigated a deployable mechanism consisted of four 6R linkages, as shown in Fig. 1.6. The mechanism can be spread onto a plane and be deployed into one dimension.

Fig. 1.6 Deployable mechanism consisted of four 6R linkages [15]

Fig. 1.7 Deployable cube mechanism based on 8R linkages [17]

Deng et al. [16] discussed a series of foldable single-loop linkages such as 4R, 5R, 6R, 7R and 8R linkages. The linkages can be spread onto a plane and be folded into a bundle. A deployable cube mechanism based on 8R linkages was introduced in [17], as shown in Fig. 1.7.

Li et al. [18] fabricated a deployable ring mechanism composed of many deployable modules. The mechanism is driven by torsion springs and controlled by cables. It has a high folding ratio, as shown in Fig. 1.8. Another deployable antenna was presented in [19]. Qi et al. [20] designed a novel double-layer truss deployable mechanism, which can be folded into a bundle or spread onto a large volume, as shown in Fig. 1.9. The DOFs of
the large-ratio mechanisms are all one. Single-layer deployable mechanisms that can be folded were also introduced in [21]. They are constructed by Myard linkages and have 1-DOF as well. Deployable mechanisms based on Bricard linkages were constructed in [22-23]. A family of deployable mechanisms was also presented in [24].

![Fig. 1.8 Deployable ring mechanism in [18]](image)

![Fig. 1.9 The double-layer deployable mechanism in [20]](image)

Lu et al. [25] proposed a family of deployable prism mechanisms, which can be folded and stretched in a single dimension, as shown in Fig. 1.10. The 1-DOF mechanisms with similar folding characteristics were also discussed in [26-27]. Lu et al. [28] also connected a 2-DOF deployable mechanism which can be deployed in two directions independently, as shown in Fig. 1.11. The mechanism is obtained by combining the units that are assembled by scissor linkages and Sarrus linkages.

![Fig. 1.10. 1-DOF deployable prism mechanism in [25]](image)
A novel class of rigid-panel deployable antennas was introduced in [29], as shown in Fig. 1.12. The antenna surface is divided into several panels which are connected by R joints. The mechanism can be folded and spread around the centre line of the mechanism.

In [30], the folding process of mechanisms with thick panels was presented, as shown in Fig. 1.13. The method of offsetting panels away from the plane that is defined by two joints on the panels was used. The mechanisms are spread onto a plane in the initial state and can be folded into several layers.
Hoberman [31] designed the foldable panels comprised of four, six or nine links, by offsetting the axes of the hinges. A panel with nine links is shown in Fig. 1.14, the panel can be spread onto a plane and folded into a compact cuboid.

![Fig. 1.14 Foldable panel made of thick panels in [31]](image)

An approach for accommodating for the thickness of the panels was also investigated in [32]. A foldable habitat was integrated, of which the folding ratio between the stowed and deployed configurations is 85% (Fig. 1.15).

Kang and Yi [33] provided the design and analysis of two new foldable mechanisms without parasitic motion. The two mechanisms can be folded to save space, as shown in Fig. 1.16. They can be applied to flat panel TV mounting.

A 6-DOF foldable PM for the ship-based stabilized platform, which is driven by four-bar linkages was designed in [34]. Specific R joints are replaced by four-bar linkages to facilitate folding. The limbs of the mechanism can be folded onto parallel planes, as shown in Fig. 1.17.
Jacobsen et al. [35] introduced the Lamina emergent mechanisms that can be fabricated using planar materials (laminae) and have motions out of the plane (Fig. 1.18). The mechanisms belong to the compliant mechanisms. The advantage of these type of mechanisms is that they can be manufactured one time and can work directly off the machine tools. After finishing work, they can be folded into the initial state. Several credit-card-sized products were fabricated based on lamina emergent mechanisms [36]. Some other applications such as board games were presented in [37].
In [38], a design method was proposed to create foldable quadrilateral meshes. The obtained foldable structures have only 1-DOF and one of them is given in Fig. 1.19. Wood was cut to fabricate the mechanism with rigid panels and flexible joints.

![Fig. 1.19 A foldable wooden structure in [38]](image1.jpg)

Lang et al. also created a family of 1-DOF deployable cut origami flashers, as shown in Fig. 1.20. These mechanisms can be spread on a plane and be folded into a bunch. Examples of 4-fold, 5-fold and 6-fold mechanisms were presented in [39]. A cut is necessary in the process of folding and the panels are regarded as zero-thickness.

Nelson et al. [40] utilized the lamina emergent arrays (networks of lamina emergent joints) to design deployable structures. One example is given in Fig. 1.21. The mechanism can be spread onto a plane and deployed into a spatial structure.

![Fig. 1.20 A foldable cut origami flasher [39]](image2.jpg)

![Fig. 1.21 A deployable structure using lamina emergent arrays [40]](image3.jpg)
An origami waterbomb base, which is a single-loop 8R compliant mechanism, was proposed by Hanna et al. [41], as shown in Fig. 1.22. Since the joints are flexible, the mechanism can be folded and has two stable states. In [42], membrane-enhanced lamina emergent torsional joints were designed, and the novel joints were also applied to the waterbomb base.

![Fig. 1.22 A prototype of the waterbomb base [41]](image)

Hoberman [43] assembled a family of deployable structures using loop assemblies, which are comprised of polygonal links. One mechanism with triangle links is presented in Fig. 1.23.

![Fig. 1.23 Deployable structures with polygon links [43]](image)

In [44], Hoberman used angulated strut elements to form scissors-pairs. Then the scissors-pairs were joined to form a closed-loop, comprised of which sphere structures such as the one in Fig. 1.24 were constructed. A geared expanding structure was also designed in [45].
Wei and Dai [46] investigated the geometry and kinematics of the Hoberman switch-pitch ball. The ball is equivalent to a multiple-loop mechanism composed of single-loop eight-bar linkages, as shown in Fig. 1.25.

In [47], a dual-plane-symmetric spatial eight-bar linkage was proposed, and gears were added to convert the 8R linkage into a 1-DOF mechanism. Then the mechanisms are inserted into faces of the polyhedron to obtain deployable Platonic mechanism, as shown in Fig. 1.26.

By embedding planar linkages into faces of the polyhedron, 1-DOF deployable polyhedral mechanisms were obtained in [48], as shown in Fig. 1.27. The same approach
was also adopted in [49]. The cube mechanism obtained is presented in Fig. 1.28 and the icosahedron mechanism is shown in Fig. 1.29.

In [50], a family of deployable mechanisms was also designed. Chen et al. [51] proposed a kinematic method to design DPMs with 1-DOF. The obtained mechanisms can achieve transitions between a truncated octahedron and cube, as shown in Fig. 1.30.
1.2 Mechanisms with Multiple Modes

A mechanism with multiple modes refers to the mechanism that can change its configuration according to the tasks or the environments, to achieve the optimal choice of the movement and the highest efficiency. The self-reconfigurable robots that change configurations by disconnecting and reassembling the modules are flexible but have too large DOFs [52-53]. In [54], a metamorphic robotic system composed of hexagonal or square modules was proposed, as shown in Fig. 1.31.

During the past decades, more and more researchers were attracted by mechanisms that can change modes without disassembling the mechanisms. The mechanisms have fewer DOFs and switch motion modes through singular positions. The present research is mainly divided into: metamorphic mechanisms, kinematotropic mechanisms, discontinuity moveable mechanisms and the mechanisms with multiple modes.
1.2.1 Metamorphic Mechanisms

Metamorphic mechanisms in which ‘the total number of effective links changes as they move from one configuration to another’ were put forward by Dai and Jones [55]. This kind of mechanisms can change structure when spread or folded. The basic principle is the origami and decorative gifts boxes which comprise of flat cards and creases. A metamorphic robotic hand was designed in [56-57], as shown in Fig. 1.32. Several metamorphic PMs were then introduced, such as in [58-59]. A reconfigurable PM with planar five-bar metamorphic linkages was analysed in [60].

![Fig. 1.32 Metamorphic robotic hand [57]](image)

1.2.2 Discontinuously Movable Mechanisms

Two novel discontinuously movable 8R mechanisms were proposed by Lee and Hervé [61]. One of them is constructed using two planar 4R linkages while the other one is connected by one planar linkage and one spherical linkage. They both have two modes (two planar 4R linkage modes, or a planar 4R linkage mode and a spherical linkage mode)
with discontinuous DOFs (Fig. 1.33). Discontinuously 6R and seven-bar mechanisms were also introduced in [62-63].

1.2.3 Kinematotropic Mechanisms

Parikian introduced the first known single-loop kinematotropic mechanism, which is comprised of eight links and eight R joints (see Fig. 1.34), at the Sixth International Symposium on Advances in Robot Kinematics in 1996. The DOF of the mechanism is two. The mechanism behaves as a spatial mechanism and turns into the planar six-bar linkage, in which the DOF is three, through a singular position. Galletti and Fanghella [64] gave the detailed analysis for kinematotropic chains with DOFs range from one to three. Multi-loop kinematotropic mechanisms were provided in [65]. One example is shown in Fig. 1.35. The difference between the kinematotropic and discontinuously mechanisms is the DOF of the former changes after transforming while the latter doesn’t.

![Fig. 1.34 Single-loop 8R kinematotropic mechanism [64]](image1)

![Fig. 1.35 Multiple-loop kinematotropic mechanism [65]](image2)

Wohlhart [66] also presented a kinematotropic linkage which changes its mode through specific positions. The mechanism is a spatial mechanism whose DOF can be changed between one and two, as shown in Fig. 1.36.
Ye et al. [67] introduced a family of reconfigurable PMs with diamond kinematotropic chains. The chain is constructed by three links and a 4-bar parallelogram, as shown in Fig. 1.37. The mechanism acts in different modes such as 3T, 2T1R, 2R1T and 3R when the 4-bar linkage is in specific postures (general position or singular position).

1.2.4 Mechanisms with Multiple Modes

Kong [68-76] investigated the type synthesis of the mechanisms with multiple motion modes, by utilizing the transition (singular) positions. The single-loop 7R mechanism in Fig. 1.38 was proposed by Kong [68]. The mechanism has variable DOFs and has five motion modes, including a 2-DOF planar 5R mode, two 1-DOF spatial 6R modes, and two 1-DOF spatial 7R modes.
The type synthesis of single-DOF single-loop mechanisms with multiple operation modes was given in [69]. Huang et al. [70] further analysed a 1-DOF 7R linkage obtained by combining two Bennett linkages with a common joint. Removing the common joint, disconnecting both Bennett linkages and reassembling the linkages, a 7R linkage can be obtained. Since the mechanism is composed of two Bennett linkages, it has the motion modes of two Bennett linkages, as well as the 7R linkage mode. The linkage can deform between different motion modes through transition configurations.

Kong and Wang [71] analysed a variable-DOF 8R mechanism with four modes, including two 3-DOF modes and two 2-DOF modes, as shown in Fig. 1.40. The work is based on the mechanism proposed by Parikian in 1996, additional modes were found through the reconfiguration analysis.
In [72-73], a novel method for the type synthesis for PMs with multiple modes was proposed. Type synthesis is to find all the possibility of the types of mechanisms. The definition of the PM with multiple modes is:

*PM with multiple modes is referred to the PM with the same DOF but different motion modes. For example, planar motion, spherical motion, and spatial translation are all 3-DOF motions. f-DOF PMs with multiple modes have several motion modes with f-DOF.*

-Kong [72], 2007

In [72], a 3-DOF PM that can switch between the 3-DOF spherical operation mode and the 3-DOF spatial translational operation mode through transition configurations was discussed as an example to illustrate the type synthesis method, as shown in Fig. 1.41. The transition configuration is when the intersection of the axes of the joints on the upper platform is coincident with those on the lower platform.

Replacing the upper platform in [72] with a Bricard linkage, another PM with multiple modes can be obtained, which was analysed in [74]. The axis of the joints within the chains can be adjusted by changing the posture of the Bricard linkage and locking specific joints. The mechanism can undergo several 3-DOF motion modes, including the planar mode, spherical mode, spatial mode, zero-torsion mode and general 3-DOF mode.

Fig. 1.41 3-DOF PM with two modes by Kong [72]: (a) translational mode; (b) spherical mode
In [75], the modes analysis of a 3-DOF PM with multiple modes (Fig. 1.42) was carried out, using the Euler parameter quaternions and algebraic geometry approach. Using the proposed method, it is obtained that the mechanism has fifteen modes in total, including four translational modes, six planar modes, four zero-torsion-rate motion modes and one spherical mode. A 4-DOF 3-RER PM with two modes was also put forward in [76].

![Prototype of the 3-RER PM by Kong](image)

Fig. 1.42 Prototype of the 3-RER PM by Kong [76]

### 1.2.5 Deployable/Foldable Mechanisms with Multiple Modes

Combining the merits of deployable mechanisms and mechanisms with multiple modes, deployable mechanisms with multiple modes were proposed. 1-DOF deployable mechanisms which have four assembly modes were discussed in [77]. Using variable R joints, a group of reconfigurable and deployable mechanisms was designed by Wei and Dai [78].

A reconfigurable lift mechanism, composed of many planar linkages with R joints, which can be spread onto a plane and further folded to a bundle was fabricated in [79], as shown in Fig. 1.43.
A rolling mechanism based on a deployable spatial 8R linkage was built in [80]. The linkage is composed of eight links and eight R joints, whose axes are perpendicular to
those of the adjacent R joints. It has multiple motion modes, including two 1-DOF spherical 4R linkage modes in which $R_1$, $R_3$, $R_5$ and $R_7$ (or $R_2$, $R_4$, $R_6$ and $R_8$) intersect at a point while $R_2$, $R_4$, $R_6$ and $R_8$ (or $R_1$, $R_3$, $R_5$ and $R_7$) are locked [Figs. 1.44(a-b)]; a 1-DOF planar 4R linkage mode, by locking $R_2$, $R_4$, $R_6$ and $R_8$ when $R_1$, $R_3$, $R_5$ and $R_7$ are parallel [Fig. 1.44(c)]; a 2-DOF spatial 8R linkage mode [Fig. 1.44(d)], two 1-DOF spatial 6R linkage modes, in which $R_4$ and $R_8$, or $R_1$ and $R_5$ are immobile [Figs. 1.44(e-f)]; and two 1-DOF 2R folding modes, in which $R_1$, $R_3$, $R_4$, $R_5$, $R_7$ and $R_8$, or $R_1$, $R_2$, $R_3$, $R_5$, $R_6$ and $R_7$ are immobile [Figs. 1.44(g-h)].

Wang et al. [81] designed a 16-bar mechanism with two motion modes, including a spherical linkage mode and a planar linkage mode, as shown in Fig. 1.45. When deployed, the mechanism works in the spherical 4R linkage mode and when folded, it turns into the planar 4R linkage mode.

Li et al. [82] constructed a family of reconfigurable DPMs using parallelogram mechanisms. In [83], a reconfigurable angulated element and reconfigurable generalized angulated elements were designed and then used as modules to connect deployable mechanisms. The mechanisms can switch between the Hoberman sphere motion mode [Figs. 1.46(a-b)] and the radially reciprocating motion mode [Figs. 1.46(c-d)].

![Fig. 1.45](image1.png)

**Fig. 1.45** The 16-bar mechanism with two modes [81]: (a) planar linkage configuration; (b) planar linkage mode to spherical linkage mode; (c) spherical linkage configuration

![Fig. 1.46](image2.png)

**Fig. 1.46** Reconfigurable and deployable mechanism by Li [83]: (a-b) Hoberman sphere motion modes; (c-d) radially reciprocating motion modes
However, the structures of most of the deployable mechanisms with multiple modes in the literature are very complicated. Mechanisms with simple structures will be proposed in this thesis.

1.3 Statically Balanced Mechanisms

Energy is one of the important topics all over the world. As we are consuming more and more resources, energy-saving becomes an important issue. Energy-efficient means reducing the energy that is used in the products or services. In the field of robotics, there are many approaches to save energy of the robots. One of the approaches to achieve energy-efficiency is to apply the principle of static balancing and to reduce the energy used to eliminate the effect of gravity or inertia force and moment.

To achieve the balance of the biped human-like robots [84-85], force control methods can be used. The forces and torques can be computed for the robot, according to the CM (centre of mass) and external forces. Then the robots can adjust their postures according to their current states.

The balanced robots using the control method are flexible, but the robots are high-cost, compared with using the method of structure design. Counterweights and springs can be adopted to compensate for the gravity. A system is statically balanced if the total potential energy or torque of the system is constant. Static balancing leads to low actuation forces required to move the devices, and therefore helps improve the efficiency of the mechanism.

![Fig. 1.47 Applications of balanced devices using counterweight [87]: (a) crane; (b) bridge](image)

The principle of force-balance has been applied to our daily life. For example, in the elevators, a heavy counterweight is used to balance the load of the elevator carriage. As
a result, the actuator lifts much less of the weight of the carriage [86]. As shown in Fig. 1.47, counterweights are used on crane or bridge to counterbalance the payload [87].

The Anglepoise lamp is a balanced-arm lamp designed in 1932 by George Carwardine [88], as shown in Fig. 1.48(a). Springs are used to balance the lamp and make sure the lamp head can keep its positions in any postures. Detailed analysis of the design was given in [89]. Another balanced lamp was proposed in [90] by Carwardine. Scissors fork mechanism and springs are adopted to counterbalance the weight of the lamp head, as shown in Fig. 1.48(b).

Bell et al. [91] proposed a statically balanced device that can support the surgical light-head of a surgical light apparatus. The force balanced surgical light was also discussed in [92-93].

In [94], the statically balanced mechanism was applied to arm support for the people suffering from the problems of lifting their arms (Fig. 1.49). The arm support can adjust
the balancer to another payload without external force or energy. The same application can also be found in [95-96].

In [97], a gravity-balanced sit-to-stand assist device was designed using the principle of static balancing (Fig. 1.50). Banala et al. [98] built a device assisting people to walk by reducing or eliminating the effects of gravity using springs, as shown in Fig. 1.51.

![Fig. 1.50 A gravity-balanced sit-to-stand assist device [97]](image)

![Fig. 1.51 A force-balanced walking assist device [98]](image)

![Fig. 1.52 A seating unit with a spring compensation mechanism [99]](image)
In [99], the balancing mechanism with springs is used as a supporting part for a seating unit to support a body, as shown in Fig. 1.52.

Kuo and Lai [100] proposed a novel laparoscope holder used for minimally invasive surgery, as shown in Fig. 1.53. The mechanism is constructed using a parallelogram linkage (positioning arm) and a PM (orientating wrist). The device has a remote centre of motion (RCM) characteristics.

![Fig. 1.53 Statically balanced laparoscope holder in [100]](image)

There are various methods can be adopted to design the force balance mechanisms, such as using mass and lever, zero-free-length springs and torsion beam [101].

1.3.1 Statically Balanced Mechanisms Using Zero-free-length Springs

![Fig. 1.54 Design of the zero-free-length spring using pulley and strings [104]](image)

The zero-free-length springs can be used to design statically balanced mechanisms. The total potential energy of the system is constant when statically balanced. The zero-free-length spring can be achieved using the pulley and strings [102], as shown in Fig. 1.54.

Wongratanaphisan and Cole [103] analysed the gravity-compensated 4R linkage with two springs. The result also shows that there are two stable and two unstable states for the four-bar linkage with springs, as shown in Fig. 1.55.

25
Several statically balanced 1-DOF planar linkages were presented in [105], such as the Stephenson mechanism and Watt mechanism, using only springs. Lin [106] designed the statically balanced SSS (spherical joint) arm and RSR arm using springs, as shown in Fig. 1.56. Dunning and Herder [107] assembled three springs for the gravity balancing of the 2-link arm.

Lustig et al. [108] analysed the statically balanced serial planar linkage using a stiffness matrix approach. Walsh et al. [109] put forward a general methodology to design $n$-spring balancers for the 2-DOF manipulator with yaw-pitch rotation. Robertson et al. [110] investigated the static balancing of a single-loop linkage with two modes using one spring, by attaching the spring right above the intersection of rotation lines in the two modes.
The systems mentioned above are all balanced using only springs. In other literature, to facilitate identifying the CM of the system or attaching the springs, auxiliary parallelograms were added. Rahman et al. (Fig. 1.57) [102] and Herder [104] introduced the statically balanced planar serial manipulators or spatial manipulators composed of planar linkages using auxiliary parallelograms and springs.

Fig. 1.57 Statically balanced planar mechanism with multiple DOFs [104]

Agrawal [112] designed the gravity-balanced 2-link and 3-link leg orthoses using auxiliary parallelograms and non-zero-free-length springs. The static balancing of planar

Fig. 1.58 Statically balanced PMs with springs: (a) spatial 3-DOF PM [114]; (b) spatial 4-DOF PM [115]; (c) spatial 6-DOF PM [116]; (d) spherical 3-DOF PM [117]
3-DOF PMs was addressed in [113]. Gosselin and Wang [114-117] proposed a series of statically balanced mechanisms, including spatial 3-DOF PMs [Fig. 1.58(a)], spatial 4-DOF PMs [Fig. 1.58(b)], spatial 6-DOF PMs [Fig. 1.58(c)] and spherical 3-DOF PMs [Fig. 1.58(d)]. Auxiliary parallelograms were adopted to attach the springs.

In [118], the CM of the system was identified by using auxiliary parallelograms, then springs were used to connect the CM of the manipulator and to balance the spatial manipulators, as shown in Fig. 1.59.

![Fig. 1.59 Gravity-balanced spatial manipulator using auxiliary parallelograms [118]](image)

**1.3.2 Statically Balanced Mechanisms Using Counterweights**

In [119-120], planar 4R linkages were balanced using counterweights. Van der Wijk [121] investigated the statically balanced manipulators, including the 4R linkage and PMs composed of RRR chains, using counterweights. Laliberté discussed the statically balanced 3-DOF planar PM with counterweights in [113]. In [114-117], a family of spatial PMs was designed and balanced using counterweights, including spatial 6-DOF PMs, spatial 3-DOF PMs, and spatial 4-DOF UPS/RUS PMs, as shown in Fig. 1.60.

Russo et al. [122] studied the static balancing of the 6-DOF PM using counterweights or pantograph counterweight. The force balancing of spatial metamorphic 6R linkage was investigated in [123]. Kuo et al. [124] presented the statically balanced design of the reconfigurable mechanism with variable DOFs, using only one counterweight.
1.3.3 Statically Balanced Mechanisms Using Other Methods

Except for balancing the mechanisms using springs or counterweights, there are other methods to balance the mechanisms, such as using gears or cams. In [121], the mechanism was balanced by identifying the CM of the manipulator based on the method of principal vectors proposed by Fischer [125]. Then the CM of the manipulator was mounted on the base. As a result, the mechanism is balanced at any positions since the heights of the CM...
of the manipulator keeps constant, as shown in Fig. 1.62. The similar method was also adopted in [126].

![Fig. 1.62 Statically balanced 5R manipulator in [121]](image1)

In [127], a balancing method was proposed for the 4R linkage using non-circular gears. Gallego and Herder [128], Rijff et al. [129] and Radaelli et al. [130] adopted the torsion springs to develop the gravity-balanced mechanisms. Boisclair et al. [131] addressed the gravity compensation of robotic manipulators using cylindrical Halbach arrays.

### 1.3.4 Statically Balanced Mechanisms Using Combined Methods

The balancing approaches mentioned above can also be combined to design the statically balanced system. The static balancing of the planar 3-DOF PM was addressed in [113], using a combination of both counterweights and springs (Fig. 1.63).

![Fig. 1.63 Static balancing of the planar 3-DOF PM using both counterweights and springs [113]](image2)
Fig. 1.64 Statically balanced 1-link manipulators with spring and cam [132-133]

In [132-133], both cams and springs were adopted to balance the 1-link manipulator. Four solutions were provided, as shown in Fig. 1.64. Koser [134] also achieved the static balancing of a 1-DOF manipulator by using cam and spring.

Van der Wijk and Herder [135] presented the reactionless 4R linkage by using counter-rotary counter-masses, which were also adopted in [136]. Arakelian and Smith [137] dealt with the balancing of the 4R linkage using the counterweight formed by gears.

Cho et al. [138] designed a gravity compensator for the arm with roll-pitch rotation. The system is comprised of two 1-DOF gravity compensators (with springs) and a bevel differential (using gears), as shown in Fig. 1.65. Based on the work in [138], a 4-DOF incomplete balanced manipulator was developed in [139]. Bijlsma et al. [140] designed a gravity equilibrator using geared transmission and torsion bars, as shown in Fig. 1.66.

Fig. 1.65 A gravity compensator for the arm with roll-pitch rotation [138]
1.3.5 Statically Balanced Mechanisms with Variable Payloads

In the cases that the weights of the payloads change, for example for the arm support, the weight of the support changes when picking up or dropping objects. Hence, various static balancing methodologies for the mechanisms with variable payloads were proposed.

In [141], five strategies are described to adapt the balancing under varying payload conditions (Fig. 1.67): (a) changing the positions of the counterweights; (b) changing the joint locations; (c) changing the amount of counterweights; (d) adding additional linkages; (e) adding redundant joints.

Chu and Kuo [142] built a statically balanced 1-DOF manipulator, in which the change of the payload can be sensed by the spring. Cam was adopted to adjust the system to adapt to variable payloads, as shown in Fig. 1.68(a).

Van Dorsser et al. [94] designed an adjustable arm support, based on the method of rearranging the springs without changing their lengths. In [143], the system was adjusted by changing the stiffness of the spring, as shown in Fig. 1.68(b). Wisse et al. [144] utilized the virtual spring concept to design the system with variable payloads. Barents et al. [145] designed a statically balanced cabinet using the spring to spring balancing, as shown in Fig. 1.68(c). In [146-148], the statically balanced mechanisms with variable payloads were also proposed.
1.3.6 Methods of Calculating Positions of Springs or Counterweights

When calculating the positions of the attachment points of springs or counterweights, different methods can be adopted, such as using the algebraic method, geometric method or numerical method. Lin et al. [149] demonstrated several statically balanced spatial
Manipulators, whose design parameters were obtained by diagonalizing the stiffness matrix. This method was also applied in [108]. In [150], a screw-based balancing methodology was introduced to derive the balancing conditions for 4R linkages. In [151], two methods, including the algebraic method and the geometric method, were presented for the balancing of Bennett linkage. Meanwhile, several optimization methods were used to obtain the positions of springs and counterweights. In [152], the static balancing of a medical parallel robot with tension spring, torsion spring or counterweight was achieved by minimizing the ‘motor torque root-mean-square value’. Haines [153] put forward a method for optimizing the ‘root-mean-square shaking moment and/or driving torque’. In [154-155], the ‘linear combination of the resultant bearing force and the input-torque required to drive the linkage’ was set as the objective function to optimize the 4R linkage. The same objective function was also adopted in [156-157]. In [158], a planar PM was optimized by minimizing the ‘sum-squared values of the elements of the position or velocity or acceleration sensitivity matrix’. A ‘mean-square root of the sum squared discrete values of all the reaction forces in the manipulator’ was adopted as the objective function in [159]. The average force was minimized to design the statically balanced mechanism in [160]. Schwarzfischer et al. [161] minimized the difference of the output motion compared with the desired output motion, to design an energy-efficient six bar Watt-II-Mechanism.

Nevertheless, the static balancing method for general spatial mechanisms, especially the ones with multiple modes is still an open issue. The approach proposed in the literature for the mechanisms with multiple modes can only be applied to specific mechanisms, and the masses of the links were neglected and only the mass of the end-effector was considered in most of the references. The optimization methods, however, are more suitable for the systems using counterweights.

1.4 Objectives and Layout of the Thesis

This thesis focuses on the type synthesis and static balancing of a class of deployable mechanisms with multiple modes. A construction approach to the foldable 8R linkages with multiple modes will be introduced first. Then a novel construction method will be proposed to design deployable mechanisms using a symmetric triad. Several mechanisms obtained have multiple motion modes and can switch modes through transition configurations. The mechanisms have simpler structures and fewer DOFs, compared with the mechanisms in the literature. Finally, the deployable mechanisms will be developed.
into statically balanced mechanisms. Novel static balancing approaches will be proposed, including the algebraic method, the geometric method and the optimization method. Using the proposed geometric approaches, the planar mechanisms, spherical mechanisms, the variations of spherical mechanisms and spatial mechanisms will be balanced, with almost no calculation. The mechanisms in this thesis are all composed of links whose weights cannot be neglected, while only the weights of the payloads on the end effectors were considered in the literature. A numerical optimization method will also be introduced, which is suitable for the system with springs. Based on the proposed static balancing methods, the deployable mechanisms with multiple modes can be developed into statically balanced mechanisms.

The thesis is organized as follows. Chapter 2 summarizes the theoretical tools and the fundamentals used in this thesis. The construction method for the single closed-loop foldable mechanisms is described in Chapter 3. Chapter 4 focuses on the design of multi-loop deployable mechanisms constructed using S joints. The S joints in Chapter 4 are replaced by RRR chains in Chapter 5. Chapter 6 describes the statically balanced methods for planar mechanisms, spherical mechanisms and spatial mechanisms using springs, and develops the deployable mechanisms obtained in Chapters 3, 4 and 5 into the statically balanced mechanisms. Finally, conclusions are drawn.
CHAPTER 2 – THEORETICAL TOOLS AND FUNDAMENTALS

In this chapter, the theoretical tools used in this thesis will be introduced. The fundamentals of the static balancing of the mechanisms will also be discussed.

2.1 Mathematical Basis

2.1.1 Distance Between a Point and a Plane Defined by Three Points

Suppose there are three points $P_1$, $P_2$ and $P_3$ that define a plane, $P_4$ is a point out of the plane. The normal vector of the plane $N = \{l \ m \ n\}^T$ can be calculated by

$$N = (P_1 - P_2) \times (P_2 - P_3)$$

(2.1)

The plane equation can be obtained as

$$l(x - P_{1x}) + m(y - P_{1y}) + n(z - P_{1z}) = 0$$

(2.2)

The distance between $P_4$ and the plane is

$$H = \frac{|l(P_{4x} - P_{1x}) + m(P_{4y} - P_{1y}) + n(P_{4z} - P_{1z})|}{\sqrt{l^2 + m^2 + n^2}}$$

(2.3)

The distance equation will be used in Chapter 4 to calculate the deploying ratio of the deployable mechanism.

2.1.2 DOF Analysis

The DOF is the mobility of the mechanism which is constrained by joints. DOF analysis is the basic problem in the process of mechanism design. The DOF of the serial manipulator is the sum of the mobility of all the joints:

$$M = \sum_{i=1}^{p} f_i$$

(2.4)

The DOF of the closed-loop spatial mechanism can be calculated using the conventional formula for DOF. There are several types of formulas, and this thesis will use the formula proposed by Hunt [162-163].

$$M = d(q - p) + \sum_{i=1}^{p} f_i$$

(2.5)

For the single-loop linkages, the formula is simplified as $M = \sum_{i=1}^{p} f_i - 6 + \lambda$. $q$, $p$, and $f_i$ respectively represent the number of mobile links, the number of joints and the freedom of the $i^{th}$ joint. $d$ is the rank of the mechanism, and is equal to $6 - \lambda$, where $\lambda$ is the number of common constraints. For a general spatial mechanism, $d = 6$, and for planar
mechanisms and spherical mechanisms, $d = 3$. In some special cases, the rank of the system can be obtained using screw theory.

Suppose that $s$ is a unit vector along the screw axis and $r$ is the position vector of a point on the screw axis [164]. The unit screw of an R joint or a force [Fig. 2.1(a)] is in the format of

$$s_0 = (s; r \times s) = (l \ m \ n; a \ b \ c)$$

(2.6)

The unit screw of a prismatic (P) joint or a couple [Fig. 2.1(b)] is in the format of

$$s_\infty = (0; s) = (0 \ 0 \ 0; l \ m \ n)$$

(2.7)

Fig. 2.1 Screws: (a) the screw of an R joint or a force; (b) the screw of a P joint or a wrench; (c-d) reciprocal screws

The joint twist and the constraint wrench [Fig. 2.1(c-d)] of a system are reciprocal.

$$s_1 \cdot s_2 = 0$$

(2.8)

The relationship between twist system and its wrench system is [73]:

a) The axis of $s_0$ is coplanar with the axis of any $s_{r0}$.

b) The axis of $s_\infty$ is perpendicular to the axis of any $s_{r0}$.

c) The axis of $s_0$ is perpendicular to the axis of any $s_{r\infty}$.

Hence, when given a mechanism, the twist system can be presented, then the constraint system can be obtained. The number of the common constraints of the mechanism is equal to the rank of the constraint system. Finally, the mobility of the mechanism is obtained.

2.1.3 Notations of D-H Convention

To obtain the positions of the links of the mechanism, the Denavit-Hartenberg (D-H) notation [165] will be introduced. In the D-H convention, the local coordinate frames (Fig. 2.2) are defined as:
a) The $z_i$ axis is along the $i^{th}$ joint axis.

b) The $x_i$ axis points to the $(i+1)^{th}$ joint. If the axes of the $i^{th}$ joint and the $(i+1)^{th}$ joint intersect, $x_i$ is perpendicular to the plane defined by the two joint axes.

c) The $y_i$ axis can be obtained using the right-hand rule.

![Fig. 2.2 D-H parameters](image)

Four D-H parameters below are used to describe each link of the mechanism:

- $a_i =$ the distance from $z_i$ to $z_{i+1}$ measured along $x_i$.
- $\alpha_i =$ the angle between $z_i$ and $z_{i+1}$ measured about $x_i$.
- $d_i =$ the distance from $x_{i-1}$ to $x_i$ measured along $z_i$.
- $\theta_i =$ the angle between $x_{i-1}$ to $x_i$ measured about $z_i$.

The transfer matrix $^{i-1}T_i$ from $(i-1)^{th}$ local frame to $i^{th}$ local frame can be obtained by moving $d_i$ along $z_i$, rotating $\theta_i$ about $z_i$, moving $a_{i-1}$ along $x_i$ and rotating $\alpha_{i-1}$ about $x_i$.

$$^{i-1}T_i = T_x(a_{i-1})R_x(\alpha_{i-1})R_z(\theta_i)T_z(d_i)$$

(2.9)

where

$$T_x(a_{i-1}) = \begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$R_x(\alpha_{i-1}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\alpha_{i-1} & -S\alpha_{i-1} & 0 \\ 0 & S\alpha_{i-1} & C\alpha_{i-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z(\theta_i) = \begin{bmatrix} C\theta_i & -S\theta_i & 0 & 0 \\ S\theta_i & C\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
where \( C \) and \( S \) stand, respectively, for the cosine and the sine of the angles. The transfer matrix is obtained as:

\[
T = \begin{bmatrix}
  C\theta_i & -S\theta_i & 0 & a_{i-1} \\
  C\alpha_{i-1}S\theta_i & C\alpha_{i-1}C\theta_i & -S\alpha_{i-1} & -d_iS\alpha_{i-1} \\
  S\alpha_{i-1}S\theta_i & S\alpha_{i-1}C\theta_i & C\alpha_{i-1} & d_iC\alpha_{i-1} \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]

(2.10)

The position vector of the \( i^{th} \) link in the \( (i-1)^{th} \) local coordinate frame \( ^{i-1}P_i \) can be obtained by multiplying the position vector of the \( i^{th} \) link in the \( i^{th} \) local coordinate frame \( ^{i}P_i \) and the transfer matrix \( ^{i-1}T \) in homogeneous form as

\[
\begin{bmatrix}
  ^{i-1}P_i \\
  1
\end{bmatrix} = ^{i-1}T \begin{bmatrix}
  ^{i}P_i \\
  1
\end{bmatrix}
\]

(2.11)

The position vector of the \( i^{th} \) link in the global coordinate frame can be calculated as

\[
\begin{bmatrix}
  ^{i}P_i \\
  1
\end{bmatrix} = ^{0}T^{1}T^{2}T^{3}T ... ^{i-1}T \begin{bmatrix}
  ^{i}P_i \\
  1
\end{bmatrix}
\]

(2.12)

It is noted that in a closed-loop mechanism, the product of the transfer matrices equals the identity matrix, which means

\[
^{i}T^{1}T^{2}T^{3}T ... ^{i-1}T = I
\]

(2.13)

### 2.1.4 Optimization Toolbox of MATLAB

To obtain the minimum value of a constrained nonlinear function with multiple variables, the optimization toolbox ‘fmincon’ of MATLAB is adopted. The objective function is in the format of \( \text{fun} = f(x(1), x(2) ... x(n)) \). The optimization results can be yielded by using the programming solver of

\[
[x, fval] = \text{fmincon}(\text{fun}, x0, [], [], [], [], lb, ub, \text{noncon}, \text{options})
\]

where \( x \) starts from \( x0 \) and attempts to find a minimizer \( x \) of the function. \( lb \) and \( ub \) are a set of lower and upper bounds on the design variable. The obtained result is a set of variables that satisfies the constraints, and the minimum value of the objective function.

### 2.2 Mass Moment Substitution

In the process of balancing mechanisms using the geometric method, the mass of certain links can be replaced by several equivalent masses located on the axes of the joints. As a consequence, the mass of the link is distributed onto its adjacent links. Using the mass
moment substitution methods, one can reduce the static balancing of mechanisms to that of several serial or tree-like kinematic chains. Three cases in which the link has two R joints with parallel axes, intersecting axes or skew axes will be presented respectively.

2.2.1 Mass Moment Substitution of an RR Link with Parallel Joint Axes

The mass moment substitution of an RR link with two parallel joint axes is shown in Fig. 2.3. The joint axes of \( R_i \) and \( R_{i+1} \) are parallel. The \( i \)th local coordinate frame is set at \( O \), which is a point on the axis of \( R_i \). \( z_i \) is along the joint axis of \( R_i \) and \( x_i \) is perpendicular to the plane defined by \( R_i \) and \( R_{i+1} \). Suppose that the distance between \( R_i \) and \( R_{i+1} \) is \( t_0 \), the position vectors of the two point-masses expressed in the \( i \)th local frame are

\[
\begin{align*}
{P_{i1}} &= \begin{bmatrix} 0 \\ 0 \\ t_1 \end{bmatrix}^T \\
{P_{i2}} &= \begin{bmatrix} 0 \\ t_0 \\ t_2 \end{bmatrix}^T
\end{align*}
\]  

(2.14)

![Fig. 2.3 Mass moment substitution of an RR link with parallel joint axes](image)

The mass and mass moment (about \( O \)) of link \( i \) should be equal to those of the two point-masses. Therefore

\[
\begin{align*}
m_i &= m_{i1} + m_{i2} \\
m_i g {P_i} &= m_{i1} g {P_{i1}} + m_{i2} g {P_{i2}}
\end{align*}
\]  

(2.15)

The following conditions can be obtained by solving Eq. (2.15).

\[
\begin{align*}
a_i &= 0 \\
t_1 &= (c_i t_0 - b_i t_2) / (t_0 - b_i) \\
m_{i2} &= m_i b_i / t_0 \\
m_{i1} &= (m_i c_i - m_{i2} t_2) / t_1
\end{align*}
\]  

(2.16)

where \( \{a_i \ b_i \ c_i\}^T \) represents the position of the CM of the \( i \)th link with respect to the \( i \)th local coordinate frame.

The mass of an RR link with parallel joint axes can be replaced by two equivalent masses on the two R joints when the CM of the link is on the plane defined by the two
axes of the $R$ joints. Besides, the CM of the link and the two equivalent masses should be collinear. The position of the CM of the augmented $(i-1)^{th}$ link in the $(i-1)^{th}$ local frame can be obtained by combining the mass moment of the $(i-1)^{th}$ link and the first mass-point.

$$i-1 P_{i-1}' = (m_{i-1} g i-1 P_{i-1} + m_{1} g i-1 P_{1})/(m_{i-1} + m_{1}) g \quad (2.17)$$

Similarly, the CM of the augmented $(i+1)^{th}$ link in the $(i+1)^{th}$ local frame can be calculated by

$$i+1 P_{i+1}' = (m_{i+1} g i+1 P_{i+1} + m_{2} g i+1 P_{2})/(m_{i+1} + m_{2}) g \quad (2.18)$$

The positions of the CMs of the augmented links in the local frames are then obtained.

### 2.2.2 Mass Moment Substitution of an RR Link with Intersecting Joint Axes

The RR link with intersecting joint axes is shown in Fig. 2.4, the joint axes of $R_i$ and $R_{i+1}$ intersect at $O$. The angles between $R_i$ and $R_{i+1}$ are noted as $\alpha$, the position vectors of the two mass-points in the $i^{th}$ local frame can be yielded as [167]

$$\begin{align*}
    i P_{i1} &= \{0 \quad 0 \quad t_1\}^T \\
    i P_{i2} &= \{0 \quad t_2 S \alpha \quad t_2 C \alpha\}^T
\end{align*} \quad (2.19)$$

![Fig. 2.4 Mass moment substitution of an RR link with intersecting joint axes](image)

The following conditions are obtained by substituting Eq. (2.19) into Eq. (2.15).

$$\begin{align*}
a_i &= 0 \\
t_1 &= t_2 C \alpha (c_i \tan \alpha - b_i)/(t_2 S \alpha - b_i) \\
m_{i2} &= m_i b_i / t_2 S \alpha \\
m_{i1} &= (m_i c_i - m_{i2} t_2 C \alpha)/t_1
\end{align*} \quad (2.20)$$

It is observed that the mass moment substitution conditions are: the CM of the link and the two R joints are coplanar and the CM of the link is on the line defined by the two mass-points.
2.2.3 Mass Moment Substitution of an RR Link with Skew Joint Axes

Moving $R_{i+1}$ by $t_0$ along $x_n$ in Section 2.2.2, the link with two skew R joints is obtained, as shown in Fig. 2.5. The angle and distance between $R_i$ and $R_{i+1}$ are denoted as $\alpha$ and $t_0$ respectively. $O_{2P}$, $P_{12P}$ and $R_{ij}$ are the projections of $O_2$, $P_{12}$ and $R_i$ on the $z_jy_i$ plane.

![Fig. 2.5 Mass moment substitution of an RR link with skew joint axes](image)

The position vectors of the two mass-points relative to the $i^{th}$ local coordinate frame are represented by [168-169]

\[
\begin{align*}
^{i}P_{t1} & = \{0 \quad 0 \quad t_1\}^T \\
^{i}P_{t2} & = \{t_0 \quad t_2S\alpha \quad t_2C\alpha\}^T
\end{align*}
\] (2.21)

The masses and the positions of the two point-masses are obtained as

\[
\begin{align*}
m_{t2} & = m_i a_i/t_0 \\
m_{t1} & = (m_i c_n - m_t2 C\alpha)/t_1 \\
t_2 & = b_i t_0/a_i S\alpha \\
t_1 & = t_0(c_i \tan \alpha - b_i)/\tan(\alpha - a_i)
\end{align*}
\] (2.22)

The CM of the link and the two masses point should be collinear.

2.3 Static Balancing Method and Its Extensions

In this section, the basic principle of static balancing will be introduced. The statically balanced 1-link manipulator will be presented and then the static balancing method will be extended to apply to the manipulators with multi-link and multi-DOF.

2.3.1 One-link Manipulator
According to [117], a link manipulator mounted to the base using an S joint can be statically balanced using only one zero-free-length spring. The spring connecting point \( H \) on the base is right above the S joint and the height is assumed to be \( h \). \( b \) is the position vector of the spring connecting point on the manipulator and \( r \) is the position vector of the CM of the manipulator. Vectors \( b \) and \( r \) must be proportional [117], i.e.,

\[
\mathbf{r} = (k h / m g) \mathbf{b}
\]

where \( m \), \( g \) and \( k \) represent the mass of the manipulator, the gravitational acceleration, and the stiffness of spring respectively. The spring connecting point on the manipulator should be on the line defined by the CM of the manipulator and the centre of the S joint [Fig. 2.6(a)]. \( h = m g / k \) when attaching the spring to the CM of the manipulator directly, for the sake of convenience of calculation and description [Fig. 2.6(b)]. This condition can also apply to one-link manipulators with an R joint [104] or a universal (U) joint.

\[ h = \| \mathbf{b} \| h \]

where \( \mathbf{b} \) is the position vector of the spring connecting point on the manipulator and \( r \) is the position vector of the CM of the manipulator.

For the 1-link manipulator mounted to the base using an R joint, the spring can be attached to an arbitrary point right above the axis of the R joint. As shown in Fig. 2.7(a), the angle between the axis of the R joint and the vertical axis is denoted by \( \alpha \) (with constant value), and the position of the CM of the manipulator in the local coordinate frame is \( \{ a \ b \ c \}^T \). \( T \) is a point on the axis of the R joint, \( OT = t \). \( H \) is right above \( T \), with a distance of \( h \). Now the special cases that no spring is required to balance the manipulator will be discussed.

The position of \( T \), \( H \) and the CM of the manipulator \( P \) are computed as

\[
\mathbf{T} = \{ 0 \ -t S\alpha \ t C\alpha \}^T
\]

\[
\mathbf{H} = \{ 0 \ -t S\alpha \ t C\alpha + h \}^T
\]

\[
\mathbf{P} = \{ a C\theta - b S\theta \ \ b C\theta C\alpha - c S\alpha + a C\alpha S\theta \ \ c C\alpha + b C\theta S\alpha + a S\theta S\alpha \}^T
\]

43
Fig. 2.7 Statically balanced 1-DOF 1-link manipulator: (a) the 3D model of the manipulator; (b) the CM of the link is on the joint axis; (c) the axis of the R joint is vertical.

One spring is adopted, with one end attached to $H$, and the other end to the CM of the manipulator. The potential energy of the link consists of two parts, including the potential energy associated with gravity ($V_{mi}$) and the elastic potential energy stored in the springs ($V_{si}$). The total potential energy of the manipulator is

$$V = \frac{1}{2} k|P - H|^2 + mg P_2 = \frac{1}{2} k[a^2 + b^2 + h^2 + (c - t)^2] + (-chk + cmg + hkt)C\alpha - (hk - mg)S\alpha(bC\theta + aS\theta)$$

(2.27)

When the manipulator is statically balanced, the total potential energy should be constant. One can obtain that

$$h = \frac{mg}{k} \quad \text{or}$$

$$a = 0 \quad \text{and} \quad b = 0 \quad \text{or} \quad \alpha = 0^\circ$$

(2.28) (2.29)

Eq. (2.29) indicates that the system is balanced if the CM of the link is on the joint axis [Fig. 2.7(b)], or the axis of the first R joint is vertical [Fig. 2.7(c)].

2.3.2 Two-link Manipulators with 2-DOF

Based on the static balancing method of the 1-link manipulator above, the multi-link manipulator with multiple DOFs constructed using the 1-link manipulator can also be balanced. In this section, the static balancing approach of several 2-link manipulators with 2-DOF will be introduced.
The 2-link manipulator with yaw-roll rotation is presented in Fig. 2.8. The manipulator is constructed using the two 1-link manipulators in Fig. 2.7(a) and Fig. 2.7(c) respectively. It has been proved that the manipulator in Fig. 2.7(c) has no need to balance, so only one spring is used to balance the 2-link manipulator with yaw-roll rotation. One end of the spring is attached to $H$, which is fixed on link 1 and is right above $R_2$ with a height of $h$ ($h = m_2 g/k$), the other end is attached to $P_2$.

A fixed coordinate frame is attached to the base, with its origin at the intersection of the axes of the two R joints, and the $z$-axis is pointing vertically upward. Suppose the position vector of the CM of the $i$th link in the local frame $i$ is represented by

$$i \mathbf{P}_i = \begin{bmatrix} a_i & b_i & c_i \end{bmatrix}^T \tag{2.30}$$

The position vectors of the global CMs of the two links are calculated using the D-H approach as

$$\mathbf{P}_1 = \begin{bmatrix} a_1 C \theta_1 - b_1 S \theta_1 & b_1 C \theta_1 + a_1 S \theta_1 \end{bmatrix}^T + \begin{bmatrix} 0 \quad 0 \quad c_1 \end{bmatrix} \tag{2.31}$$

$$\mathbf{P}_2 = \begin{bmatrix} P_{2x} \quad P_{2y} \end{bmatrix} - b_2 C \theta_2 - a_2 S \theta_2 \tag{2.32}$$

where

$$P_{2x} = a_2 C \theta_1 C \theta_2 - b_2 C \theta_1 S \theta_2 - c_2 S \theta_1$$

$$P_{2y} = a_2 C \theta_2 S \theta_1 - b_2 S \theta_1 S \theta_2 + c_2 C \theta_1$$

The spring connecting point $H$ on the base is given by

$$\mathbf{H} = \begin{bmatrix} 0 & 0 & m_2 g/k \end{bmatrix}^T \tag{2.33}$$

The total potential energy of the manipulator is obtained as
\[ V = V_s + V_m = \frac{1}{2} k \lvert \mathbf{P}_2 - \mathbf{H} \rvert^2 + m_2 g \mathbf{P}_{2z} + m_1 g \mathbf{P}_{1z} = (a_2^2 k^2 + b_2^2 k^2 + c_2^2 k^2 + m_2^2 g^2) / 2k + m_1 g c_1 \]  

(2.34)

Eq. (2.34) verifies the system is constant in any configurations. Similarly, the manipulator in Fig. 2.9 in which the first link is horizontal is balanced. Instead of attaching the spring to the CM of the link, the attachment point \( B_2 \) can be any point on the line defined by \( P_2 \) and \( R_2 \).

Fig. 2.9 Statically balanced 2-link manipulator with yaw-roll rotation II

The manipulator with pitch-roll rotation is shown in Fig. 2.10. It consists of two moving links connected by two R joints with intersecting and orthogonal axes. The first link rotates around the \( y \)-axis which is horizontal and the axis of the second R joint is perpendicular to the axis of the first R joint.

Two springs are used to balance the two links of the manipulator respectively. The spring connecting point on the base \( \mathbf{H}_1 \) for balancing of the first link can be any point right above the axis of first R joint. The other end is attached to the CM of the first link. One end of the second spring \( \mathbf{H}_2 \) is located on the vertical axis passing through the intersection of the two R joints, the other end is fixed on the CM of the second link.

Fig. 2.10 Statically balanced 2-link manipulator with pitch-roll rotation: (a) the sketch of the manipulator; (b) the 3D model of the manipulator
The position vectors of the CMs of the links in the global coordinate frame can be obtained as

\[
P_1 = \{-b_1 S \theta_1 + a_1 C \theta_1 \quad c_1 \quad -a_1 S \theta_1 - b_1 C \theta_1\}^T
\]

\[
P_2 = \{P_{2x} \quad -b_2 C \theta_2 - a_2 S \theta_2 \quad P_{2z}\}^T
\]

where

\[P_{2x} = a_2 C \theta_2 C \theta_1 - b_2 C \theta_1 S \theta_2 - c_2 S \theta_1\]

\[P_{2z} = -a_2 S \theta_1 C \theta_2 + b_2 S \theta_1 S \theta_2 - c_2 C \theta_1\]

The springs connecting points \(H_1\) and \(H_2\) on the base are given by

\[
H_1 = \{0 \quad -s \quad m_1 g / k\}^T
\]

\[
H_2 = \{0 \quad 0 \quad m_2 g / k\}^T
\]

The total potential energies of both links are expressed as

\[
V_1 = V_{s1} + V_{m1} = \frac{1}{2} k |P_1 - H_1|^2 + mg P_{1z}
\]

\[
= (a_1^2 k^2 + b_1^2 k^2 + c_1^2 k^2 + m_1^2 g^2 + 2 c_1 s k^2 + s^2 k^2)/2k
\]

\[
V_2 = V_{s2} + V_{m2} = \frac{1}{2} k |P_2 - H_2|^2 + mg P_{2z} = (a_2^2 k^2 + b_2^2 k^2 + c_2^2 k^2 + m_2^2 g^2)/2k
\]
2.3.3 Multi-link Manipulators with 3-DOF

Based on the results in Section 2.3.1 and 2.3.2, multi-link manipulators with 3-DOF can also be balanced. In this section, statically balanced manipulators with two links assembled on one R joint and one universal (U) joint, and those with three links connected using three R joints are presented. The axes of the R joints, including the ones within the U joint, are always perpendicular to each other in the initial state.

![Diagram](image)

**Fig. 2.12** Statically balanced 2-link manipulator with yaw-pitch-roll rotation: (a) the sketch of the manipulator; (b) the 3D model of the manipulator

The manipulator with yaw-pitch-roll rotation is constructed using a 1-link manipulator in Fig. 2.7(c) and a 1-link manipulator mounted on a U joint, as shown in Fig. 2.12. The first R joint is along \( z \)-axis which is pointing vertically upward and the two axes of the U joint are on the plane that perpendicular to the R joint in the initial state. The position vectors of the global CMs of the links are

\[
P_1 = \{a_1 C \theta_1 - b_1 S \theta_1 \quad b_1 C \theta_1 + a_1 S \theta_1 \quad c_1\}^T
\]

\[
P_2 = \{P_{2x} \quad P_{2y} \quad -a_2 C \theta_2 C \theta_3 + b_2 C \theta_2 S \theta_3 - c_2 S \theta_2\}^T
\]

where

\[
P_{2x} = S \theta_1 (b_2 C \theta_3 + a_2 S \theta_3) + C \theta_1[-c_2 C \theta_2 + S \theta_2(a_2 C \theta_3 - b_2 S \theta_3)]
\]

\[
P_{2y} = -C \theta_1 (b_2 C \theta_3 + a_2 S \theta_3) + S \theta_1[-c_2 C \theta_2 + S \theta_2(-a_2 C \theta_3 + b_2 S \theta_3)]
\]

The springs connecting point \( H \) on the base is

\[
H = \{0 \quad 0 \quad m_2 g / k\}^T
\]
Only one spring is needed to balance the manipulator. The total potential energy of the system is expressed as

\[
V = V_e + V_m = \frac{1}{2} k |\mathbf{P}_2 - \mathbf{H}|^2 + m_2 g P_{2z} + m_1 g P_{1z}
\]

\[
= (a_2^2 k^2 + b_2^2 k^2 + c_2^2 k^2 + m_2^2 g^2) / 2k + m_1 g c_1
\]

which is a constant. Similarly, the manipulator in Fig. 2.13 is balanced.

![Fig. 2.13 S...er with yaw-pitch-roll rotation](image)

The manipulator with pitch-yaw-roll rotation is shown in Fig. 2.14. The axis of the R joint is on the horizontal plane and the two axes of the U joint are on the plane that is perpendicular to the axis of the R joint in the initial state.

![Fig. 2.14 S...er with pitch-yaw-roll rotation: (a) the sketch of the manipulator; (b) the 3D model of the manipulator](image)

Two springs are used to balance the two links of the manipulator respectively. Like the 2-DOF 2-link manipulator, one end of each spring is attached right above the joint, while the other end is fixed on the CM of the link.

The position vectors of the CMs of the two links are obtained as

\[
\mathbf{P}_1 = \{a_1 c\theta_2 - b_1 S\theta_1 \quad c_1 \quad -a_1 S\theta_1 - b_1 C\theta_1\}^T
\]
\[
\mathbf{P}_2 = \{ P_{2x} \ b_2 C \theta_2 S \theta_3 - a_2 C \theta_2 C \theta_3 - c_2 S \theta_2 \ P_{2z} \}^T
\]  
(2.46)

where

\[
P_{2x} = S \theta_1 (b_2 C \theta_3 + a_2 S \theta_3) + S \theta_2 (a_2 C \theta_2 - b_2 S \theta_3)
\]

\[
P_{2z} = C \theta_1 (b_2 C \theta_3 + a_2 S \theta_3) + S \theta_2 (a_2 C \theta_2 + b_2 S \theta_3)
\]

The spring connecting points \( \mathbf{H}_1 \) and \( \mathbf{H}_2 \) on the base are

\[
\mathbf{H}_1 = \{ 0 \ -s \ m_1 g/k \}^T
\]
(2.47)

\[
\mathbf{H}_2 = \{ 0 \ 0 \ m_2 g/k \}^T
\]
(2.48)

The potential energies of the two links are

\[
V_1 = V_{s1} + V_{m1} = \frac{1}{2} k |\mathbf{P}_1 - \mathbf{H}_1|^2 + mg \ P_{1z}
\]

\[
= (a_1^2 k^2 + b_1^2 k^2 + c_1^2 k^2 + m_1^2 g^2 + 2c_1sk^2 + s^2k^2)/2k
\]

\[
V_2 = V_{s2} + V_{m2} = \frac{1}{2} k |\mathbf{P}_2 - \mathbf{H}_2|^2 + mg \ P_{2z} = (a_2^2 k^2 + b_2^2 k^2 + c_2^2 k^2 + m_2^2 g^2)/2k
\]

(2.49)

The results imply that the manipulator with pitch-yaw-roll rotation is statically balanced through its range of motion.

The manipulators in Figs. 2.15 and 2.16 are constructed using the 1-link manipulator in Fig. 2.7 (c) and the 2-link manipulator in Fig. 2.10. Similarly, the first link has no need to balance, and the second link and the third link can be balanced using two springs.

![Fig. 2.15 Statically balanced 3-link manipulator with yaw-roll-pitch rotation I](image-url)
2.4 Summary

In this chapter, the theoretical tools, including the calculation of the distance, DOF and kinematic analysis and the optimization tools, have been introduced. The optimization tools will be used on the calculations of the minimum of the potential energy variance to obtain the positions of the attachment of the springs. The fundamentals of static balancing of the manipulators, including the 1-link, 2-link and 3-link manipulators, have been discussed. The static balancing approaches proposed in this thesis will be based on the above statically balanced manipulators.
CHAPTER 3 – TYPE SYNTHESIS OF DEPLOYABLE SINGLE-LOOP 8R LINKAGES WITH MULTIPLE MODES

In this section, a novel construction method for the single-loop foldable mechanisms with multiple modes will be introduced. Two types of 8R mechanisms can be obtained using the proposed method.

3.1 Construction Method

The mechanisms can be fully folded by offsetting the panels away from the planes defined by the joints (offset distance $d$ is equal to one half of the thickness of the panel). The modified planar 4R linkage and spherical 4R linkage are presented in Fig. 3.1.

![Fig. 3.1 Mono-mode foldable mechanisms: (a) planar 4R linkage; (b) spherical 4R linkage](image)

A spatial foldable 8R mechanism 1-5-2-6-3-7-4-8 can be obtained by connecting two foldable 4R linkages 1-2-3-4 and 5-6-7-8. The first type of 8R mechanisms is constructed by combining two spherical 4R linkages, as shown in Fig. 3.2(a). The second type of 8R mechanisms is designed by connecting a planar 4R linkage and a spherical 4R linkage, as shown in Fig. 3.2(b). The two 4R linkages are staggered rather than connecting consecutively for two reasons: the former configuration is symmetric and the latter one cannot be spread onto a plane.

The two types of 8R mechanisms are listed in Table 3.1. The orders of wrench systems and the DOF analysis are also given, using the formula described in Section 2.1.2. It is noted that the DOF obtained is the instantaneous DOF. The singular positions will be identified.
Table 3.1. The obtained foldable 8R mechanisms

<table>
<thead>
<tr>
<th>Type</th>
<th>Mechanism</th>
<th>Order of wrench system</th>
<th>M</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spherical 4R</td>
<td></td>
<td>λ=4 (singular positions)</td>
<td>M=6</td>
</tr>
<tr>
<td></td>
<td></td>
<td>λ=3 (general positions)</td>
<td>M=5</td>
</tr>
<tr>
<td>-planar 4R mechanism</td>
<td></td>
<td>λ=2 (singular positions)</td>
<td>M=4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>λ=1 (general positions)</td>
<td>M=3</td>
</tr>
<tr>
<td>Spherical 4R</td>
<td></td>
<td>λ=1 (singular positions)</td>
<td>M=3</td>
</tr>
<tr>
<td>-planar 4R mechanism</td>
<td></td>
<td>λ=0 (general positions)</td>
<td>M=2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>λ=0</td>
<td>M=2</td>
</tr>
</tbody>
</table>

3.2 Double-centred 8R Mechanism

Figure 3.3 shows an origami that we see frequently. We regard the panels as rigid bodies and the creases as revolute joints, then the origami turns into a 16-bar linkage. An 8R mechanism which is the first mechanism in Table 3.1 can be achieved by removing the outer eight links. All the revolute joints of the mechanism are on the same plane and intersect at one point when spread. The instantaneous DOF of the mechanism is 5 and is 6 when spread.
Fig. 3.3 Design process of the mechanism: (a) 16-link origami; (b) single-centred 8R mechanism evolved from origami

To have better foldability, several approaches can be adopted, such as offsetting the R joints or using flexible joints. A prototype of the single-centred 8R mechanism of which the revolute joints are designed with soft materials is shown in Fig. 3.4. It can be seen from Figs. 3.4(b-c) that the mechanism can be fully folded. Another merit of the prototype with flexible joints is that they can recover to the initial state after working. The structure is the same as the waterbomb base in [41].

Fig. 3.4 Single-centred 8R mechanism with compliant joints: (a) spread state; (b-c) folded states

Another method is to offset the R joints. The resulting mechanism is a double-centred 8R mechanism. As shown in Fig. 3.5, the panels are designed in a shape of the isosceles right triangle and are connected to another two panels by two joints that are mounted at the base and the hypotenuse of the triangle. The angle between the two joints is 45 degrees and the distance between them is equal to the thickness of the panels. As a consequence, the axes of joints 1, 2, 3 and 4 intersect at a point, while those of 5, 6, 7 and 8 intersect at another point. The projections of all the joints onto the ground plane intersect at a point. The initial state of the mechanism, which is a singular configuration is referred to the position when spread onto a plane. In the initial state, the two planes that are generated by the axes of the joints of the two spherical 4R linkages respectively are parallel with
the plane defined by the panels. The line defined by the two centres of the two spherical 4R linkages are perpendicular to the plane defined by the joint axes of each spherical 4R linkage.

![Fig. 3.5 Double-centred 8R mechanism: (a-b) spherical 4R linkage modes; (c-d) planar 4R linkage modes](image)

The mechanism can behave as: (a) spherical 4R linkages when driving joints 5, 6, 7 and 8 (or 1, 2, 3 and 4) synchronously and then locking the joints of one of the spherical 4R linkages [Figs. 3.5(a-b)]; (b) planar 4R linkages when actuating joints 5, 6, 7 and 8 (1, 2, 3 and 4) to be parallel and locking joints 1, 2, 3 and 4 (5, 6, 7 and 8), as shown in Figs. 3.5(c-d); and (c) spatial 8R linkage. The DOF of the mechanism is 4 in the initial state (singular position) and reduces to 3 in other positions. The mechanism can be folded into two layers as shown in Fig. 3.6(b) when rotating joints 6 and 8 (or 5 and 7). In this position, \( R_1//R_4 \) (the axes of R joint 1 and 4 are parallel), \( R_5//R_7 \) and \( R_2//R_3 \). It can also be folded into four layers as shown in Fig. 3.6(c) in which state \( R_5//R_6//R_7//R_8 \). Following a similar procedure, when folding along axes of joints 1 and 3 (or 2 and 4), the mechanism deforms into the position in Fig. 3.6(d) and can be deployed into the posture in Fig. 3.6(e).

The pins for the axes of the revolute joints are manufactured with soft material in order to easily assemble and disassemble the mechanism without breaking the rigid part. This flexible pin provides a novel method to design easy-assemble revolute joints without changing the kinematic characteristics of the mechanism, considering the pins barely have deformation along the directions that are perpendicular to the axis of the pin.
Combining a spherical 4R linkage and a planar 4R linkage, a perpendicular-axis 8R mechanism shown in Fig. 3.7(a) can be obtained. This mechanism has been discussed in [80]. The two joints on the same panels are designed at the base and the vertex of the triangle links and are perpendicular to each other. In the initial configuration, \( R_1 || R_2 || R_3 || R_4 \), joints 5, 6, 7 and 8 lie on the same plane and intersect at a point. The axes of joints 5 and 7 are collinear, those of 6 and 8 are also collinear, and \( R_5 \) is perpendicular to \( R_6 \). The instantaneous DOF of the mechanism is three in this position.

To improve the foldability, a novel mechanism which is a variation of the perpendicular-axis 8R mechanism is designed as follows. Starting from the configuration when the mechanism is spread onto a plane [Fig. 3.7(a)], lift modules A and C by a distance \( h \), which is the thickness of the panels, apart from B and D along the direction that is perpendicular to the panels [Fig. 3.7(b)]. As a consequence, the axes of joints 5, 6, 7 and 8 are on the same plane and intersect at two points in the initial configuration, \( R_5 || R_7 \), and \( R_6 || R_8 \). The order of the wrench system in this configuration is zero and the DOF of the mechanism is two. The configuration can be further modified into the one shown in Fig. 3.8(a), in which joints 1, 2, 3 and 4 are offset away from the central planes of the panels and the panels are cut to enable the links connected by \( R_5 \) to be embedded into the links connected by \( R_7 \) in the folding process. An additional part is assembled to assure concentric of joint \( R_7 \).
The mechanism has three modes: (a) planar 4R linkage mode [Fig. 3.8(a)] when locking joints 5, 6, 7 and 8 in the initial configuration, (b) spherical 4R linkage mode when the axes of joints 5, 6, 7 and 8 intersect at a point and joints 1, 2, 3 and 4 are locked [Fig. 3.8(f)], (c) spatial 8R linkage mode. In the spherical linkage mode, the axes of joints 1, 2, 3 and 4 of the mechanism in [80] have a common point, so do joints 5, 6, 7 and 8. While only the axes of joints 5, 6, 7 and 8 intersect at a point in the proposed mechanism.

When rotating $R_6$ and $R_8$ in the initial configuration, the mechanism turns into the position in which $R_1 \perp R_4$ (the axes of joints 1 and 4 are perpendicular to each other), $R_2 \perp R_3$ and $R_5 \perp R_7$, as shown in Fig. 3.8(b). Then locking $R_6$ and $R_8$, the mechanism deforms into a 1-DOF triple-centred six-bar linkage. By rotating $R_5$ clockwise or anticlockwise, the 6R linkage can be spread into one layer, when $R_5//R_6//R_7//R_8$, joints 1, 2, 3 and 4 are on two parallel planes and their projections on any one of planes that are perpendicular to joints 5, 6, 7 and 8 intersect in a point, as shown in Fig. 3.8(g), or be folded into two layers in which the projections of joints 1, 2, 3 and 4 on the ground plane generate a square, joints 5, 6, 7 and 8 are parallel without the consideration of interference [see Fig. 3.8(c)].
Fig. 3.8 Folding process of variation of perpendicular-axis 8R mechanism
The mechanism can be folded into two layers through two approaches by alternatively fixing joints 5, 6, 7 and 8 and rotating any one of joints 1, 2, 3 and 4 in the initial configuration [Fig. 3.8(e)] or rotating any one of joints 5, 6, 7 and 8 when the mechanism is in its spread state [Figs. 3.8(h) and (i)]. In the two folded states, joints 1, 2, 3 and 4 are parallel with each other, joints 5, 6, 7 and 8 are also parallel. The configuration in Fig. 3.8(i) deforms to the posture in Fig. 3.8(d) by driving $R_5$ and $R_7$ 90 degrees. Then the configuration in Fig. 3.8(e) can be turned into the ones in Figs. 3.8(k) and (l) and further be folded into four layers which is shown in Fig. 3.8(j) by actuating joints 5, 6, 7 and 8. In this state, joints 1, 2, 3 and 4 are on a plane that is perpendicular to the planes of the panels, as well as the plane defined by the axes of joints 5, 6, 7 and 8.

![Fig. 3.9 32-bar mechanism](image)

Since the side of the triangle is longer than its height, the process of folding into four layers of this prototype is achieved using clearance. If we change the shapes of the panels, for example rounding the base angle of the triangles, the mechanism can be folded into four layers without utilizing the clearance. The folding ratio between the maximum volume [the cube into which the mechanism would exactly fit, shown in Fig. 3.8(a)] and the minimum volume [shown in Fig. 3.8(j)] is $r \approx 9.54$. This shows that the mechanism has excellent folding performance.

Cutting the links of the mechanism in Fig. 3.7(a) in half, reassembling the mechanism, and overlapping four mechanisms, then the 32-bar mechanism in Fig. 3.9 is obtained. The
folding ratio between the maximum area [shown in Fig. 3.9(a)] and the minimum area [shown in Fig. 3.9(j)] of the mechanism \( r = 32 \).

### 3.4 Summary

Two types of single-loop foldable 8R mechanisms with multiple modes have been presented. The mechanisms are composed of eight thick triangle panels and eight R joints that are mounted on the side or vertex of the triangles. The first type is constructed by connecting two spherical 4R linkages; the second type is the mechanisms combining a planar 4R linkage and a spherical 4R linkage. The mechanisms have the modes that inherited from the two original 4R linkages and an additional spatial 8R linkage mode.

The mechanisms can spread onto a plane and emerge out of the plane. In addition, they can be folded onto two planes and four planes in different ways to facilitate storage and transportation. The mechanisms have the potential to be applied to roofs [Fig. 3.10(a)], solar panels [Fig. 3.10(b)] and space mirrors, and are suitable for 3D-printing. This characteristic also provides an opportunity to develop 4D printing mechanisms, which can transform from one state to another directly off the build plate.

![Fig. 3.10 The applications of the 8R mechanisms: (a) roofs; (b) solar panels](image)
Even though the two types of mechanisms have different configurations, they have common motion modes and similar folded states, as shown in Table 3.2. Both of them can be spread onto a plane. In addition, they can be folded into two layers in which posture $R_1//R_4$, $R_2//R_3$ or $R_1//R_3$, $R_2$ and $R_4$ are collinear and then further folded into four layers in which the axes of joints 1, 2, 3 and 4 are on the same plane and parallel with each other. Both of the mechanisms have multiple modes, including planar 4R modes when joints 1, 2, 3 and 4 are parallel with each other, spherical 4R modes when joints 5, 6, 7 and 8 intersect and spatial 8R modes.

### Table 3.2. Comparisons of the two types of mechanisms

<table>
<thead>
<tr>
<th></th>
<th>One layer</th>
<th>Two layers</th>
<th>Two layers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spherical-Spherical</td>
<td><img src="image1.png" alt="Image" /></td>
<td><img src="image2.png" alt="Image" /></td>
<td><img src="image3.png" alt="Image" /></td>
</tr>
<tr>
<td>Spherical-Planar</td>
<td><img src="image4.png" alt="Image" /></td>
<td><img src="image5.png" alt="Image" /></td>
<td><img src="image6.png" alt="Image" /></td>
</tr>
<tr>
<td>Four layers</td>
<td><img src="image7.png" alt="Image" /></td>
<td><img src="image8.png" alt="Image" /></td>
<td><img src="image9.png" alt="Image" /></td>
</tr>
<tr>
<td>Planar mode</td>
<td><img src="image10.png" alt="Image" /></td>
<td><img src="image11.png" alt="Image" /></td>
<td><img src="image12.png" alt="Image" /></td>
</tr>
<tr>
<td>Spherical mode</td>
<td><img src="image13.png" alt="Image" /></td>
<td><img src="image14.png" alt="Image" /></td>
<td><img src="image15.png" alt="Image" /></td>
</tr>
<tr>
<td>Spherical-Planar</td>
<td><img src="image16.png" alt="Image" /></td>
<td><img src="image17.png" alt="Image" /></td>
<td><img src="image18.png" alt="Image" /></td>
</tr>
</tbody>
</table>
In the previous chapter, single-loop 8R linkages have been constructed. This chapter is to design deployable multiple-loop mechanisms by connecting orthogonal single-loop linkages. An orthogonal single-loop linkage refers to a linkage with an even number of identical links with orthogonal R joint axes and no offset. A novel construction method will be proposed to obtain DPMs using S joints.

4.1 DPMs Based on Identical Bricard Linkages

In this section, deployable mechanisms based on identical orthogonal Bricard linkages will be designed.

4.1.1 Analysis of Bricard Linkage

Fig. 4.1 Orthogonal Bricard linkage: (a) the sketch of the Bricard linkage; (b) the 3D model of the Bricard linkage; (c-d) the spatial triads
Kong et al. introduced the plane-symmetric spatial triad shown in Figs. 4.1 (c-d) in [71]. In this triad, the axes of R joints 1 and 3 are symmetric about a plane passing through the axis of R2 and so are the centres of S joints 1 and 2. A single-loop composed of three such units is called the three-fold plane-symmetric Bricard linkage [13]. The analysis of the orthogonal Bricard linkage has been given in several references [12-13] using the D-H approach. The links of the orthogonal Bricard linkage are all identical and the adjacent joints are perpendicular to each other (Fig. 4.1). Let l represent the link length of the Bricard linkage. The D-H parameters are given as

Table 4.1 D-H parameters of the Bricard linkage

<table>
<thead>
<tr>
<th>i</th>
<th>a_{i-1}</th>
<th>d_i</th>
<th>\theta_i</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>l</td>
<td>270°</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>l</td>
<td>90°</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>l</td>
<td>270°</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>l</td>
<td>90°</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>l</td>
<td>270°</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>l</td>
<td>90°</td>
<td>0</td>
</tr>
</tbody>
</table>

According to Ref. [13], the output angle \( \phi \) can be represented by input angle \( \theta \)

\[
\phi = \arccos \left( -\frac{c\theta}{c\theta+1} \right) \quad (4.1)
\]

Let frame 1 be the reference frame, the positions of \( R_1 \) and \( R_2 \) are

\[
\mathbf{P}_1 = [0 \quad 0 \quad 0]^T \quad (4.2)
\]
\[
\mathbf{P}_2 = [l \quad 0 \quad 0]^T \quad (4.3)
\]

Suppose the position vector of joint \((i+1)\) in frame \(i\) \((O_i-x_iy_iz_i)\) is represented by \(\mathbf{iP}_{i+1}\). We have

\[
2\mathbf{P}_3 = 3\mathbf{P}_4 = 4\mathbf{P}_5 = 5\mathbf{P}_6 = [l \quad 0 \quad 0]^T \quad (4.4)
\]

The positions of \( R_3, R_4, R_5 \) and \( R_6 \) in the global frame can be calculated as

\[
\begin{align*}
\{ \mathbf{P}_3 \} &= \frac{1}{2} T^{23} \{ 3 \mathbf{P}_4 \} = \left\{ \frac{l}{c\theta+1} \quad 0 \quad l \sqrt{\frac{2\theta+1}{(c\theta+1)^2}} \quad 1 \right\}^T \quad (4.5a) \\
\{ \mathbf{P}_4 \} &= \frac{1}{2} T^{23} T^{34} \{ 3 \mathbf{P}_4 \} = \left\{ -l(c\theta-1) \quad l\theta \quad l(c\theta+1) \sqrt{\frac{2\theta+1}{(c\theta+1)^2}} \quad 1 \right\}^T \quad (4.5b) \\
\{ \mathbf{P}_5 \} &= \frac{1}{2} T^{34} T^{45} \{ 4 \mathbf{P}_5 \} = \left\{ -\frac{lc\theta}{c\theta+1} \quad \frac{l\theta}{c\theta+1} \quad l \sqrt{\frac{2\theta+1}{(c\theta+1)^2}} \quad 1 \right\}^T \quad (4.5c) \\
\{ \mathbf{P}_6 \} &= \frac{1}{2} T^{34} T^{45} T^{56} \{ 5 \mathbf{P}_6 \} = \left\{ -lc\theta \quad l\theta \quad 0 \quad 1 \right\}^T \quad (4.5d)
\end{align*}
\]
where \( t - \frac{1}{2} \) are given in Appendix (A). Now the characteristics of the orthogonal Bricard linkage will be revealed in order to construct DPMs. Define \( S_i \) \((i = 1, 2, \ldots 6)\) on link \( i \) with its position vector \( \mathbf{s}_i \) in the coordinate frame \( O_i-x_iy_iz_i \) as (Fig. 4.1(b))

\[
\mathbf{s}_i = \begin{cases} 
(l/2 & 0 & e)^T \quad \text{for } i = 1, 3 \text{ and } 5 \\
(l/2 & e & 0)^T \quad \text{for } i = 2, 4 \text{ and } 6
\end{cases} \quad (4.6)
\]

where \( e \) is called the offset of point \( S_i \).

The positions of the \( S_2, S_3, S_4, S_5 \) and \( S_6 \) in the global frame can be calculated as

\[
\begin{align*}
\{S_2\} &= \frac{1}{2}r_1 \begin{pmatrix} 2s_2 \end{pmatrix} = \left\{ l - e \sqrt{\frac{2C\theta + 1}{(C\theta + 1)^2}} - \frac{lC\theta}{2(C\theta + 1)} \ 0 \ l \sqrt{\frac{2C\theta + 1}{(C\theta + 1)^2}} - \frac{eC\theta}{C\theta + 1} \right\}^T \quad (4.7a)
\{S_3\} &= \frac{1}{2}r_2 \begin{pmatrix} 3s_3 \end{pmatrix} = \left\{ l - e \sqrt{\frac{2C\theta + 1}{(C\theta + 1)^2}} - \frac{lC\theta}{2(C\theta + 1)} \ - \frac{lC\theta^2}{2(C\theta + 1)} \ S_3z \right\}^T \quad (4.7b)
\{S_4\} &= \frac{1}{2}r_3 \begin{pmatrix} 4s_4 \end{pmatrix} = \left\{ S_4x \ lS\theta - eS\theta \sqrt{\frac{2C\theta + 1}{(C\theta + 1)^2}} - \frac{lC\theta S\theta}{2(C\theta + 1)} \ S_4z \right\}^T \quad (4.7c)
\{S_5\} &= \frac{1}{2}r_4 \begin{pmatrix} 5s_5 \end{pmatrix} = \left\{ S_5x \ lS\theta - eS\theta \sqrt{\frac{2C\theta + 1}{(C\theta + 1)^2}} - \frac{lC\theta S\theta}{2(C\theta + 1)} \ S_5z \right\}^T \quad (4.7d)
\{S_6\} &= \frac{1}{2}r_5 \begin{pmatrix} 6s_6 \end{pmatrix} = \left\{ - \frac{C\theta}{2} \ S_6x \ e \ 1 \right\}^T \quad (4.7e)
\end{align*}
\]

The normal vector to the plane defined by \( S_1, S_2 \) and \( S_3 \) can be obtained by \( \mathbf{N} = (\mathbf{S}_2 - \mathbf{S}_1) \times (\mathbf{S}_3 - \mathbf{S}_2) \)

\[
\mathbf{N} = \left\{ \frac{lS\theta(2e+4eC\theta-t\sqrt{2C\theta+1})}{4(C\theta+1)} \ \frac{l(2e+4eC\theta-t\sqrt{2C\theta+1})}{4} \ \frac{lS\theta(1-2e\sqrt{2C\theta+1})}{4(C\theta+1)} \right\}^T \quad (4.8)
\]

The plane defined by \( S_i \) \((i = 1, 2, 3)\) is calculated as

\[
N_x(x - l/2) + N_yy + N_z(z - e) = 0 \quad (4.9)
\]

Substituting the positions of \( S_4, S_5 \) and \( S_6 \) into Eq. (4.9), it can be verified that \( S_i \) \((i = 1, 2, \ldots 6)\) are always on the same plane, which is referred to the mirror plane in the following
sections. As shown in Fig. 4.1(b), the lines defined by \( S_1 \) and \( S_2 \), \( S_3 \) and \( S_4 \), and \( S_5 \) and \( S_6 \) respectively form a regular triangle. \( S_1S_2S_3S_4S_5S_6 \) is a semi-regular hexagon. The edges of the hexagon vary with the deformation of the Bricard linkage. In addition, both triangles \( S_1S_3S_5 \) and \( S_2S_4S_6 \) are regular triangles.

Based on the above characteristics of the orthogonal Bricard linkage, DPMs will be constructed by connecting orthogonal Bricard linkages through S joints located at \( S_i \) on each link.

### 4.1.2 DPMs Based on Bricard Linkages

By connecting two orthogonal Bricard linkages, which are mirrored versions of each other about the mirror plane, using six S joints located at \( S_i \) \((i = 1, 2, \ldots 6)\), a DPM in the shape of a triangular prism can be obtained (Fig. 4.2). Since \( S_i \) \((i = 1, 2, \ldots 6)\) of each orthogonal Bricard linkage remain on the same plane, the prism DPM has the same DOF of each orthogonal Bricard linkage, i.e., 1-DOF.

![Fig. 4.2 Prism DPM based on orthogonal Bricard linkages: (a) initial posture; (b) outward deploying; (c) inward deploying](image)

The prism DPM is axisymmetric about the line defined by the intersection of \( R_2 \), \( R_4 \) and \( R_6 \) and the intersection of \( R_2' \), \( R_4' \) and \( R_6' \) and plane-symmetric about three planes defined by joints \( R_i \), \( R_{i+3} \), \( R_i' \) and \( R_{i+3}' \) \((i = 1, 2 \text{ and } 3)\). In the initial state [Fig. 4.2(a)], \( R_1 \), \( R_3 \) and \( R_5 \) of the two Bricard linkages are parallel with each other and \( R_2 \), \( R_4 \) and \( R_6 \) lie on the same plane and intersect at a point. \( R_1 \) and \( R_1' \) are collinear, and so are \( R_3 \) and \( R_3' \) and \( R_5 \) and \( R_5' \). \( R_2//R_2' \), \( R_4//R_4' \) and \( R_6//R_6' \). When deployed, the two Bricard linkages deform...
synchronously and are always symmetrical about the mirror plane. The axes of the R joints of each Bricard linkage intersect at two points, and these four intersection points are always collinear. The mechanism has two deploying states, which are referred to as the outward state [Fig. 4.2(b)] and the inward state [Fig. 4.2(c)]. In the outward state, the distance between joints $R_1$ and $R_1'$ increases with the distance between joints $R_2$ and $R_2'$ decreasing, while in the inward state, the distance between joints $R_1$ and $R_1'$ decreases with the distance between joints $R_2$ and $R_2'$ increasing.

To define the ratio of the stowed-to-deployed diameter of the prism PM, the circumscribed cylinder is used to represent the volume of the DPM (Fig. 4.3).

![Fig. 4.3 Circumscribed cylinder of the mechanism](image)

The side lengths of the equilateral triangles defined by joints $R_1$, $R_3$ and $R_5$ as well as $R_2$, $R_4$ and $R_6$ can be calculated, respectively, as:

$$D_1 = |P_3 - P_1|$$  \hspace{1cm} (4.10)

$$D_2 = |P_4 - P_2|$$  \hspace{1cm} (4.11)

The plots of $D_1$ and $D_2$ with respect to the variation of input angle ($R_1$) $\theta \in [0^\circ, 120^\circ]$ are displayed in Fig. 4.4 for the case when the link length $l = 0.05m$. $D_1$ and $D_2$ vary from $0.05m$ to $0.1m$. $D_1$ obeys to monotonic decreasing, while $D_2$ obeys monotonic increasing. It can be seen that when $\theta = 90^\circ$, $D_1 = D_2$; when $\theta > 90^\circ$, $D_1 > D_2$; and when $\theta < 90^\circ$, $D_1 < D_2$.
The distance, $H_1$, between joints $R_1$ and $R_1'$ is twice the distance between $R_1$ and the mirror plane (Fig. 4.2). We have

$$H_1 = \frac{2}{t^2 + m^2 + n^2}$$  \hspace{1cm} (4.12)

Similarly, the distance, $H_2$, between $R_2$ and $R_2'$ is (shown in Fig. 4.2)

$$H_2 = \frac{2}{t^2 + m^2 + n^2}$$  \hspace{1cm} (4.13)

When deployed inward, $H_1 < H_2$, and when deployed outward, $H_1 > H_2$. The volume of the circumscribed cylinder of the mechanism $V$ can be described by

$$V = \begin{cases} 
\theta \geq 90^\circ & \left( \frac{\pi}{3} \right)^2 \times H_1 \text{ outward deploying} \\
\theta < 90^\circ & \left( \frac{\pi}{3} \right)^2 \times H_2 \text{ inward deploying} 
\end{cases}$$  \hspace{1cm} (4.14)
Let $e = 0.02\, \text{mm}$, the variation of the volume of the circumscribed cylinder of the DPM with respect to the input angle $\theta$ within the range from $0^\circ$ to $120^\circ$ is depicted in Fig. 4.5. It can be observed that the volume reaches the maximum value of $4.985 \times 10^{-4}\, \text{m}^3$ when $\theta = 117.17^\circ$, as shown in Fig. 4.6(a). For an ideal zero-thickness model, the minimum volume of the circumscribed cylinder is zero when $\theta = 0^\circ$ [Fig. 4.6(c), when the two Bricard linkages overlap].

A 3D printed prototype (Fig. 4.7) is fabricated to verify the feasibility of the mechanism. A compliant prototype is also built with its links printed using rigid material and its joints with soft material (Fig. 4.8). Since the initial state is a stable state, the mechanism can
recover to the initial state after deploying. In the initial state, the mechanism is in the shape of a regular prism. When deployed, it can be spread onto parallel planes through two ways shown in Figs. 4.8(b) and (c) respectively. Suppose the thickness of the links is $h = 0.005m$, the minimum volume of the mechanism [Fig. 4.8(c)] is $2\sqrt{3}l^2h = 4.33\times10^{-5}m^3$. The ratio between the maximum volume and the minimum volume is $r = 11.5$.

![Fig. 4.8 Compliant prototype of the prism DPM based on orthogonal Bricard linkages: (a) initial posture; (b) outward deploying; (c) inward deploying](image1)

![Fig. 4.9 The multiple-layer prism DPM constructed using Bricard linkages: (a) the construction method; (b) initial posture; (c) outward deploying; (d) inward deploying](image2)

By overlaying the prism DPMs, the mechanism can be extended along the axial direction. The 1-DOF $n$-layer prism DPM in Fig. 4.9 is connected using $n$ prism DPM proposed above. Let $R'_1$, $R'_2$, $R'_3$, $R'_4$ and $R'_5$ of the first DPM be coincident with $R_1$, $R_2$, $R_3$, $R_4$, $R_5$ and $R_6$ of the second DPM respectively, a double layers DPM is obtained. Using the construction method, multiple-layer DPM can be constructed. The mechanisms can be deployed outward [Fig. 4.9(b)] and inward [Fig. 4.9(c)]. It can be applied to sun
shield, or other aerospace mechanisms.

A three-layer prototype is 3D printed with rigid links and flexible joints. The compliant prototype has a higher folding ratio than the rigid mechanism in Fig. 4.9. The mechanism is in the shape of a prism in its stable position [refers to the initial state in Fig. 4.10(a)] and can be folded into four layers through two approaches [Figs. 4.10(b) and (c) respectively]. The flexible joints also provide a method to develop origami mechanisms into thin panel foldable mechanisms. It is noted that the mechanisms with offset rigid joints can also be folded into several layers. The folding ratio can be expressed as

\[ r = \frac{2e}{(n + 1)h} \]  

(4.15)

The folding ratio of the prototype is 6. The ratio will be larger when increasing the value of \( e \) and decreasing the value of \( h \), and the ratio can reach to infinity when \( h \) tends towards 0.

![Fig. 4.10 The prototype of the multi-layer prism DPM constructed using Bricard linkages: (a) initial posture; (b) outward deploying; (c) inward deploying](image)

A 1-DOF DPM in the shape of a tetrahedron can also be obtained (Fig. 4.11) by connecting four orthogonal Bricard linkages using twelve S joints. Each Bricard linkage is connected with another Bricard linkage by two S joints. In the initial state [Fig. 4.11(a)], \( R_1, R_3 \) and \( R_5 \) of each Bricard linkages are parallel with each other and \( R_2, R_4 \) and \( R_6 \) lie on the same plane and intersect at a point. The planes defined by joints \( R_2, R_4 \) and \( R_6 \) of each Bricard linkage generate a regular tetrahedron and the planes defined by the six S joints of each Bricard linkage generate a similar tetrahedron.

The DOF of the mechanism is one and the tetrahedron DPM can be deployed outward [Fig. 4.11(b)], which refers to the case that the distance between \( R_5 \) and \( R_5' \) increases from the initial configuration, or inward [Fig. 4.11(c)], which refers to the case that the distance between \( R_5 \) and \( R_5' \) decreases from the initial configuration.
A tetrahedron DPM with compliant joints is fabricated with a high ratio of stowed-to-deployed diameter. In the initial state as presented in Fig. 4.12(a), the mechanism is in a shape of a regular tetrahedron and the angle $\tau$ between the plane defined by $R_3$ and $R_5$ in the first Bricard linkage and the plane defined by $R_1'$ and $R_3'$ in the adjacent Bricard linkage $\tau = \pi - \arccos(1/3) \approx 109.5^\circ$. The angle $\tau$ becomes zero when deployed inward or $\pi$ when deployed outward. The mechanism is stable in its initial state, so it will return to the initial state after the deploying process.

Similarly, a 1-DOF DPM in the shape of an octahedron is constructed by connecting eight orthogonal Bricard linkages with twenty-four S joints (Fig. 4.13). The mechanism can be deployed outward or inward, as shown in Figs. 4.13(b) and (c). When connecting twenty orthogonal Bricard linkages with sixty S joints, a deployable 1-DOF DPM in the shape of an icosahedron is obtained (Fig. 4.14).
Fig. 4.14 Octahedron DPM based on orthogonal Bricard linkages: (a) initial posture; (b) outward deploying; (c) inward deploying

Fig. 4.14 Icosahedron DPM based on orthogonal Bricard linkages: (a) initial posture; (b) outward deploying; (c) inward deploying

4.1.3 Variations of the Mechanisms Based on Bricard Linkages

Fig. 4.15 The triad unit of the single-loop linkages

Using the above approach for constructing DPMs using orthogonal single-loop linkages, we can also construct similar DPMs by connecting single-loop linkages in which the axes
of two adjacent joints are not perpendicular. Such single-loop mechanisms are composed of three to five identical spatial plane-symmetric triad as shown in Fig. 4.15.

However, these DPMs may not be as practical as the mechanisms based on orthogonal single-loop mechanisms. For example, the prism DPM based on the three-fold plane-symmetric Bricard linkage with a twist angle of $60^\circ$ is either not a regular prism in the initial position [configuration I in Figs. 4.16(a-b)] or cannot be folded onto two planes when using flexible joints [configuration II in Figs. 4.16(c-d)].

![Fig. 4.16](image)

Fig. 4.16 Prism DPMs based on Bricard linkages with a twist angle of $60^\circ$: (a-b) configuration I; (c-d) configuration II

![Fig. 4.17](image)

Fig. 4.17 Variation of prism DPM based on Bricard linkages: (a) initial posture; (b) outward deploying; (c) inward deploying

It is noted that the S joints can be set at arbitrary positions on the links as long as the mechanism is symmetric about the mirror plane. As shown in Fig. 4.17 and Fig. 4.18, the S joint has an offset from the plane defined by the triangle link, as well as the median of
the triangle, since the hexagons formed by the S joints of the two Bricard linkages respectively always coincide. The DOFs of these mechanisms are also one.

![Fig. 4.18 Variation of tetrahedron DPM based on Bricard linkages: (a) initial posture; (b) outward deploying; (c) inward deploying](image)

4.2 DPMs Based on Identical 8R/10R Linkages

In this section, 8R and 10R linkages will be adopted to connect DPMs, using the proposed construction method.

4.2.1 Analysis of 8R Linkage

By connecting four spatial triads [Fig. 4.19(a)], an orthogonal 8R linkage is obtained, as shown in Figs. 4.19(b-c). The 8R linkage is composed of eight identical links and eight R joints and has two DOF in a general position [80]. The adjacent joint axes of the 8R linkages are perpendicular to each other.

![Fig. 4.19 Construction of orthogonal 8R linkage: (a) the triad; (b-c) the orthogonal 8R linkage](image)
In a general position, the S joints of the 8R linkage are not on the same plane, as shown in Fig. 4.20(a). To construct DPMs using 8R linkages, the S joints should be coplanar [Fig. 4.20(b)]. Now it will be verified that the DOF of the 8R linkage reduces to one when the S joints are constrained to be coplanar.

![Fig. 4.20 Variation of orthogonal 8R linkage: (a) the general configuration; (b) the virtual-plane-constrained configuration](image)

The D-H parameters of the 8R linkage are given in Table 4.2. In the deployable mode, suppose that $\theta' = \theta$, for the sake of conciseness. $\varphi_1 = \varphi_2$ if the DOF of the virtual-plane-constrained 8R linkage is one, similar to the Bricard linkage. Otherwise, the DOF is still two. The position vectors of the centers of the S joints with respect to the local coordinate are

\[
\begin{align*}
\mathbf{s}_i &= \begin{bmatrix} l/2 & 0 & e \end{bmatrix}^T \\
& \text{for } i = 1, 3, 5 \text{ and } 7 \\
\mathbf{s}_i &= \begin{bmatrix} l/2 & e & 0 \end{bmatrix}^T \\
& \text{for } i = 2, 4, 6 \text{ and } 8
\end{align*}
\]

(4.16) (4.17)

<table>
<thead>
<tr>
<th>$i$</th>
<th>$a_{i-1}$</th>
<th>$\alpha_{i-1}$</th>
<th>$d_i$</th>
<th>$\theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$l$</td>
<td>$270^\circ$</td>
<td>0</td>
<td>$\theta$</td>
</tr>
<tr>
<td>2</td>
<td>$l$</td>
<td>$90^\circ$</td>
<td>0</td>
<td>$\varphi_1$</td>
</tr>
<tr>
<td>3</td>
<td>$l$</td>
<td>$270^\circ$</td>
<td>0</td>
<td>$\theta'$</td>
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<tr>
<td>4</td>
<td>$l$</td>
<td>$90^\circ$</td>
<td>0</td>
<td>$\varphi_2$</td>
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<tr>
<td>5</td>
<td>$l$</td>
<td>$270^\circ$</td>
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<tr>
<td>6</td>
<td>$l$</td>
<td>$90^\circ$</td>
<td>0</td>
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<tr>
<td>7</td>
<td>$l$</td>
<td>$270^\circ$</td>
<td>0</td>
<td>$\theta'$</td>
</tr>
<tr>
<td>8</td>
<td>$l$</td>
<td>$90^\circ$</td>
<td>0</td>
<td>$\varphi_2$</td>
</tr>
</tbody>
</table>
Link 1 is fixed on the ground, and the global coordinate frame is attached to $R_1$. The position vectors of the centres of the S joints in the global coordinate frame can be calculated as

$$\{S_2\}_1 = \frac{1}{2} T_1 \begin{bmatrix} {2S_2} \end{bmatrix} = \begin{bmatrix} l + lC\varphi_1/2 - eS\varphi_1 & 0 & eC\varphi_1 + lS\varphi_1/2 \end{bmatrix}^T$$  \hspace{1cm} (4.18a)

$$\{S_3\}_1 = \frac{1}{2} T_3 \begin{bmatrix} {3S_3} \end{bmatrix} = \begin{bmatrix} lS\theta/2 & eC\varphi_1 + lS\varphi_1 + lC\theta S\varphi_1/2 \end{bmatrix}^T$$  \hspace{1cm} (4.18b)

where

$S_{3x} = l + lC\varphi_1 - eS\varphi_1 + lC\theta C\varphi_1/2$

$$\{S_4\}_1 = \frac{1}{2} T_4 \begin{bmatrix} {4S_4} \end{bmatrix} = \begin{bmatrix} S_{4x} & S\theta[l(2 + C\varphi_2) - 2eS\varphi_2]/2 & S_{4z} \end{bmatrix}^T$$  \hspace{1cm} (4.18c)

where

$S_{4x} = l - S\varphi_1(2eC\varphi_2 + lS\varphi_2)/2 + C\varphi_1[l + C\theta(l + lC\varphi_2/2 - eS\varphi_2)]$

$S_{4z} = C\varphi_1(eC\varphi_2 + lS\varphi_2/2) + S\varphi_1[2l + C\theta(l(2 + C\varphi_2) - 2eS\varphi_2)]/2$

$$\{S_5\}_1 = \frac{1}{2} T_5 \begin{bmatrix} {5S_5} \end{bmatrix} = \begin{bmatrix} S_{5x} & S\theta[(2 + C\theta)(1 + C\varphi_2) - 2eS\varphi_2]/2 & S_{5z} \end{bmatrix}^T$$  \hspace{1cm} (4.18d)

The equation of the mirror plane defined by joint centres of $S_1$, $S_2$ and $S_3$ is

$$N_x(x - l/2) + N_yy + N_z(z - e) = 0$$  \hspace{1cm} (4.20)

The condition that the joint centre of $S_4$ is on the plane defined by $S_1$, $S_2$ and $S_3$ is

$$N_x(S_{4x} - l/2) + N_yS_{4y} + N_z(S_{4z} - e) =$$

$$-\frac{1}{2} lS\theta[lC(\varphi_1^2/2 - 2eS(\varphi_1^2/2))S(\varphi_1^2/2)lC(\varphi_2^2/2) - 2eS(\varphi_2^2/2)] = 0$$  \hspace{1cm} (4.21)

which leads to

$$\varphi_2 = \varphi_1 \quad \text{or} \quad \varphi_2 = 2\tan^{-1}(2e/l)$$  \hspace{1cm} (4.22)

When $\varphi_2 = 2\tan^{-1}(2e/l)$, $S_3$ and $S_4$ are concentric, in which position $S_5$ is not on the mirror plane unless $S_1$ and $S_2$ are also concentric. In this case, $\varphi_1$ is also equal to
$2\tan^{-1}(2e/l)$ and the linkage is in spherical 4R linkage mode (discussed in the following section). Substituting the position vectors of $S_5$, $S_6$, $S_7$ and $S_8$ into Eq. (4.20), it is verified that all the S joints are on the same plane when $\varphi_2 = \varphi_1$. Hence, $\varphi_2 = \varphi_1$ is the unique solution.

The 8R linkage is a single closed-loop, the product of the transfer matrices equals the identity matrix, which means

$$\frac{1}{2}T_{\frac{1}{2}}T_{\frac{3}{4}}T_{\frac{5}{6}}T_{\frac{7}{8}}T_{\frac{8}{7}}T_{\frac{6}{5}}T_{\frac{4}{3}}T_{\frac{1}{2}} = I$$

(4.24)

$\varphi_1 (\varphi_1 = \varphi_2)$ can be represented by $\theta$ as

$$\varphi_2 = \varphi_1 = \arccos\left(\frac{1-c\theta}{c\theta+1}\right)$$

(4.25)

![Sketch of the virtual-plane-constrained orthogonal 8R linkage](image)

Fig. 4.21 Sketch of the virtual-plane-constrained orthogonal 8R linkage: (a) four spheres are constrained on the plane; (b) five spheres are constrained on the plane

Then the DOF of the virtual-plane-constrained orthogonal 8R linkage will be verified using the conventional formula. When four spheres are constrained on the same plane, assume that the center of $S_1$ is fixed on the plane, $S_3$ has a translational DOF on the plane and $S_5$ and $S_7$ have two translational DOFs on the plane [Fig. 4.21(a)]. The DOF of the linkage can also be computed by

$$M = 6(q - p) + \sum_{i=1}^{p} f_i = 6(8 - 12) + 8 + 3 + 4 + 5 \times 2 = 1$$

(4.26)

When $s$ additional spheres are constrained to be on the plane, the 8R linkage is over-constrained [Fig. 4.21(b)].

$$M = 6(q - p) + \sum_{i=1}^{p} f_i = 6 \times (8 - 12 - s) + 8 + 3 + 4 + 5 \times (2 + s) = 1 - s$$

(4.27)

The DOF of the linkage is still one, due to the symmetry characteristics of the linkage. It can be readily proved that the virtual-plane-constrained 10R/12R linkages also have one DOF when deployed.

The virtual-plane-constrained orthogonal 8R linkage has three 1-DOF modes: spatial
mode [Fig. 4.22(a)], planar 4R mode [Fig. 4.22(b)], and spherical 4R mode [Fig. 4.22(c)]. In the spatial mode, the lines defined by $S_1$, $S_2$, $S_3$, and $S_4$, $S_5$ and $S_6$, respectively, form a square [red one in Fig. 20(b)], $S_1S_3S_5S_7$ is a square [green one in Fig. 20(b)] too. $S_1S_2S_3S_4S_6S_7S_8$ is a semi-regular octagon. $S_1S_2 = S_3S_4 = S_5S_6 = S_7S_8$ and $S_2S_3 = S_4S_5 = S_6S_7 = S_8S_1$. In the planar 4R mode, the mechanism moves like a 4-bar mechanism composed of joints $R_1$, $R_3$, $R_5$, and $R_7$, whose joint axes are perpendicular to the plane defined by $S_i$ ($i = 1, 2, \ldots, 8$). In the spherical 4R mode, the mechanism moves like a 4-bar mechanism composed of joints $R_1$, $R_3$, $R_5$, and $R_7$, whose joint axes intersect at a point.

![Fig. 4.22](image)

Fig. 4.22 The virtual-plane-constrained orthogonal 8R linkage: (a) spatial mode; (b) planar 4R linkage mode; (d) spherical 4R linkage mode.

### 4.2.2 DPMs Based on 8R/10R Linkages

By connecting two 8R orthogonal linkages, which are mirrored versions of each other about the mirror plane, using eight $S$ joints located at $S_i$ ($i = 1, 2, \ldots, 8$), a quadrangular DPM can be obtained (Fig. 4.23). Since $S_i$ ($i = 1, 2, \ldots, 8$) of each orthogonal 8R linkage remain on the same plane, the DPM has the same DOF of each constrained orthogonal 8R linkage, i.e., 1-DOF.

The DPM based on 8R orthogonal linkages has three modes: deployable mode; planar 4R mode and spherical 4R mode. In the deployable mode, the mechanism can be deployed outward or inward, when all the eight $S$ joints move backwards or towards, as shown in Figs. 4.23(b) and (c). In the planar 4R mode [Fig. 4.23(d)], the two 8R linkages deform synchronously and each moves as a planar 4R linkage composed of joints $R_1$, $R_3$, $R_5$, and $R_7$. In the spherical 4R mode [Fig. 4.23(f)], the two 8R linkages also deform synchronously and each moves as a spherical 4R linkage composed of joints $R_1$, $R_3$, $R_5$, and $R_7$ as well.
The deployable mode and the planar 4R mode can be switched through the transition configuration I (singular configuration) [Fig. 4.23(a)]. In this transition configuration, the axes of joints $R_1$, $R_3$, $R_5$ and $R_7$ are parallel, and $R_2$ and $R_4$ are collinear, as well as $R_4$ and $R_8$. The deployable mode and the spherical 4R mode can be transformed through the transition configuration II, which is shown in Fig. 4.23(e). $S_1$ and $S_2$ are adjusted to be concentric, as well as $S_3$ and $S_4$, $S_5$ and $S_6$, $S_7$ and $S_8$, and $S_1S_3S_5S_7$ is a square. Compared with the multi-mode deployable mechanisms in the references, the mechanism has a very simple structure and is lightweight.

Fig. 4.24 Rigid prototype of the quadrangular DPM based on orthogonal 8R linkages: (a) initial posture; (b) outward deploying; (c) inward deploying; (d) planar 4R mode
A rigid prototype is fabricated to verify the DOF of the mechanism, as shown in Fig. 4.24. It is noted that the spherical 4R mode can only be achieved if there is no interference between the S joints. The mechanism has one DOF in each mode.

![Fig. 4.25 Pentagonal prism DPM based on orthogonal 10R linkages: (a) initial posture; (b) outward deploying; (c) inward deploying; (d) planar 5R linkage mode](image)

![Fig. 4.26 Rhombohedron DPM based on orthogonal 8R linkages: (a) initial posture; (b)-(c) outward deploying; (d) inward deploying; (e)-(f) 3-DOF parallelepiped mechanism mode.](image)

Using the construction method proposed, mechanisms based on other orthogonal single-loop linkages can be obtained as well. Figure 4.25 illustrates the pentagonal prism DPM constructed by two orthogonal 10R linkages using ten S joints. The mechanism has
two modes, including a 1-DOF deployable mode [Fig. 4.25(b-c)] and a 2-DOF planar 5R linkage mode [Fig. 4.25(d), in which the mechanism behaves as a planar 5R linkage and has two DOFs]. Unlike the DPM based on orthogonal 8R linkages, this DPM has no spherical mode. This DPM is a new variable-DOF mechanism and therefore enriches the types of variable-DOF mechanisms.

Connecting six orthogonal 8R linkages, leads to the construction of a DPM in the shape of a rhombohedron in the initial state (Fig. 4.26). Each 8R linkage connects with four 8R linkages using two S joints between each pair of 8R linkages. In the initial state, the planes defined by \( R_2, R_4, R_6 \) and \( R_8 \) in each 8R linkage generate a rhombohedron and the planes defined by the centers of S joints in each 8R linkage generate a smaller rhombohedron. The angle \( \tau' \) between the plane defined by \( R_3 \) and \( R_5 \) in the first 8R linkage and the plane defined by \( R_{i'} \) and \( R_{j'} \) in the adjacent 8R linkage is 90°.

The mechanism has two modes. In the first mode, it can be deployed outward and inward, as shown in Figs. 4.26(b-d). When deployed outward, \( \tau' \) ranges from 90° [Fig. 4.26(a)] to 270° [Fig. 4.26(c)]; when deployed inward [Fig. 4.26(d)], \( \tau' \) ranges from 90° to 10°, considering the link interference. In the second mode, \( R_2, R_4, R_6 \) and \( R_8 \) in each 8R linkage lose their DOFs, and the mechanism behaves as a parallelepiped mechanism [170]. The mechanism in this mode has three DOFs, as shown in Figs. 4.26(e) and (f).

A 1-DOF DPM in the shape of a dodecahedron is also designed as shown in Fig. 4.27. It is constructed using twelve 10R orthogonal linkages and sixty S joints. The mechanism can also be deployed outward [Fig. 4.27(b)] and inward [Fig. 4.27(c)]. It is noted the deploying ratio depends on the shape of the links which affects the interference between the links.

![Fig. 4.27 Dodecahedron DPM based on orthogonal 10R linkages: (a) initial posture; (b) outward deploying; (c) inward deploying](image)

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4.3 DPMs Based on Different Loops

Instead of constructing DPMs using identical loops, the same type of loops with different sizes or different types of loops can be adopted to connect DPMs.

4.3.1 DPMs Based on the Same Type of Linkages with Different Sizes

By connecting two single-loop linkages with the same type but different sizes, frustum DPMs are constructed. The deployable triangular frustum mechanism based on two Bricard linkages will be addressed as an example to illustrate the construction method. As shown in Fig. 4.28, the S joints of the bigger Bricard linkage are on the medians of the triangle links, while the S joints of the smaller one have offset from the medians.

Fig. 4.28 The relationship between the two Bricard linkages with different sizes: (a) the positions of the S joints; (b) semi-regular hexagons defined by the S joints

Let \( l \) and \( l' \) represent the link lengths of the two triads respectively. The distances between the medians of the triangles within the two loops along the direction of the joint axis of \( R_2 \) (or \( R_4, R_6 \)) and \( R_1R_3 \) (or \( R_3R_5, R_5R_1 \)) are noted as \( D \) and \( T \) respectively [Fig. 4.28(a)]. \( D \) and \( T \) are calculated as

\[
D = l/\sqrt{3} - l'/\sqrt{3} \\
T = l/2 - l'/2
\]
Fig. 4.29 The two Bricard linkages with different sizes: (a) the smaller one; (b) the bigger one

Unlike the DPM constructed by \( nR \) orthogonal linkages with the same size using \( n S \) joints, the loop can only be connected to the other loop with different size by \( n/2 S \) joints. As shown in Fig. 4.28(a), only three \( S \) joints are used to connect the two Bricard linkages with different sizes. The ratio of one side length and the adjacent side length of the semi-regular hexagons defined by the \( S \) joints of the two Bricard linkages respectively are distinct. The smaller loop rotates when deploying [Fig. 4.28(b)].

Now it will be verified that the three spheres of the bigger Bricard linkage and those of the smaller Bricard linkage form two regular triangles respectively during the deploying process (Fig. 4.29). The position vectors of the spheres of the bigger Bricard linkage have been given in Eq. (4.7), and those of the smaller Bricard linkage are to be calculated.

The joint centre of first \( S \) joint of the smaller Bricard linkage is obtained as

\[
S'_1 = \begin{pmatrix} \frac{l'}{2} - T - D' & e \end{pmatrix}^T \quad (4.30a)
\]

The position vectors of the other two \( S \) joints are calculated as

\[
\begin{pmatrix} S'_2 \\ 1 \end{pmatrix} = \frac{1}{2}T_3^2 T \begin{pmatrix} S'_1 \\ 1 \end{pmatrix} = \begin{pmatrix} S'_{2x} \\ (l' - l)C\theta'/\sqrt{3} - (l - 2l')S\theta'/2 \\ S'_{2z} \\ 1 \end{pmatrix}^T \quad (4.30b)
\]

where

\[
S'_{2x} = -3l'C\theta' + 3(l - 2l')C^2\theta'/2 + 3l'b - 3eab - \sqrt{3}(l - l')C\theta'S\theta'/3b
\]

\[
a = \frac{\sqrt{2C\theta'+1}}{(C\theta'+1)^2}
\]

\[
b = 1 + C\theta'
\]

\[
S'_{2z} = C\theta'\{-e/b - (l - 2l'a/2) + a[3l' + \sqrt{3}(l - l')S\theta']/3
\]

\[
\begin{pmatrix} S'_3 \\ 1 \end{pmatrix} = \frac{1}{2}T_3^2 T_4^3 T_5^4 T_6^5 T \begin{pmatrix} S'_1 \\ 1 \end{pmatrix} = \begin{pmatrix} S'_{3x} \\ S'_{3y} \\ la/2 + e(1/c - 1) \\ 1 \end{pmatrix}^T \quad (4.30c)
\]

where

\[
S'_{3x} = \{3C\theta'[-2l' + (lC\theta')/c + 2ea] + 2\sqrt{3}(l' - l)S\theta'/6
\]

\[
S'_{3y} = (l' - ea)S\theta' + C\theta'[l'(l' - l)/\sqrt{3} - ltan(\theta'/2)/2]
\]
\[ c = \sqrt{\frac{s^2 \theta'}{(c \theta' + 1)^2}} \]

The three spheres form a triangle, whose squared side lengths are

\[
|S'_{2} - S'_{1}|^2 = |S'_{3} - S'_{2}|^2 = |S'_{1} - S'_{3}|^2 = \left[ c \theta' D - D + S \theta' \left( \frac{l'}{2} - l' \right) \right]^2 + \left\{ \frac{l'^2}{2} + T - ea - \frac{c \theta'}{2b} \left[ \left( \frac{l'}{2} - \frac{l}{2} \right) C \theta' + 2(l' + S \theta' D) \right] \right\} + \left[ e - l' a + \frac{e \theta'}{b} + S \theta' a C + c \theta' a \left( \frac{l}{2} - l' \right) \right]^2 \tag{4.31}
\]

Equation (4.31) indicates that the triangle formed by the three spheres of the smaller Bricard linkage is a regular triangle. The squared side lengths of the triangle defined by the spheres of the bigger Bricard linkage are yielded as

\[
|S_{2} - S_{1}|^2 = |S_{3} - S_{2}|^2 = |S_{1} - S_{3}|^2 = \left\{ -6el\sqrt{d} - 2elC\theta\sqrt{d} + 4e^2d + l^2 \left[ 3 + C\theta(3 + 3\theta) \right] \right\} / [2(1 + C\theta)] \tag{4.32}
\]

where

\[ d = 1 + 2C\theta \]

The triangle defined by the three spheres of the bigger Bricard linkage also keeps as a regular triangle. There always exist \( \theta \) and \( \theta' \) to equalise the two side lengths. Let \( l = 0.05m \), \( e = 0.02m \) and \( l' = 0.03m \), \( |S_{2} - S_{1}|^2 = |S'_{2} - S'_{1}|^2 = 0.0044m^2 \) in the initial position, when \( \theta = 120^\circ \).

![Fig. 4.30 Prism mechanism constructed using two Bricard linkages with different sizes: (a) initial posture; (b) outward deploying; (c) inward deploying](image)

Based on the results, a triangular frustum mechanism is constructed using the two Bricard linkages with different sizes, as shown in Fig. 4.30. The mechanism can be deployed outward [Fig. 4.30(b)] and inward [Fig. 4.30(c)].

A prototype of the triangular frustum mechanism is fabricated to verify the feasibility of the mechanism (Fig. 4.31). The links are 3D printed with magnet disc inside, and the S joints are designed using steel balls.
Fig. 4.31 Prototype of the triangular frustum mechanism constructed using two Bricard linkages with different sizes: (a) initial posture; (b) outward deploying; (c) inward deploying

A 1-DOF three-layer triangular frustum DPM based on Bricard linkages with different sizes is also constructed with a high stowed-to-deployed ratio (Fig. 4.32). It is measured in the CAD software that the height of the DPM is 143.87 mm in the initial state [Fig. 4.32(a)] and can reach 74.99 mm [Fig. 4.32(c)] when deployed inward. The deploying ratio is 1.92.

Fig. 4.32 The multiple-layer DPM constructed using Bricard linkages with different sizes: (a) the construction method; (b) initial posture; (c) outward deploying; (d) inward deploying

Similarly, a quadrilateral frustum mechanism based on two 8R linkages with different sizes is built, as shown in Fig. 4.33. Unlike the DPM with identical loops, the mechanism has no planar 4R linkage mode or spherical 4R linkage mode. The distances between the medians of the triangles within the two loops along the direction of the joint axis of $R_4R_8$ and $R_1R_7$ are noted as D and T respectively. D and T (shown in Fig. 4.34) are calculated
as

\[ D = l - l' \quad (4.33) \]

\[ T = l/2 - l'/2 \quad (4.34) \]

Fig. 4.33 Quadrilateral frustum mechanism constructed using two 8R linkages with different sizes: (a) initial posture; (b) outward deploying; (c) inward deploying

Fig. 4.34 Parameters of the two 8R linkages with different sizes

4.3.2 DPMs with Double Layers

Extending the mechanisms along the radial direction refers to constructing polyhedrons whose faces have double layers. First, a 1-DOF double-layer unit is designed, as shown in Fig. 4.35. The unit is obtained by deforming the triangular frustum mechanism in Fig. 9, through the deforming process presented in Fig. 4.36. The bigger Bricard linkage deploys while rotating. In the initial state, \( R_1//R_1' \) and \( R_2 \) and \( R_2' \) are collinear [Fig. 4.35(a)].

A double-layer prism mechanism is obtained by connecting two double-layer units, as shown in Fig. 4.37. The two units are symmetric about the mirror plane defined by the S joints. The DPM has one DOF and can be deployed outward [Fig. 4.37(b)] and inward [Fig. 4.37(c)]. The interference problem of the S joints can be solved by using steel balls and links with magnet disc.
Fig. 4.35 The double-layer deployable unit constructed using Bricard linkages: (a) initial posture; (b) outward deploying; (c) inward deploying

Fig. 4.36 The deforming process: (a) triangular frustum mechanism configuration; (b-e) deforming process; (f) double-layer configuration

Fig. 4.37 The double-layer prism mechanism constructed using Bricard linkages: (a) initial posture; (b) outward deploying; (c) inward deploying
To construct DPMs using the double-layer units, three additional S joints are inserted on the bigger Bricard linkage, as shown in Fig. 4.38. Based on the modified units, 1-DOF double-layer tetrahedron DPM and octahedron DPM are obtained, as shown in Figs. 4.39 and 4.40 respectively.

![Fig. 4.38 Variation of the double-layer deployable unit](image)

![Fig. 4.39 Double-layer tetrahedron DPM constructed using Bricard linkages: (a) initial posture; (b) outward deploying; (c) inward deploying](image)

![Fig. 4.40 Double-layer octahedron DPM constructed using Bricard linkages: (a) initial posture; (b) outward deploying; (c) inward deploying](image)

DPMs constructed using 8R/10R-based double-layer units can also be built, which can
be deployed but have no planar or spherical 4R/5R linkage mode.

4.3.3 DPMs Based on Different Types of Linkages

In this section, different types of loops will be used to connect DPMs. 1-DOF DPMs can be obtained by connecting the virtual-plane-constrained 8R/10R linkages and Bricard linkages. When connecting additional loops, the degree of over-constraint increases but the mechanism still has 1-DOF [49]. As shown in Fig. 4.41, a 1-DOF rectangular pyramid mechanism is built by connecting four Bricard linkages and one 8R linkage. In the initial position, the Bricard linkages and the 8R linkage are in the shapes of regular triangles and square respectively, and the joint axis of $R_1$ in the 8R linkage and $R_1'$ in the Bricard linkage are on the same plane. The mechanism can be deployed outward, which refers to the case that the distance between $R_1$ and $R_1'$ increases [Fig. 4.41(b)], and inward, which refers to the case in which the distance between $R_1$ and $R_1'$ decreases [Fig. 4.41(c)].

![Fig. 4.41 The rectangular pyramid mechanism: (a) initial posture; (b) outward deploying; (c) inward deploying](image)

A triangular prism mechanism is constructed using two Bricard linkages and three 8R linkages, as shown in Fig. 4.42. The mechanism can also be deployed outward [Fig. 4.42(b)] and inward [Fig. 4.42(c)]. Apart from the 1-DOF deployable mode, the mechanism has an additional 2-DOF translation mode, in which the Bricard linkages are immobile and the 8R linkages move as planar 4R linkages [Fig. 4.42(d)]. The mechanism in the translation mode is equal to the 14-bar mechanism in Fig. 4.42(e), and can translate along the direction that is perpendicular to $R_2R_6$. It can switch modes from deployable mode to translation mode through transition configuration, which is referred to the initial position shown in Fig. 4.42(a). In this transition configuration, the Bricard linkages and
8R linkages are in the shapes of regular triangles and squares respectively. The joint axis of $R_1'$ in the upper Bricard linkage and that in the lower Bricard linkage are collinear, and perpendicular to the joint axes of $R_1$ and $R_7$ of the 8R linkage.

Fig. 4.42 The triangular prism mechanism: (a) initial posture; (b) outward deploying; (c) inward deploying; (d) 2-DOF translation mode; (e) the equilateral 2-DOF mechanism

Fig. 4.43 The multiple-layer DPM constructed using Bricard linkages and 8R linkages: (a) the construction method; (b) initial posture; (c) outward deploying; (d) inward deploying; (e) 4-DOF translation mode
By joining \( n \) DPMs in Fig. 4.42 and \( n-1 \) DPMs in Fig. 4.2 along the axial direction, multiple-layer DPMs are obtained. The DPM has one DOF when deployed and the DOF is \( 2n \) in the translation mode. For example, the two-layer DPM in Fig. 4.43 has four DOFs in the translation mode. The mechanisms have the potential to be applied to robot arms.

Fig. 4.44 The cuboctahedron mechanism: (a) initial posture; (b, d) outward deploying; (c, e) inward deploying; (f) 3-DOF cuboctahedron mechanism mode; (g) the equilateral 3-DOF mechanism

A cuboctahedron mechanism is also constructed, using six 8R linkages and eight Bricard linkages, as shown in Fig. 4.44. The mechanism has 1-DOF when deployed and
can switch to 3-DOF cuboctahedron mechanism [171] mode [Fig. 4.44(f)] through the transition configuration (initial position). The mechanism in the cuboctahedron mechanism mode is equal to the mechanism shown in Fig. 4.44(g). As observed from the 3D model in CAD, the radius of the sphere generated by the mechanism reaches 144.63 mm when deployed outward [Fig. 4.44(d)] and can be 105.27 mm when deployed inward [Fig. 4.44(e)]. The deploying ratio of the mechanism is \( r = \pi r_o^3 / \pi r_i^3 = 144.63^3 / 105.27^3 = 2.59 \). It is noted the deploying ratio will increase if the structure is optimized to avoid interference.

A 1-DOF icosidodecahedron mechanism is built based on twelve 10R linkages and twenty Bricard linkages, as shown in Fig. 4.45. The deploying ratio of the mechanism is \( r = \pi r_o^3 / \pi r_i^3 = 221.85^3 / 156.65^3 = 2.84 \).

![Fig. 4.45 The icosidodecahedron mechanism: (a) initial posture; (b, d) outward deploying; (c, e) inward deploying](image)

Similarly, we can construct the following 15 DPMs (shown in Table 4.3): rhombicuboctahedron, truncated tetrahedron, truncated octahedron, truncated cube, snub cube, great rhombicuboctahedron, truncated icosahedron, truncated dodecahedron,
rhombicosidodecahedron, snub dodecahedron, pentagonal pyramid, triangular cupola, square cupola, pentagonal cupola, and pentagonal rotunda. It is noted that the DPMs based on the rhombicuboctahedron, truncated octahedron, great rhombicuboctahedron, rhombicosidodecahedron, triangular cupola, square cupola, pentagonal cupola have an additional motion mode with 6-DOF, 5-DOF, 5-DOF, 3-DOF, 1-DOF, 2-DOF and 3-DOF [171] respectively.

<table>
<thead>
<tr>
<th>Rhombicuboctahedron</th>
<th>Truncated tetrahedron</th>
<th>Truncated octahedron</th>
<th>Truncated cube</th>
<th>Snub cube</th>
</tr>
</thead>
<tbody>
<tr>
<td>6-DOFa</td>
<td>5-DOF</td>
<td>5-DOF</td>
<td>3-DOF</td>
<td>1-DOF</td>
</tr>
<tr>
<td>Rhombicuboctahedron</td>
<td>Truncated icosahedron</td>
<td>Truncated dodecahedron</td>
<td>Rhombicosidodecahedron</td>
<td>Snub dodecahedron</td>
</tr>
<tr>
<td>5-DOF</td>
<td>3-DOF</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pentagonal pyramid</td>
<td>Triangular cupola</td>
<td>Square cupola</td>
<td>Pentagonal cupola</td>
<td>Pentagonal rotunda</td>
</tr>
<tr>
<td>1-DOF</td>
<td>2-DOF</td>
<td>3-DOF</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a. The DOFs are those of the mechanisms in the polyhedral mechanism modes, when the two links of the triads are on the same plane.

4.4 Summary

In this chapter, a construction method for designing DPMs has been addressed. Spatial single-loop linkages, such as Bricard linkages, 8R linkages and 10R linkages composed of several symmetric spatial triad units are connected using S joints. There are two types of DPMs, including the prism mechanisms and polyhedral mechanisms. The DPMs have
only one DOF when deployed and can be deployed inward and outward. Several mechanisms involving 8R/10R/12R linkages have multiple modes and can switch modes through transition positions. The variations of the DPMs, in which the S joints have offset, and the DPMs with different types of loops, the same type of loops but different sizes, double-layer and multiple-layer DPMs have been discussed respectively.

The single-layer DPMs have potential applications for education, entertainment and decorations and the multiple-layer DPMs can be used in applications that require a large folding ratio, such as sunshield or other aerospace mechanisms.
CHAPTER 5 – TYPE SYNTHESIS OF DEPLOYABLE POLYHEDRAL MECHANISMS WITH MULTIPLE MODES CONNECTED USING RRR CHAINS

DPMs connected using S joints have been described in Chapter 4. However, S joints are not as practical as R joints. They are expensive and are not easy to assemble. Hence, in this chapter, RRR chains with perpendicular joints will be used to replace the S joint.

5.1 Single-loop Linkages

In this chapter, there are two types of triad units adopted, including unit I with triangle links which are used to construct single-loop linkages, and unit II with straight links to connect the single-loop linkages [Figs. 5.1(a) and (b) respectively]. As described in the previous chapter, the adjacent R joints can be with arbitrary twist angles. The orthogonal unit, which is the most practical one, is just an example to illustrate the construction method.

![Fig. 5.1 RRR triad: (a) unit I with triangle links; (b) unit II with straight links](image)

Figure 5.2 shows one case of Bricard linkages obtained by connecting three units I. The linkage is axisymmetric about the line that passes through the intersection of the axes of \(R_2, R_4\) and \(R_6\) and is perpendicular to the plane defined by the three R joints in the initial posture. The mechanism has 1-DOF and can be deployed outward [Fig. 5.2(b)] and inward [Fig. 5.2(c)].
Fig. 5.2 Orthogonal Bricard linkage constructed using RRR units: (a) initial posture; (b) outward deploying; (c) inward deploying

Fig. 5.3 Orthogonal 8R linkage constructed using RRR units: (a) initial posture; (b) outward deploying; (c) inward deploying; (d) planar 4R linkage mode

Fig. 5.4 Orthogonal 10R linkage constructed using RRR units: (a) initial posture; (b) outward deploying; (c) inward deploying; (d) planar 5R linkage mode

When connecting four units I, an 8R linkage can be obtained (Fig. 5.3). In the rest of this chapter, additional constraints are imposed on the linkage. In a deployable
mechanism composed of the 8R kinematic chain, there only exists a 1-DOF deploying mode [Figs. 5.3(b-c)], a 1-DOF planar 4R linkage mode [Fig. 5.3(d)] and a 1-DOF spherical 4R linkage mode (see Section 5.3 for details) due to the constraints imposed on the 8R kinematic chain by the deployable mechanism.

Similarly, the 10R kinematic chain in a deployable mechanism has only a 1-DOF deployable mode [Figs. 5.4(b-c)] and a 2-DOF 5R linkage mode [Fig. 5.4(d)]. In the next section, DPMs will be constructed using the above single-loop linkage chains.

5.2 DPMs Based on Bricard Linkages

The prism mechanism in Fig. 5.5 is constructed by connecting two identical Bricard linkages using six RRR kinematic chains (unit II, represented by $C_i$ in Fig. 5.5). The construction process is shown in Fig. 5.6(a). In the initial state [Fig. 5.5(a)], the joint axes of $R_1$ and $R'_1$ coincide and all the chains are parallel. The mechanism has 1-DOF and can be deployed outward and inward, as shown in Figs. 5.5(b-c). $R_2, R'_2$ and the three R joints in chain 1 and those in chain 2 form a spatial 8R linkage, so do $R_1, R'_1$, the three joints of chain 1 and those in chain 6. The joint axes of $R_{11}, R_{22}, R_{33}, R_{44}, R_{55}$ and $R_{66}$ (the second R joints within the chains) are always on the same plane (refers to the mirror plane) and the six R joints form a semi-regular hexagon, as shown in Fig. 5.6(b). Among the six R joints, $R_{22}, R_{44}$ and $R_{66}$ form an equilateral triangle, whose side length varies when deploying the mechanism.

![Fig. 5.5 Prism mechanism connected by Bricard linkages using six RRR chains: (a) initial posture; (b) outward deploying; (c) inward deploying](image)

A double-layer prism mechanism is also constructed using two prism mechanisms in Fig. 5.5(a), as shown in Fig. 5.7. The upper prism mechanism deploys outward with the lower prism mechanism deploying inward [Fig. 5.7(b)] and the upper prism mechanism
deploys inward with the lower prism mechanism deploying outward [Fig. 5.7(c)]. A four-layer prism mechanism is also assembled, as shown in Fig. 5.8.

Fig. 5.6 Prism mechanism connected by Bricard linkages using six RRR chains: (a) construction process; (b) the symmetry of the mechanism

Fig. 5.7 Double-layer prism mechanism based on Bricard linkages: (a) initial posture; (b) outward deploying; (c) inward deploying

Fig. 5.8 Four-layer prism mechanism
Fig. 5.9 Tetrahedron DPM based on Bricard linkages: (a) initial posture; (b) outward deploying; (c) inward deploying

By inserting the Bricard linkages into the faces of the tetrahedron, a tetrahedron mechanism constructed using four Bricard linkages and twelve RRR chains is obtained, as presented in Fig. 5.9(a). The mechanism can deploy inward and outward, as shown in Figs. 5.9(b-c).

5.3 DPMs Based on 8R/10R Linkages

Joining two 8R linkages using eight RRR chains, leads to a quadrangular prism mechanism, as shown in Fig. 5.10. In the initial state, the mechanism is in the shape of a regular quadrangular prism, the joint axes of \( R_1, R_3, R_5 \) and \( R_7 \) of the upper 8R linkage and the corresponding R joints of the lower 8R linkage are collinear, and all the RRR chains are parallel.

The mechanism has a 1-DOF deployable mode, as shown in Figs. 5.10(b-c). Through the singular position (the initial posture), the mechanism can switch into the 1-DOF planar 4R linkage mode, as shown in Fig. 5.10(d).

Now it will be verified that the quadrangular prism mechanism has only 1-DOF. Since the mechanism is always symmetric about the mirror plane, the mechanism is cut in half to facilitate analysis (Fig. 5.11). The cut mechanism is called the virtual-plane constrained 8R linkage. First, the joint axes of \( R_{11} \) and \( R_{33} \) are constrained to be on the mirror plane, the linkage is equivalent to the mechanism in Fig. 5.12(a). Assume that \( R_{11} \) is fixed on the plane, \( R_{33} \) has a translational DOF on the plane. The DOF of the linkage can be calculated using the conventional formula for DOF.

\[
M = 6(q - p) + \sum_{i=1}^{p} f_i = 6 \times (11 - 13) + 8 + 2 + 3 = 1 \quad (5.1)
\]

When \( s \) additional \( R_{ii} \) are constrained to be on the plane, the 8R linkage is over-constrained. Figure 5.12(b) shows the configuration in which \( R_{55} \) and \( R_{77} \) are also
constrained to be on the mirror plane ($s = 2$). The two joints have two translational DOFs on the plane respectively. The DOF of the virtual-plane-constrained orthogonal 8R linkage is still one, due to the symmetry characteristics of the linkage.

Fig. 5.10 Quadrangular prism mechanism connected by 8R linkages using eight RRR chains: (a) initial posture; (b) outward deploying; (c) inward deploying; (d) planar 4R linkage mode

Fig. 5.11 The virtual-plane constrained 8R linkage: (a) spatial mode; (b) planar 4R linkage mode; (c) spherical 4R linkage mode
The virtual-plane-constrained orthogonal 8R linkage has three 1-DOF modes, including 1) spatial mode [deployable mode, as shown in Fig. 5.11(a)], in which $R_{11}, R_{22}, R_{33}, R_{44}, R_{55}, R_{66}, R_{77}$ and $R_{88}$ form a semi-regular octagon, and $R_{11}, R_{33}, R_{55}$ and $R_{77}$ form a square, 2) planar 4R linkage mode [Fig. 5.11(b)], when $R_{11}/R_{22}/R_{33}/R_{66}$, and 3) spherical 4R linkage mode [Fig. 5.11(c), discussed in Section 5.4].

Similar to constrained 8R linkages, it can be readily proved that the virtual-plane-constrained orthogonal 10R/12R linkages also have one DOF when deployed. The sketch of the virtual-plane-constrained orthogonal 10R/12R linkages with five and six R joints constrained on the plane respectively are shown in Fig. 5.13. The virtual-plane-constrained orthogonal 10R and 12R linkages are both over-constrained.
Using the construction method, countless prism mechanisms can be obtained by connecting single-loop linkages using RRR chains. Figure 5.14 shows the pentagonal prism mechanism constructed by two 10R linkages using ten RRR chains. The mechanism has two modes, including a 1-DOF deployable mode [Figs. 5.14(b-c)] and a 2-DOF planar 5R mode [Fig. 5.14(d)].

![Pentagonal prism mechanism](image)

**(a)** Initial posture; **(b)** outward deploying; **(c)** inward deploying; **(d)** planar 5R linkage mode

Using six 8R linkages and twenty-four RRR chains, a rhombohedron mechanism is assembled. Apart from the 1-DOF deploying mode [Figs. 5.15(b-c)], the mechanism has a 3-DOF parallelepiped mechanism mode [Fig. 5.15(d)]. The two modes can achieve transitions through the singular position, i.e., the initial state.
Fig. 5.15 Rhombohedron mechanism based on 8R linkages: (a) initial posture; (b) outward deploying; (c) inward deploying; (d) 3-DOF parallelepiped mechanism mode

Fig. 5.16 Dodecahedron mechanism based on 10R linkages: (a) initial posture; (b) outward deploying; (c) inward deploying

A dodecahedron mechanism is also designed by connecting twelve 10R linkages, as shown in Fig. 5.16.

5.4 DPMs Connected Using Half the Number of RRR Chains
In this section, specific RRR chains of the prism mechanism in Section 5.2 (5.3) will be removed, and the mechanism obtains an additional rotation mode and has less interference.

Fig. 5.17 Prism mechanism connected by Bricard linkages using three RRR chains: (a) construction process; (b) the symmetry of the mechanism

The mechanism constructed by two Bricard linkages using three RRR chains (removing chains 1, 3 and 5 in Fig. 5) is shown in Fig. 5.17(a). $R_{22}$, $R_{44}$ and $R_{66}$ are always on the same plane and form an equilateral triangle [Fig. 5.17(b)]. Apart from the deploying mode [Figs. 5.18(b-c)], the mechanism has a 1-DOF rotation mode, in which
the upper platform rotates around the central line of the mechanism while deploying, as shown in Fig. 5.18(d) (it is noted the lower Bricard linkage also deploys within a small range). In this case, the mechanism is not symmetric about the mirror plane anymore. The mechanism can further deform into the configurations shown in Figs. 5.18(e) and (f) by rotating the lower platform around the opposite direction. Since there is no interference between the adjacent RRR chains, the mechanism can be folded by deploying outward, as shown in Fig. 5.18(g). Then it can be further deployed into the posture in Fig. 5.18(h).

To my best knowledge, among all the deployable mechanisms with multiple modes in the literature, this mechanism has the simplest structure. Besides, it is the first 1-DOF mechanism that is composed of Bricard linkages and has multiple modes.

Fig. 5.19 Quadrangular prism mechanism constructed by 8R linkages using four RRR chains: (a) initial state; (b) deploying mode; (c) planar 4R linkage mode; (d) rotation mode; (e) deploying mode↔spherical 4R linkage mode; (f) spherical 4R linkage mode

Connecting two 8R linkages using four RRR chains, leads to a quadrangular prism mechanism with four motion modes. The same as the mechanism in Fig. 5.10, the mechanism with four RRR chains can also be deployed through two ways and has a planar 4R linkage mode [Figs. 5.19(b-c)]. Besides, it has an additional rotation mode, in which the two 8R linkages don’t deploy synchronously [Fig. 5.19(d)]. In addition, during the
deploying process, the mechanism can turn into the configuration in Fig. 5.19(e). Through this position, the mechanism can switch into the spherical 4R mode, as shown in Fig. 5.19(f). The two 8R linkages move synchronously and both behave as spherical 4R linkages.

Similarly, connecting two 10R linkages with five RRR chains, as shown in Fig. 5.20, the pentagonal prism mechanism also obtains an additional 1-DOF rotation mode [Fig. 5.20(b)], apart from the deploying mode and the planar 5R linkage mode [Figs. 5.20(b-e)].

When removing chains 2, 3 and 5 in Fig. 5.5, the prism mechanism obtained (Fig. 5.21) also has 1-DOF. Apart from the deployable mode [Fig. 5.21(b)], the mechanism has a bending mode, as shown in Figs. 5.21(c-d). The mechanism in this mode is symmetric about the mirror plane and is not line-symmetric. Similarly, removing chains 1, 2 and 3, results in another mechanism with 1-DOF deployable mode [Fig. 5.22(b)] and 1-DOF bending mode [Figs. 5.22(c-d)].
5.5 Summary

In this chapter, a novel construction method has been proposed to construct DPMs with multiple modes. Using symmetric spatial RRR compositional units, single-loop linkages such as Bricard linkage, 8R linkage and 10R linkage are first built, and DPMs are then obtained by connecting these single-loop linkages using type II symmetric spatial RRR compositional units. The DPMs have only 1-DOF when deployed and can be deployed inward and outward. The mechanisms based on 8R and 10R linkages and the one constructed by connecting two Bricard linkages and three RRR chains have multiple modes and can switch modes through transition positions.
The polyhedral mechanisms can be used in applications for decorations, entertainment and education. The prism mechanisms have potential applications on grippers, as shown in Fig. 5.23. Since there are two layers of grippers, they are more stable than the normal grippers [Fig. 5.23(a-c)]. Especially the prism mechanisms with half the number of RRR limbs, they can achieve 1-DOF rotation motions [Fig. 5.23(d-e)]. The one with multiple layers can be applied to the fall protection system (Fig. 5.24). When the upper prism mechanism deploys outward, the object is stuck in the first layer. When the upper prism mechanism deploys inward, the object falls into the second layer and is stuck again. It can also be used for the other types of transmission.
CHAPTER 6 – STATICALLY BALANCED DEPLOYABLE
POLYHEDRAL MECHANISMS WITH MULTIPLE MODES

In Chapters 3, 4 and 5, deployable mechanisms with multiple modes have been designed. In this chapter, the mechanisms obtained will be developed into statically balanced mechanisms.

The static balancing methods, including the ones for planar 4R parallelograms, general planar mechanisms, spherical mechanisms and spatial mechanisms, and the approach by using optimization tools will be addressed first.

6.1 Static Balancing Methods of Planar 4R Parallelogram

The 4R parallelogram mechanism is defined as the 4-bar mechanism consisting of two cranks, with the same link lengths, connected by a coupler. It has been widely applied to the railway engine wheels and lifting mechanisms. In this section, the balancing conditions for a planar 4R parallelogram linkage will be derived using both an algebraic method and a geometric method. The aim of this design is to use as few springs as possible to reduce the interference. We start with the case in which at least three springs are needed. After achieving balancing using three springs, the stiffness of specific springs is set as zero to reduce the number of the springs. We assume all the springs are zero-free-length springs.

6.1.1 Algebraic Method for Balancing Using External Springs

6.1.1.1 Static balancing of 4R parallelogram mechanism with three springs

A planar 4R parallelogram linkage is shown in Fig. 6.1(a). The four R joints of the mechanism are denoted by $R_1$, $R_2$, $R_3$, and $R_4$. The length of links 1 and 3 are $L$, and the length of link 2 and the frame are $L'$. Link 2 remains horizontal during the motion. A coordinate system is fixed to the base with its $z$-axis pointing vertically upward and its origin located at $R_1$. $x$-axis is along $R_4R_1$ and $y$-axis can be obtained by the right-hand rule.

Let the position vectors of the CM of the $i$th link $^iP_i$ and the spring connecting point $^iA_i$ expressed in the $i$th local coordinate frame be

$$
^iP_i = \begin{bmatrix} a_{pi} \\ b_{pi} \\ c_{pi} \end{bmatrix}^T
$$

$$
^iA_i = \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix}^T
$$

(6.1)
Fig. 6.1 Static balancing of planar 4R parallelogram linkage using three springs (algebraic method): (a) the sketch of the 4R parallelogram linkage; (b) the spring attachment point; (c) special case a; (d) special case b; (e) special case c

The position vectors of the CMs of the links and the spring connecting points in the global coordinate frame are calculated as

\[
P_1 = \left\{-\sqrt{a_{p1}^2 + b_{p1}^2 c(\theta + \arctan \frac{-b_{p1}}{a_{p1}})} - c_{p1} \sqrt{a_{p1}^2 + b_{p1}^2 s(\theta + \arctan \frac{-b_{p1}}{a_{p1}})}\right\}^T
= \{-a_{p1}c\theta - b_{p1}s\theta - c_{p1} c_{p1}s\theta - b_{p1}c\theta\}^T
\]

\[
P_2 = P_{R2} + 2P_2 = \{-Lc\theta - a_{p2} - c_{p2} \quad LS\theta - b_{p2}\}^T
\]

\[
P_3 = \left\{\sqrt{(l - a_{p3})^2 + b_{p3}^2 c(\theta - \beta) - L'} - c_{p3} \sqrt{(l - a_{p3})^2 + b_{p3}^2 s(\theta - \beta)}\right\}^T
= \{-Lc\theta - L' + a_{p3}c\theta + b_{p3}s\theta - c_{p3} Ls\theta + b_{p3}c\theta - a_{p3}s\theta\}^T
\]

where

\[
\beta = \arctan \frac{-b_{p3}}{l - a_{p3}}
\]

\[
A_1 = \{-a_1c\theta - b_1s\theta - c_1 a_1s\theta - b_1c\theta\}^T
\]
\[ A_2 = (-LC\theta - a_2 \quad -c_2 \quad LS\theta - b_2)^T \]
\[ A_3 = (-LC\theta - L' + a_3 C\theta + b_3 S\theta \quad -c_3 \quad LS\theta + b_3 C\theta - a_3 S\theta)^T \]

According to [102, 104], the spring connecting points on the base, \( H_1 \) and \( H_3 \), for links 1 and 3, which are mounted on the base, should be right above the associated R joints [Fig. 6.1(b)]. Position vectors \( H_1 \) and \( H_3 \) are given by
\[ H_1 = \{ 0 \quad 0 \quad h_1 \}^T \]
\[ H_3 = \{ -L' \quad 0 \quad h_3 \}^T \]

Suppose the position vector of the spring connecting point on the base, \( H_2 \), for link 2 is
\[ H_2 = \{ s \quad 0 \quad h_2 \}^T \]

The total potential energy is
\[ V_{123} = \frac{1}{2} k_1 |A_1 - H_1|^2 + \frac{1}{2} k_2 |A_2 - H_2|^2 + \frac{1}{2} k_3 |A_3 - H_3|^2 + m_1 g P_{1z} + m_2 g P_{2z} + m_3 g P_{3z} = \frac{1}{2} \{(a_1^2 + b_1^2 + c_1^2 + h_1^2)k_1 + a_2^2 k_2 + b_2^2 k_2 + c_2^2 k_2 + 2b_2 k_2 h_2 + h_2^2 k_2 + k_2 L^2 + k_3 L^2 + k_3 (a_3^2 + b_3^2 + c_3^2 + h_3^2 - 2L a_3) - 2b_p g m_2 + 2a_2 k_2 s + k_2 s^2 + 2[b_1 h_1 k_1 - b_3 h_3 k_3 + m_3 g b_p_3 - m_1 g b_p_1 + k_2 L (a_2 + s)] C\theta + 2[a_1 h_1 k_1 - a_3 h_3 k_3 + L[h_3 k_3 + (b_2 + h_2) k_2] - m_1 g a_p_1 - m_2 g L + m_3 g a_p_3 - m_3 g L] S\theta \]

where \( m_i \) is the mass of the \( i \)th link, \( g \) is the gravitational acceleration, and \( k_i \) is the stiffness of the \( i \)th spring.

In a statically balanced system, the potential energy of the system is constant, and the coefficients of the variable \( \theta \) must vanish [117]. The conditions for the static balancing of the 4R linkage using three springs are obtained as:
\[
\begin{align*}
b_1 h_1 k_1 - b_3 h_3 k_3 + m_3 g b_{p_3} - m_1 g b_{p_1} + k_2 L (a_2 + s) &= 0 \quad \text{(6.8)} \\
a_1 h_1 k_1 - a_3 h_3 k_3 + L [h_3 k_3 + (b_2 + h_2) k_2] - m_1 g a_{p_1} - m_2 g L + m_3 g a_{p_3} - m_3 g L &= 0 \quad \text{(6.7)}
\end{align*}
\]

Equation (6.7) indicates that the balancing conditions for the 4R parallelogram linkage are independent of the length and the local position of the CM of link 2 (coupler), because \( a_{p_2}, b_{p_2} \) and \( L' \) have vanished.

In the following, we will discuss three specific cases:

Case a: \( m_1 a_{p_1} = m_3 a_{p_3}, m_1 b_{p_1} = m_3 b_{p_3} \) [Fig. 6.1(c)]. In this case the balancing conditions are
\[
\begin{align*}
b_1 h_1 k_1 - b_3 h_3 k_3 + k_2 L (a_2 + s) &= 0 \\
a_1 h_1 k_1 - a_3 h_3 k_3 + L [h_3 k_3 + (b_2 + h_2) k_2] - (m_2 + m_3) g L &= 0
\end{align*}
\]

The balancing conditions are only related to the weights of links 2 and 3.
Case b: \(m_1a_p1 - m_1 = m_3a_p3 - m_3L, m_1b_p1 = m_3b_p3\) [Fig. 6.1(d)]. In this case, the balancing conditions are

\[
\begin{align*}
\{ & b_1h_1k_1 - b_2h_3k_3 + k_2L(a_2 + s) = 0 \\
& a_1h_1k_1 - a_3h_3k_3 + L[h_3k_3 + (b_2 + h_2)k_2] - (m_1 + m_2)gL = 0
\}
\]

(6.9)

The balancing conditions are only related to the weights of links 1 and 2.

Case 3: \(m_3a_p3 - m_3L = m_1a_p1, m_1b_p1 = m_3b_p3\) [Fig. 6.1(e)]. In this case the balancing conditions are

\[
\begin{align*}
\{ & b_1h_1k_1 - b_2h_3k_3 + k_2L(a_2 + s) = 0 \\
& a_1h_1k_1 - a_3h_3k_3 + L[h_3k_3 + (b_2 + h_2)k_2] - m_2gL = 0
\}
\]

(6.10)

Links 1 and 3 are balanced by each other, one only needs to balance link 2.

From Eqs. (6.8-6.10), it is observed that the balancing conditions for the 4R parallelogram linkage are independent of the positions of the CMs of the links in the above three cases \(a_p1, a_p3, b_p1\) and \(b_p3\) have vanished). This is important, because it shows there is no need to consider the length of link 2 and the positions of CMs of the links when balancing these particular 4R parallelogram linkages. One can set \(L' = L\) and \(^iP_t = \{0 \ 0 \ 0\}^T\) when analyzing the 4R parallelogram linkage, for the convenience of calculation.

6.1.1.2 Static balancing of 4R parallelogram mechanism with two springs

Fig. 6.2 Static balancing of planar 4R parallelogram linkage using two springs (algebraic method): (a) the springs are attached to links 1 and 2; (b) the springs are attached to links 1 and 3; (c) the springs are attached to links 2 and 3;

Now let us investigate how to balance the linkage using two springs. Let the stiffness of one of the three springs of the system in Fig. 6.1(b) be zero, the balancing conditions for the parallelogram with two springs are readily obtained.
If the planar 4R parallelogram linkage is statically balanced using two springs attached to links 1 and 2 respectively [Fig. 6.2(a)], the total potential energy of the system can be obtained from Eq. (6.6) by setting $k_1 = 0$ as:

$$V_{12} = \frac{1}{2} k_1 |\mathbf{A}_1 - \mathbf{H}_1|^2 + \frac{1}{2} k_2 |\mathbf{A}_2 - \mathbf{H}_2|^2 + m_1 g P_{1z} + m_2 g P_{2z} + m_3 g P_{3z} = \frac{1}{2} \{ (a_1^2 + b_1^2 + c_1^2 + h_1^2) k_1 + a_2^2 k_2 + b_2^2 k_2 + c_2^2 k_2 + 2 b_2 k_2 h_2 + h_2^2 k_2 + k_2 L^2 - 2 b_2 g m_2 + 2 a_2 k_2 s + k_2 s^2 + 2 (b_2 h_1 k_1 + m_3 g b_{p3} - m_1 g b_{p1} + k_2 L(a_2 + s)) \} \theta + 2 (a_1 h_1 k_1 + L(b_2 + h_2) k_2 - m_1 g a_{p1} - m_2 g L + m_3 g a_{p3} - m_3 g L) \theta \}
$$

(6.11)

The balancing conditions are derived as:

$$\begin{align*}
\{ b_1 h_1 k_1 + m_3 g b_{p3} - m_1 g b_{p1} + k_2 L(a_2 + s) = 0 \\
(a_1 h_1 k_1 + L(b_2 + h_2) k_2 - m_1 g a_{p1} - m_2 g L + m_3 g a_{p3} - m_3 g L = 0
\end{align*}
$$

(6.12)

If $m_1 b_{p1} = m_3 b_{p3}$ and $a_{p3} = L$, there is no need to balance link 3. Equation (6.12) becomes

$$\begin{align*}
\{ b_1 h_1 k_1 + k_2 L(a_2 + s) = 0 \\
(a_1 h_1 k_1 + L(b_2 + h_2) k_2 - m_1 g a_{p1} - m_2 g L = 0
\end{align*}
$$

(6.13)

Let $k_2 = 0$ [in Eq. (6.6)], i.e., the two springs are connected to links 1 and 3 respectively [Fig. 6.2(b)]. The potential energy of the system and the balancing conditions are yielded as:

$$V_{13} = \frac{1}{2} k_1 |\mathbf{A}_1 - \mathbf{H}_1|^2 + \frac{1}{2} k_3 |\mathbf{A}_3 - \mathbf{H}_3|^2 + m_1 g P_{1z} + m_2 g P_{2z} + m_3 g P_{3z} = \frac{1}{2} \{ (a_1^2 + b_1^2 + c_1^2 + h_1^2) k_1 + k_3 L^2 + k_3 (a_2^2 + b_2^2 + c_3^2 + h_3^2 - 2 L a_3) - 2 b_2 g m_2 + 2 (b_1 h_1 k_1 - b_3 h_3 k_3 + m_3 g b_{p3} - m_1 g b_{p1}) \} \theta + 2 (a_1 h_1 k_1 - a_3 h_3 k_3 + L h_3 k_3 - m_1 g a_{p1} - m_2 g L + m_3 g a_{p3} - m_3 g L) \theta \}
$$

(6.14)

$$\begin{align*}
\{ b_1 h_1 k_1 - b_3 h_3 k_3 + m_3 g b_{p3} - m_1 g b_{p1} = 0 \\
(a_1 h_1 k_1 - a_3 h_3 k_3 + L h_3 k_3 - m_1 g a_{p1} - m_2 g L + m_3 g a_{p3} - m_3 g L = 0
\end{align*}
$$

(6.15)

Similarly, let $k_1 = 0$ [in Eq. (6.6)]. The springs are attached to links 2 and 3 respectively [Fig. 6.2(c)], the potential energy of the system and the balancing conditions are calculated as

$$V_{23} = \frac{1}{2} k_2 |\mathbf{A}_2 - \mathbf{H}_2|^2 + \frac{1}{2} k_3 |\mathbf{A}_3 - \mathbf{H}_3|^2 + m_1 g P_{1z} + m_2 g P_{2z} + m_3 g P_{3z} = \frac{1}{2} \{ a_2^2 k_2 + b_2^2 k_2 + c_2^2 k_2 + 2 b_2 k_2 h_2 + h_2^2 k_2 + k_2 L^2 + k_3 (a_3^2 + b_3^2 + c_3^2 + h_3^2 - 2 L a_3) - 2 b_2 g m_2 + 2 a_2 k_2 s + k_2 s^2 + 2 (b_2 h_3 k_3 + m_3 g b_{p3} - m_1 g b_{p1} + k_2 L(a_2 + s)) \} \theta + 2 (a_3 h_3 k_3 + L h_3 k_3 - m_1 g a_{p1} - m_2 g L + m_3 g a_{p3} - m_3 g L) \theta \}
$$

(6.16)

$$\begin{align*}
\{ - b_3 h_3 k_3 + m_3 g b_{p3} - m_1 g b_{p1} + k_2 L(a_2 + s) = 0 \\
(-a_3 h_3 k_3 + L h_3 k_3 - m_1 g a_{p1} - m_2 g L + m_3 g a_{p3} - m_3 g L = 0
\end{align*}
$$
If \( m_1 b_{p1} = m_3 b_{p3} \) and \( a_{p1} = 0 \), there is no need to balance link 1, as shown in Fig. 6.2(e).

The balancing conditions become

\[
\begin{aligned}
-b_3 h_3 k_3 + k_2 L (a_2 + s) &= 0 \\
-a_3 h_3 k_3 + L \left[ h_3 k_3 + (b_2 + \theta_2) k_2 \right] - m_2 g L + m_3 g a_{p3} - m_3 g \theta &= 0
\end{aligned}
\]  
(6.18)

### 6.1.1.3 Static balancing of 4R parallelogram mechanism with one spring

Now it will be verified that the linkage can be balanced using one spring. The spring is attached to links 1, 2 or 3 respectively, as shown in Fig. 6.3.

Fig. 6.3 Static balancing of planar 4R parallelogram linkage using one spring (algebraic method): (a) the spring is attached to link 1; (b) the spring is attached to link 3; (c) the spring is attached to link 2

Fig. 6.4 The solutions of static balancing of planar 4R parallelogram linkage using one springs: (a) the spring is attached to link 1; (b) the spring is attached to link 3; (c) the spring is attached to link 2

Let \( k_2 = k_3 = 0 \) [in Eq. (6.6)], there is only one spring used to connect link 1 to balance the 4R parallelogram linkage. The total potential energy of the system is
\[ V_1 = \frac{1}{2} k_1 |A_1 - H_1|^2 + m_1 g P_{1z} + m_2 g P_{2z} + m_3 g P_{3z} = \frac{1}{2} \left[ (a_1^2 + b_1^2 + c_1^2 + h_1^2) k_1 - 2b_2 g m_2 + 2(b_1 h_1 k_1 + m_3 g b p_3 - m_1 g b p_1) C \theta + 2(a_1 h_1 k_1 - m_1 g a p_1 - m_2 g L + m_3 g a p_3 - m_3 g L) S \theta \right] \]

The balancing conditions are derived as:

\[ \begin{align*}
&b_1 h_1 k_1 + m_3 g b p_3 - m_1 g b p_1 = 0 \\
&a_1 h_1 k_1 - m_1 g a p_1 - m_2 g L + m_3 g a p_3 - m_3 g L = 0
\end{align*} \]  

(6.19)

If \( m_1 a p_1 = m_3 a p_3 \) and \( m_1 b p_1 = m_3 b p_3 \) [Fig. 6.1(c)], the balancing conditions are

\[ \begin{align*}
&b_1 = 0 \\
&a_1 h_1 k_1 - (m_2 + m_3) g L = 0
\end{align*} \]  

(6.20)

Eq. (6.21) shows that the spring is attached to the line \( R_1 R_2 \), as shown in Fig. 6.4(a).

Similarly, when the spring is attached to link 3, the potential energy of the system and the balancing conditions are obtained as

\[ V_3 = \frac{1}{2} k_3 |A_3 - H_3|^2 + m_1 g P_{1z} + m_2 g P_{2z} + m_3 g P_{3z} = \frac{1}{2} \left[ k_3 L^2 + k_3 (a_2^2 + b_2^2 + c_2^2 + h_3^2 - 2La_3) - 2b_2 g m_2 + 2(-a_3 h_3 k_3 + m_3 g b p_3 - m_1 g b p_1) C \theta + 2(-a_3 h_3 k_3 + L h_3 k_3 - m_1 g a p_1 - m_2 g L + m_3 g a p_3 - m_3 g L) S \theta \right] \]

(6.22)

\[ \begin{align*}
&-b_3 h_3 k_3 + m_3 g b p_3 - m_1 g b p_1 = 0 \\
&-a_3 h_3 k_3 + L h_3 k_3 - m_1 g a p_1 - m_2 g L + m_3 g a p_3 - m_3 g L = 0
\end{align*} \]  

(6.23)

When \( m_1 a p_1 = m_3 a p_3, m_1 b p_1 = m_3 b p_3 \) [Fig. 6.1(c)], the balancing conditions are

\[ \begin{align*}
&b_3 = 0 \\
&(L - a_3) h_3 k_3 - L g (m_2 + m_3) = 0
\end{align*} \]  

(6.24)

The spring attachment point on the link should be on the line \( R_3 R_4 \) [Fig. 6.4(b)].

When letting \( k_1 = k_3 = 0 \) [in Eq. (6.6)], namely, the spring is attached to link 2. The potential energy of the system is

\[ V_2 = \frac{1}{2} k_2 |A_2 - H_2|^2 + m_1 g P_{1z} + m_2 g P_{2z} + m_3 g P_{3z} = \frac{1}{2} \left[ a_2^2 k_2 + b_2^2 k_2 + c_2^2 k_2 + 2b_2 k_2 h_2 + h_2^2 k_2 + k_2 L^2 - 2b_2 g m_2 + 2a_2 k_2 s + k_2 s^2 + 2[m_3 g b p_3 - m_1 g b p_1 + k_2 L(a_2 + s)] C \theta + 2[L(b_2 + h_2)k_2 - m_1 g a p_1 - m_2 g L + m_3 g a p_3 - m_3 g L] S \theta \right] \]

(6.25)

The following balancing conditions are derived as

\[ \begin{align*}
&m_3 g b p_3 - m_1 g b p_1 + k_2 L(a_2 + s) = 0 \\
&(L(b_2 + h_2)k_2 - m_1 g a p_1 - m_2 g L + m_3 g a p_3 - m_3 g L) = 0
\end{align*} \]  

(6.26)

Let \( m_1 a p_1 = m_3 a p_3, m_1 b p_1 = m_3 b p_3 \) [Fig. 6.1(c)], the balancing conditions become

\[ \begin{align*}
&a_2 + s = 0 \\
&(b_2 + h_2) k_2 - (m_2 + m_3) g = 0
\end{align*} \]  

(6.27)
It is observed from Eq. (6.27) that the \( x \) coordinate of the spring attachment point on the base in the global coordinate and that on link 2 in the local coordinate should be the same [Fig. 6.4(c)]. Besides, the sum of the \( z \) coordinate of the spring attachment point on the base in the global coordinate and that on link 2 in the local coordinate should be constant and is equal to \((m_2 + m_3)g/k_2\).

### 6.1.2 Algebraic Method for Balancing Using Internal Springs

![Diagram](image)

Fig. 6.5 Static balancing of planar 4R parallelogram mechanism using internal springs (algebraic method): (a) using three springs; (b-d) using two springs; (e-g) using one spring

In this section, the 4R parallelogram mechanism will be balanced using internal springs. First, three springs will be used to connect links 1 and 2, links 2 and 3, and links 1 and 3 respectively.
Let the stiffness of springs be \( k_4, k_5 \) and \( k_6 \), the total potential energy of the mechanism is obtained as

\[
V_{456} = \frac{1}{2}\{b_1^2 k_4 + b_2^2 k_4 + c_1^2 k_4 - 2c_1 c_2 k_4 + c_2^2 k_4 + 2a_1 a_3 k_5 + b_1^2 k_5 + 2b_1 b_3 k_5 + b_2^2 k_5 + c_1^2 k_5 - 2c_1 c_3 k_5 + c_2^2 k_5 + a_1^2 (k_4 + k_5) + b_2^2 k_6 + b_3^2 k_6 + c_2^2 k_6 - 2c_2 c_3 k_6 + c_3^2 k_6 + a_2^2 (k_4 + k_6) + a_3^2 (k_5 + k_6) - 2a_3 k_5 L - 2a_1 (k_4 + k_5) L + k_4 L^2 + k_5 L^2 - 2a_2 k_2 L' + (k_5 + k_6) L'^2 - 2m_2 b p_2 - 2(a_1 a_2 k_4 + b_1 b_2 k_4 - a_2 a_3 k_5 - b_2 b_3 k_5 - a_2 a_3 k_5 - k_5 L) L' + g(m_1 g b p_1 - m_3 g b p_3) \}/ \theta - 2(-a_1 b_2 k_4 + a_3 b_2 k_6 + a_2 (b_1 k_4 - b_3 k_6) + b_2 k_4 L + (b_1 + b_3) k_5 L' + b_3 k_6 L' - m_1 g a p_1 - m_2 g L + m_3 g a p_3 - m_3 g L \}
\]

(6.28)

The following balancing conditions are derived as

\[
\begin{aligned}
&\begin{cases}
 a_1 a_2 k_4 + b_1 b_2 k_4 - a_2 a_3 k_6 - b_2 b_3 k_6 - a_2 k_4 L + [a_1 k_5 + a_3 (k_5 + k_6) - k_5 L] L' + g(m_1 g b p_1 - m_3 g b p_3) = 0 \\
 -a_1 b_2 k_4 + a_3 b_2 k_6 + a_2 (b_1 k_4 - b_3 k_6) + b_2 k_4 L + (b_1 + b_3) k_5 L' + b_3 k_6 L' - m_1 g a p_1 - m_2 g L + m_3 g a p_3 - m_3 g L = 0
\end{cases}
\end{aligned}
\]

(6.29)

Let \( k_6 = 0 \), which means only two springs are used to connect links 1 and 2, and links 1 and 3, the potential energy of the mechanism is

\[
V_{45} = \frac{1}{2}\{b_1^2 k_4 + b_2^2 k_4 + c_1^2 k_4 - 2c_1 c_2 k_4 + c_2^2 k_4 + 2a_1 a_3 k_5 + b_1^2 k_5 + 2b_1 b_3 k_5 + b_2^2 k_5 + c_1^2 k_5 - 2c_1 c_3 k_5 + c_2^2 k_5 + a_1^2 (k_4 + k_5) + b_2^2 k_6 + b_3^2 k_6 + c_2^2 k_6 - 2c_2 c_3 k_6 + c_3^2 k_6 + a_2^2 (k_4 + k_6) + a_3^2 (k_5 + k_6) - 2a_3 k_5 L - 2a_1 (k_4 + k_5) L + k_4 L^2 + k_5 L^2 - 2m_2 g b p_2 - 2(a_1 a_2 k_4 + b_1 b_2 k_4 - a_2 a_3 k_5 - b_2 b_3 k_5 - a_2 a_3 k_5 - k_5 L) L' + g(m_1 g b p_1 - m_3 g b p_3) \}/ \theta - 2(-a_1 b_2 k_4 + a_3 b_2 k_6 + a_2 (b_1 k_4 - b_3 k_6) + b_2 k_4 L + (b_1 + b_3) k_5 L' + b_3 k_6 L' - m_1 g a p_1 - m_2 g L + m_3 g a p_3 - m_3 g L \}
\]

(6.30)

The balancing conditions are

\[
\begin{aligned}
&\begin{cases}
 a_1 a_2 k_4 + b_1 b_2 k_4 - a_2 k_4 L + (a_1 k_5 + a_3 (k_5 - k_5 L)) L' + g(m_1 g b p_1 - m_3 g b p_3) = 0 \\
 -a_1 b_2 k_4 + a_2 b_1 k_4 + b_2 k_4 L + (b_1 + b_3) k_5 L' - m_1 g a p_1 - m_2 g L + m_3 g a p_3 - m_3 g L = 0
\end{cases}
\end{aligned}
\]

(6.31)

When two springs are adopted to connect links 1 and 2, and links 2 and 3, the potential energy is calculated as

\[
V_{46} = \frac{1}{2}\{b_1^2 k_4 + b_2^2 k_4 + c_1^2 k_4 - 2c_1 c_2 k_4 + c_2^2 k_4 + a_1^2 k_4 + b_2^2 k_6 + b_3^2 k_6 + c_2^2 k_6 - 2c_2 c_3 k_6 + c_3^2 k_6 + a_2^2 (k_4 + k_6) + a_3^2 k_6 - 2a_3 k_4 L + k_4 L^2 - 2a_2 k_6 L' + k_6 L^2 - 2m_2 g b p_2 - 2(a_1 a_2 k_4 + b_1 b_2 k_4 - a_2 a_3 k_6 - b_2 b_3 k_6 - a_2 a_3 k_6 + a_3 k_6 L + a_3 k_6 L' +
\]

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\[ g(m_1gb_{p1} - m_3gb_{p3})\theta - 2[-a_1b_2k_4 + a_3b_2k_6 + a_2(b_1k_4 - b_3k_6) + b_2k_4L + b_3k_6L' - m_1ga_{p1} - m_2gL + m_3ga_{p3} - m_3gL]S\theta \] \tag{6.32}

The mechanism is balanced if
\[
\begin{aligned}
a_1a_2k_4 + b_1b_2k_4 - a_2a_3k_6 - b_2b_3k_6 - a_2k_4L + a_3k_6L' + g(m_1gb_{p1} - m_3gb_{p3}) &= 0 \\
-a_1b_2k_4 + a_3b_2k_6 + a_2(b_1k_4 - b_3k_6) + b_2k_4L + b_3k_6L' &= 0 \\
-m_1ga_{p1} - m_2gL + m_3ga_{p3} - m_3gL &= 0
\end{aligned}
\tag{6.33}

Similarly, if two springs are attached to links 1 and 3, and links 2 and 3 respectively, the potential energy is
\[
V_{56} = \frac{1}{2}[2a_1a_3k_5 + b_1^2k_5 + 2b_1b_3k_5 + b_2^2k_5 + c_1^2k_5 - 2c_1c_3k_5 + c_2^2k_5 + a_1^2k_5 + b_2^2k_6 + b_2^2k_6 + c_2^2k_6 - 2c_2c_3k_6 + c_3^2k_6 + a_2^2(k_5 + k_6) - a_2a_3k_5L - 2a_1k_5L + k_5L^2 - 2a_2k_6L + (k_5 + k_6)L^2 - 2m_2gb_{p2} - 2[-a_2a_3k_5 - b_2b_3k_6 + [a_1k_5 + a_3(k_5 + k_6) - k_5L]L' + g(m_1gb_{p1} - m_3gb_{p3})]\theta - 2[a_3b_2k_6 - a_2b_3k_6 + (b_1 + b_3)k_5L' + b_3k_6L' - m_1ga_{p1} - m_2gL + m_3ga_{p3} - m_3gL]S\theta \tag{6.34}
\]

The balancing conditions are
\[
\begin{aligned}
-a_2a_3k_6 - b_2b_3k_6 + [a_3k_5 + a_3(k_5 + k_6) - k_5L &= 0 \\
a_3b_2k_6 - a_2b_3k_6 + (b_1 + b_3)k_5L' + b_3k_6L' - m_1ga_{p1} - m_2gL + m_3ga_{p3} - m_3gL &= 0
\end{aligned}
\tag{6.35}

When only one spring is used to connect links 1 and 2, the potential energy is yielded as
\[
V_4 = \frac{1}{2}k_4|A_1 - A_2|^2 + m_1gp_{1z} + m_2gp_{2z} + m_3gp_{3z} = \frac{1}{2}[k_4(a_2^2 + b_1^2 + b_2^2 + (c_1 - c_2)^2 + (a_1 - L)^2] - 2m_2gb_{p2} - 2(a_1a_2k_4 + b_1b_2k_4 - a_2k_4L + m_1gb_{p1} - m_3gb_{p3})\theta + 2[-a_2b_1k_4 + a_1b_2k_4 - b_2k_4L + m_1ga_{p1} + m_2gL - m_3ga_{p3} + m_3gL]S\theta \tag{6.36}
\]

The following balancing conditions are derived as
\[
\begin{aligned}
a_1a_2k_4 + b_1b_2k_4 - a_2k_4L + m_1gb_{p1} - m_3gb_{p3} &= 0 \\
-a_2b_1k_4 + a_1b_2k_4 - b_2k_4L + m_1ga_{p1} + m_2gL - m_3ga_{p3} + m_3gL &= 0
\end{aligned}
\tag{6.37}

When the spring is attached to links 1 and 3, the potential energy is
\[
V_5 = \frac{1}{2}k_5|A_1 - A_3|^2 + m_1gp_{1z} + m_2gp_{2z} + m_3gp_{3z} = \frac{1}{2}[k_5(b_1 + b_3)^2 + (c_1 - c_3)^2 + (a_1 + a_3 - L)^2 + L_2^2] - 2m_2gb_{p2} - 2[k_5L_2(a_1 + a_3 - L) + m_1gb_{p1} - m_3gb_{p3})\theta + 2[-(b_1 + b_3)k_5L_2 + m_1ga_{p1} - m_3ga_{p3} + m_2gL + m_3gL]S\theta \tag{6.38}
\]
The mechanism is statically balanced if

$$
\begin{align*}
&k_5L_2(a_1 + a_3 - L) + m_1gb_{p1} - m_3gb_{p3} = 0 \\
&-(b_1 + b_3)k_5L_2 + m_1ga_{p1} - m_3ga_{p3} + m_2gL + m_3gL = 0
\end{align*}
$$

(6.38)

If the spring is attached to links 2 and 3, the potential energy is

$$
V_6 = \frac{1}{2}[b_2^2k_6 + b_3^2k_6 + c_2^2k_6 - 2c_2c_3k_6 + c_3^2k_6 + a_2^2k_6 + a_3^2k_6 - 2a_2k_6L' + k_6L'^2 - 2m_2gb_{p2} - 2[-a_2a_3k_6 - b_2b_3k_6 + a_3k_6L' + g(m_1gb_{p1} - m_3gb_{p3})]C\theta - 2(a_2b_2k_6 - a_2b_3k_6 + b_3k_6L' - m_1ga_{p1} - m_2gL + m_3ga_{p3} - m_3gL)S\theta] \\
$$

(6.39)

The balancing conditions are

$$
\begin{align*}
&-a_2a_3k_6 - b_2b_3k_6 + a_3k_6L' + g(m_1gb_{p1} - m_3gb_{p3}) = 0 \\
&a_2b_2k_6 - a_2b_3k_6 + b_3k_6L' - m_1ga_{p1} - m_2gL + m_3ga_{p3} - m_3gL = 0
\end{align*}
$$

(6.40)

6.1.3 Geometric Method for Balancing Using External Springs

First, the 4R parallelogram linkage is balanced using three external springs. Link 1(3) are readily balanced by connecting the CM of link 1(3) and the point right above \( R_1 (R_4) \) \( (H_1 = [0 \quad 0 \quad m_1g/k_1]^T, \quad H_3 = [-L' \quad 0 \quad m_3g/k_3]^T) \) respectively [Fig. 6.6(a)].

To balance link 2, the concept of virtual rotation centre is introduced. Draw a parallelogram based on \( R_1, R_2 \) and \( P_2 \), and the fourth vertex of the parallelogram \( B_2 \) is the virtual rotation centre of \( P_2 \) [Fig. 6.6(b)]. \( P_2 \) always rotates around \( B_2 \), and the spring attachment point on the base for link 2 should be right above \( B_2 \). The position of \( B_2 \) is obtained as \( \{-a_{p2} \quad -c_{p2} \quad -b_{p2}\}^T \), \( H_2 \) is right above \( B_2 \) and \( H_2 = \)}
\[-a_{p2} -c_{p2} m_2 g/k_2 - b_{p2}]^T.\] Link 2 is balanced by connecting \(H_2\) and \(P_2\). The parameters \(a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3, h_1, h_2\) and \(h_3\) obtained using the geometric method satisfy Eq. (6.7). This confirms that the results are correct.

![Diagram](image)

**Fig. 6.7** Static balancing of planar 4R parallelogram linkage using two external springs (geometric method): (a) the balancing method; (b) mass moment substitution of link 2

To balance the 4R parallelogram linkage using two springs, the mass of link 2 should be distributed to links 1 and 3 [Fig. 6.7(b)]. Since link 2 is always horizontal, the difference between the potential energies associated with the gravity of the CM of link 2 and an arbitrary point on the line \(R_2R_3\) keeps constant (equal to \(m_2 gb_{p2}\)), the balancer of link 2 experiences the mass as if it were fixed anywhere on the line \(R_2R_3\). Suppose that the position vector of the CM of link 2 in the 2nd local coordinate frame is \[
\{a_{p2} 0 c_{p2}\}^T.\] The position of the first equivalent mass on \(R_2\) in the 2nd local coordinate frame is \(\{0 0 0\}^T\), and that of the second equivalent mass on \(R_3\) is \(\{L' 0 t_2\}^T.\) \(t_2\) and the weights of the two equivalent masses satisfy the following conditions:

\[
\begin{align*}
m_{22} &= m_2 a_{p2}/L' \\
m_{21} &= m_2 - m_{22} \\
t_2 &= m_2 c_{p2}/m_{22}
\end{align*}
\]

\(6.42\)

The positions vector of the CMs of the augmented links 1 and 3 in the local frames are obtained as

\[
\begin{align*}
1A_1 &= 1P_1 = \left\{ \begin{array}{c} m_1 a_{p1} + m_{21} L/(m_1 + m_{21}) \\ m_1 b_{p1}/(m_1 + m_{21}) \\ m_1 c_{p1}/(m_1 + m_{21}) \end{array} \right\} \\
3A_3 &= 3P_3 = \left\{ \begin{array}{c} m_3 a_{p3}/(m_3 + m_{22}) \\ m_3 b_{p3}/(m_3 + m_{22}) \\ (m_3 c_{p3} + m_{22} t_2)/(m_3 + m_{22}) \end{array} \right\}
\end{align*}
\]

\(6.43, 6.44\)
$H_1$ and $H_3$ are points right above $R_1$ and $R_4$ respectively, with heights of $h_1 = (m_1 + m_{21})g/k_1, h_3 = (m_3 + m_{22})g/k_3$.

The 4R parallelogram linkage is statically balanced by connecting points $H_1$ and $A_1$, and $H_3$ and $A_3$ using two springs respectively [Fig. 6.7(a)]

An example is now given to illustrate the balancing method. Let $m_1 = m_2 = m_3 = m, L' = L = 100mm$, the positions of the CMs of link 1 in the 1st local coordinate frame and link 3 in the 3rd local coordinate frame are $\{10 \quad 20 \quad 0\}^T$ and the CM of link 2 is on the line $R_2R_3$ and the position vector in 2nd local coordinate frame is $\{10 \quad 0 \quad 0\}^T$. The position vector, $^1A_1$, of the CM of the augmented link 1 in the 1st local coordinate frame and the mass of the augmented link 1 are calculated as

$$^1A_1 = ^1P_1 = \{1000/19 \quad 200/19 \quad 0\}^T$$

$$m'_1 = 19m/10$$

Similarly, the position vector, $^3A_3$, of the CM of the augmented link 3 in the 3rd local coordinate frame and the mass of the augmented link 3 are

$$^3A_3 = ^3P_3 = \{100/11 \quad 200/11 \quad 0\}^T$$

$$m'_3 = 11m/10$$

The heights of the spring attachment points on the base $H_1$ and $H_3$ are $h_1 = 19mg/10k_1, h_3 = 11mg/10k_3$. The obtained positions of the spring attachment points satisfy Eq. (6.15). This proves that the balancing conditions have nothing to do with the positions of CMs of the links when the mass moments of links 1 and 3 are equal.

When balancing the system using one spring, the masses of links 1 and 3 should be distributed onto the base and link 2 [Fig. 6.8(b)]. It is observed from Eqs. (6.2a) and (6.2c) that the sum of $P_{1z}$ and $P_{3z}$ is $LC\theta$ when $a_{p1} = a_{p3}$ and $b_{p1} = b_{p3}$, i.e., the total potential energy and the balancing condition of the system when $m_1 = m_3$ are independent of the values of $a_{p1} (a_{p3})$ and $b_{p1} (b_{p3})$. Suppose that $a_{p1} = a_{p3} = a, b_{p1} = b_{p3} = 0$ and $c_{p1} = c_{p3} = c$, i.e., the position vectors of the CMs of link 1 in 1st local coordinate frame and link 3 in 3rd local coordinate frame are $\{a \quad 0 \quad c\}^T$. The position of the equivalent mass on $R_1$ is $\{0 \quad 0 \quad 0\}^T$, and that of the equivalent mass on $R_2$ is $\{L \quad 0 \quad t_1\}^T$. $t_1$ and the weights of the two equivalent masses can be calculated as:

$$\begin{cases}
m_{12} = m_1a/L \\
m_{11} = m_1 - m_{12} \\
t_1 = m_1c/m_{12}
\end{cases}$$

(6.49)

Similarly, the parameters of link 3 can be obtained as

$$\begin{cases}
m_{32} = m_3a/L \\
m_{31} = m_3 - m_{32} \\
t_3 = m_3c/m_{32}
\end{cases}$$

(6.50)
The position vector, \( 2^A_2 \), of the CM of the augmented link 2 in the 2\(^{nd} \) local frame is yielded as

\[
2^A_2 = 2^P_2 = \begin{pmatrix}
\frac{(m_2 a_{p2} + m_{31} L')}{(m_2 + m_{12} + m_{31})} \\
m_2 b_{p2}/(m_2 + m_{12} + m_{31}) \\
m_2 c_{p2}/(m_2 + m_{12} + m_{31})
\end{pmatrix}
\]  
(6.51)

Draw a parallelogram based on \( R_1, R_2 \) and \( A_2 \), and the fourth vertex of the parallelogram \( B_2 \) is the virtual rotation centre of \( A_2 \) [Fig. 6.8(a)]. The spring attachment point on the base for link 2 should be right above \( B_2 \), with a height of \( h_2 = (m_2 + m_{12} + m_{31})g/k_2 \).

A numerical example is given as follows. Suppose the positions of CMs of the links 1 and 3 in local coordinate frames are \( \{10 \ 0 \ 0\}^T \) and that of link 2 is \( \{10 \ 20 \ 0\}^T \). The position vector, \( 2^A_2 \), of the CM of the augmented link 2 in the local coordinate frame and the mass of the augmented link 2 are calculated as

\[
2^A_2 = 2^P_2 = \begin{pmatrix} 50 \\ 10 \\ 0 \end{pmatrix}^T
\]  
(6.52)
\[
m_2 = 2m
\]  
(6.53)

\( B_2 \), the position of virtual rotation centre of \( A_2 \), and \( H_2 \) (the point right above \( B_2 \)) are calculated as

\[
B_2 = \begin{pmatrix} -a_{p2} & 0 & -b_{p2} \end{pmatrix}^T = \begin{pmatrix} -50 & 0 & -10 \end{pmatrix}^T
\]  
(6.54)
\[
H_2 = \begin{pmatrix} -a_2 & 0 & 2mg/k_2 - b_2 \end{pmatrix}^T = \begin{pmatrix} -50 & 0 & 2mg/k_2 - 10 \end{pmatrix}^T
\]  
(6.55)

The 4R parallelogram linkage is statically balanced by connecting points \( H_2 \) and \( A_2 \) using one spring [Fig. 6.8(a)]. The obtained positions of the spring attachment points satisfy Eq. (6.27).
The algebraic method proposed above can also be applied to a general 4R linkage, which can be balanced using two or three springs. When balancing the general 4R linkage using the geometric method, the mass of the coupler can be distributed onto the two cranks. The mechanism is then equivalent to two 1-link manipulators with payloads. Each manipulator can be balanced by connecting a point right above the R joint and the point on the line defined by the R joint and the CM of the augmented manipulator. When distributing the masses of links 1 and 3 onto the base and link 2, the 4R parallelogram linkage turns into a 2-link manipulator (whose link masses are zero) with a payload. The manipulator then can be balanced using the method proposed in [104].

6.2 Static Balancing Method of Planar Manipulators

In this section, planar manipulators with links whose masses cannot be neglected will be balanced. This is different from the system in [102, 104] in which only the weight of the payload was considered, the weights of the links are considered in this thesis. Planar 1R, 2R and 3R manipulators are represented schematically in Fig. 6.9. For the sake of convenience, assume that the CM of each link is at the centroid of the link.

A global coordinate is attached to the ground, with the z-axis pointing in the direction opposite to the gravitational acceleration vector. The z-axis of the local coordinate frames are along the axes of the R joints and x-axis point from $R_i$ to $R_{i+1}$. Let $\theta_1$, $\theta_2$, and $\theta_3$ be the joint variables associated with the three R joints respectively and the link lengths are represented by $L_i$ ($i = 1, 2, 3$).

The positions of the CMs of the links $P_i$, and the R joints $P_{R_i}$ are:

\[
P_1 = \begin{bmatrix} L_1 C \theta_1/2 & 0 & L_1 S \theta_1/2 \end{bmatrix}^T \tag{6.56a}
\]

\[
P_{R2} = \begin{bmatrix} L_1 C \theta_1 & 0 & L_1 S \theta_1 \end{bmatrix}^T \tag{6.56b}
\]

\[
P_2 = \begin{bmatrix} L_1 C \theta_1 + [L_2 C(\theta_1 + \theta_2)]/2 & 0 & L_1 S \theta_1 + [L_2 S(\theta_1 + \theta_2)]/2 \end{bmatrix}^T \tag{6.56c}
\]

\[
P_{R3} = \begin{bmatrix} L_1 C \theta_1 + L_2 C(\theta_1 + \theta_2) & 0 & L_1 S \theta_1 + L_2 S(\theta_1 + \theta_2) \end{bmatrix}^T \tag{6.56d}
\]

\[
P_3 = \begin{bmatrix} L_1 C \theta_1 + L_2 C(\theta_1 + \theta_2) + L_3 [C(\theta_1 + \theta_2 + \theta_3)/2 & 0 \end{bmatrix}^T \tag{6.56e}
\]

where

\[
P_{3z} = L_1 S \theta_1 + L_2 S(\theta_1 + \theta_2) + L_3 [S(\theta_1 + \theta_2 + \theta_3)/2
\]

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According to [102, 104], a one-link manipulator mounted on R joint, U joint or S joint [117] can be balanced using one spring. One end of the spring is right above the joint and the other end should be on the line defined by the CM of the manipulator and the centre of the joint. It is noted that the spring attachment point on the base for the manipulator mounted on an R joint can be any point right above the axis of the R joint. The height of the spring connecting point on the base is $h = mg/k$, when attaching the spring to the CM of the manipulator directly. Hence, the first link of the manipulators can be readily balanced, by connecting the point right above $R_1$ and the CM of the first link using one spring [Fig. 6.9(a)]. Let the spring connecting point $H_1$ on the base be

$$H_1 = \begin{bmatrix} 0 & 0 & m_1g/k \end{bmatrix}^T$$  \hspace{1cm} (6.57)

The total potential energy is calculated as
which is a constant value. The result implies that the one-link manipulator is gravity-compensated.

As in [174], one spring is adopted to balance the link 2, with one end fixed at a point right above $R_2$ and the other end attached to the CM of link 2 (or any other positions, as long as they meet the requirement of similar triangles [174]). The global coordinate frame is set at $R_2$ and the system is equal to a 1-link manipulator mounted on an R joint, which can be statically balanced using one spring. However, when attaching the global coordinate frame to $R_1$, link 2 with $R_2$ is not an independent unit anymore. The spring connecting point on the base $H_2$, which is a point right above $R_2$, is given by

$$f = \begin{bmatrix} \frac{b}{2} + \frac{m_2g}{k} + P_{R2z} \end{bmatrix}$$  \hspace{1cm} (6.59)$$

The total potential energy of link 2 is calculated as

$$V_2 = V_{s2} + V_{m2} = \frac{1}{2} k|\mathbf{P}_2 - \mathbf{H}_2|^2 + mgP_{2z} = \frac{1}{2} \left( L_2^2k^2 + 4m_2^2g^2 + 8L_1km_2gS\theta_1 \right)/8k$$  \hspace{1cm} (6.60)$$

Similarly, $H_3$, which is a point right above $R_3$, and the potential energy of link 3 are yielded as:

$$H_3 = [P_{R3x} \hspace{0.5cm} P_{R3y} \hspace{0.5cm} m_3g/k + P_{R3z}]^T$$

$$V_3 = V_{s3} + V_{m3} = \frac{1}{2} k|\mathbf{P}_3 - \mathbf{H}_3|^2 + mgP_{3z}$$

$$= \frac{1}{2} \left[ L_3^2k^2 + 4m_3^2g^2 + 8L_1km_3gS\theta_1 + 8L_2km_3gS(\theta_1 + \theta_2) \right]/8k$$  \hspace{1cm} (6.62)$$

Equations (6.60) and (6.62) imply that the potential energies of links 2 and 3 are not constants, which means the two links are not statically balanced.

Now let us observe the position vectors of links 2 and 3. The position of the second link contains two components, including the translation of $R_2$ and the rotation of link 2 with respect to the global coordinate frame. The latter term can be balanced using one spring as mentioned above and for the former one, which is a variation with respect to $\theta_1$, an additional spring is needed. One end of the spring should be attached to $H_{12}$ ($[P_{R1x} \hspace{0.5cm} P_{R1y} \hspace{0.5cm} m_2g/k + P_{R1z}]^T$) with the other end to $R_2$, as shown in Fig. 6.9(b). The potential energy of the link 2 is then calculated as:

$$V_2 = V_{s2} + V_{m2} = \frac{1}{2} k|\mathbf{P}_2 - \mathbf{H}_2|^2 + \frac{1}{2} k|\mathbf{P}_{R2} - \mathbf{H}_{12}|^2 + mgP_{2z}$$

$$= \frac{1}{2} \left( 4L_2^2k^2 + L_2^2k^2 + 8m_2^2g^2 \right)/8k$$  \hspace{1cm} (6.63)$$

which is a constant value. The position vector of link 3 is composed of the translation of $R_3$ (including the movement of $R_3$ with respect to $R_2$ and the movement of $R_2$ with respect to $R_1$) and the rotation of link 3. Two additional springs are added, which connect $H_{13}$
\((\{P_{R1x} P_{R1y} m_3 g/k + P_{R1z}\})^T\) and \(R_2\), and \(H_{23}\) \((\{P_{R2x} P_{R2y} m_3 g/k + P_{R2z}\})^T\) and \(R_3\) respectively [Fig. 6.9(c)]. The total potential energy of link 3 is obtained as:

\[
V_3 = V_{s3} + V_{m3} = \frac{1}{2} k |P_3 - H_3|^2 + \frac{1}{2} k |P_{R3} - H_{23}|^2 + \frac{1}{2} k |P_{R2} - H_{13}|^2 + mg P_{3z} \\
= (4L_3^2 k^2 + 4L_2^2 k^2 + L_3^2 k^2 + 12 m_3^2 g^2)/8k
\]

which is also a constant value. Equations (6.63-6.64) verify that the method proposed in this section is valid and the \(i^{th}\) link of the manipulators can be balanced using \(i\) springs.

When adding the fourth link, another four springs are needed, which connect \(H_{14}\) \((\{P_{R1x} P_{R1y} m_4 g/k + P_{R1z}\})^T\) and one point on the axis of \(R_2\), and \(H_{24}\) \((\{P_{R2x} P_{R2y} m_4 g/k + P_{R2z}\})^T\) and one point on the axis of \(R_3\), \(H_{34}\) \((\{P_{R3x} P_{R3y} m_4 g/k + P_{R3z}\})^T\) and one point on the axis of \(R_4\) and \(H_{4}\) \((\{P_{R4x} P_{R4y} m_4 g/k + P_{R4z}\})^T\) and the CM of link 4 respectively.

### 6.3 Static Balancing Method of Spherical Manipulators

In this section, the conditions of the static balancing for the spherical manipulators will be derived. Spherical manipulators refer to the ones in which the axes of the R joints intersect at a point, and have RCM kinematics.

#### 6.3.1 Static Balancing of Spherical Manipulators

Since all the links of spherical manipulators move around the point of intersection, which is equivalent to a virtual S joint, it is hypothesized that all the spherical manipulators composed of \(n\) moving links can be balanced using \(n\) \([\text{or } (n-1)]\) springs. The general balancing method is: using one spring to balance each moving link of the manipulators. One end of the spring is attached right above the point of intersection, the other end is fixed on the CM of the link [Fig. 6.10]. It is noted that the connecting point on the base for the first link can be any point on the line right above the axis of the first R joint, as shown in Fig. 6.10.
Fig. 6.10 Static balancing of spherical manipulator: (a) the sketch of the manipulator; (b) the 3D model of the manipulator

A global coordinate system is fixed to the base with its z-axis pointing vertically upward and with its origin located at the intersection of the axes of the R joints. Suppose the position vector of the CM of the $i$th link in the $i$th local frame is represented by

$$i \mathbf{p}_i = [a_i \ b_i \ c_i]^T$$

(6.65)

The link masses and joint twist angles of the manipulator are respectively noted as $m_i$ and $\alpha_i$ ($i = 1, 2, 3\ldots$). The position vectors of the CMs of the three links in the global coordinate system are yielded as

$$\begin{align*}
\{ \mathbf{p}_1 \} &= 0^T_1 \{ 1 \mathbf{p}_1 \} = \{ -b_1 S \theta_1 + a_1 C \theta_1 \ c_1 \quad -a_1 S \theta_1 - b_1 C \theta_1 \quad 1 \}^T \quad \text{(6.66a)} \\
\{ \mathbf{p}_2 \} &= 0^T_2 \{ 2 \mathbf{p}_2 \} = \{ P_{2x} \quad (c_2 C \alpha_1 - b_2 S \alpha_1 C \theta_2 - a_2 S \alpha_1 S \theta_2)/2 \quad P_{2z} \quad 1 \}^T
\end{align*}$$

(6.66b)

where

$$
\begin{align*}
P_{2x} &= [-S \theta_1 (S \alpha_1 c_2 + b_2 C \alpha_1 C \theta_2 + a_2 C \alpha_1 S \theta_2) + C \theta_1 (a_2 C \theta_2 - b_2 S \theta_2)]/2 \\
P_{2z} &= [C \theta_1 (-S \alpha_1 c_2 + b_2 C \alpha_1 C \theta_2 + a_2 C \alpha_1 S \theta_2) + S \theta_1 (-a_2 C \theta_2 + b_2 S \theta_2)]/2
\end{align*}
$$

(6.66c)

$$
\begin{align*}
\{ \mathbf{p}_3 \} &= 0^T_3 \{ 3 \mathbf{p}_3 \} = \{ P_{3x} \quad P_{3y} \quad P_{3z} \quad 1 \}^T
\end{align*}
$$

(6.66c)

where

$$
\begin{align*}
P_{3x} &= -C \theta_1 [C \theta_2 (-a_3 C \theta_3 + b_3 S \theta_3) + S \theta_2 (b_3 C a_2 C \theta_3 + c_3 S a_2 + a_3 C a_2 S \theta_3)] - S \theta_1 [C a_1 [c_3 C \theta_2 S a_2 + a_3 C \theta_3 S \theta_2 - b_3 S \theta_2 S \theta_3 + C a_2 C \theta_2 (b_3 C \theta_3 + a_3 S \theta_3)] + S a_1 [c_3 C a_2 - S a_2 (b_3 C \theta_3 + a_3 S \theta_3)]} \\
P_{3y} &= -S \alpha_1 [c_3 C \theta_2 S a_2 + a_3 C \theta_3 S \theta_2 - b_3 S \theta_2 S \theta_3 + C a_2 C \theta_2 (b_3 C \theta_3 + a_3 S \theta_3)] + C a_1 [c_3 C a_2 - S a_2 (b_3 C \theta_3 + a_3 S \theta_3)]
\end{align*}
$$
\[
P_{3z} = c_3S\alpha_2S\theta_1S\theta_2 + C\theta_1S\alpha_1S\alpha_2(b_3C\theta_3 + a_3S\theta_3) + C\theta_2S\theta_1(-a_3C\theta_3 + b_3S\theta_3) - 
C\alpha_1C\theta_1[c_3C\theta_2S\alpha_2 + a_3C\theta_3S\theta_2 - b_3S\theta_2S\theta_3 + C\alpha_2C\theta_2(b_3C\theta_3 + a_3S\theta_3)] + 
C\alpha_2[-c_3C\theta_1S\alpha_1 + S\theta_1S\theta_2(b_3C\theta_3 + a_3S\theta_3)]
\]

The spring connecting points \( H_i \) on the base are all set to be

\[
H_i = \begin{bmatrix} 0 \\ 0 \\ m_i g / k \end{bmatrix}^T \quad (i = 1, 2, 3)
\]

(6.67)

The potential energy of each link can be obtained as

\[
V1 = \frac{1}{2} k |P_1 - H_1|^2 + m_1 g P_{1z} = (a_1^2 k^2 + b_1^2 k^2 + c_1^2 k^2 + g^2 m_1^2) / (2k)
\]

(6.68)

\[
V2 = \frac{1}{2} k |P_2 - H_2|^2 + m_2 g P_{2z} = (a_2^2 k^2 + b_2^2 k^2 + c_2^2 k^2 + g^2 m_2^2) / (2k)
\]

(6.69)

\[
V3 = \frac{1}{2} k |P_3 - H_3|^2 + m_3 g P_{3z} = (a_3^2 k^2 + b_3^2 k^2 + c_3^2 k^2 + g^2 m_3^2) / (2k)
\]

(6.70)

Based on the results, one can conclude that the total potential energy is constant and the system designed is statically balanced.

### 6.3.2 Example 1: Static Balancing of Mechanisms Constructed Using Spherical Kinematic Chain Units

![Fig. 6.11 Static balancing of spherical 5R mechanism: (a) the sketch of the mechanism; (b) the 3D model of the mechanism](image)

Based on Section 6.3.1, one can obtain that each link of the spherical manipulators can be balanced using one spring. Similarly, all the mechanisms constructed using spherical kinematic chain units can be balanced, by removing specific joints. Take the spherical 5R mechanism as an example. When removing \( R_5 \) [Fig. 6.11(a)], the mechanism turns into a spherical 4R manipulator, and each moving link of the mechanism is balanced using one spring. The spring connecting points on the base for link 1 can be any point right above
the joint axis of $R_1$, as shown in Fig. 6.11. One can also divide the spherical 5R linkage into two spherical chain units composed of $R_1$ and $R_2$, and $R_3$ and $R_4$ respectively by removing $R_3$.

![Diagram](image)

Fig. 6.12 Static balancing of Bennett 6R double-spherical mechanism: (a) the sketch of the mechanism; (b) the 3D model of the mechanism (the special case when the spring attachment points on the base for balancing links 1 and 5 are right above $O_1$ and $O_2$ respectively); (c) the practical design of the system

In fact, the above static balancing approach is not limited to spherical mechanisms in which the axes of all the R joints intersect at a point. For instance, in the 1-DOF Bennett 6R double-spherical mechanism (Fig. 6.12), the joint axes of $R_1$, $R_2$ and $R_3$ intersect at $O_1$, and the joint axes of $R_4$, $R_5$ and $R_6$ intersect at $O_2$ (double-RCM mechanism). The mechanism can be divided into two spherical kinematic chain units composed of $R_1$, $R_2$ and $R_3$, and $R_6$ and $R_5$ respectively by removing $R_4$ (or two spherical kinematic chain units composed of $R_1$ and $R_2$ and $R_6$, $R_5$ and $R_4$ respectively by removing $R_3$). $H_1$ is a point right
above the axis of \( R_1 \) and \( H_2 \) and \( H_2' \) are two points right above \( O_1 \). The masses of the links in the first spherical kinematic chain unit are balanced by connecting \( H_1 \) and the CM of link 1, \( H_2 \) and the CM of link 2, and \( H_2' \) and the CM of link 3 using three springs respectively. Similarly, the second spherical kinematic chain unit can also be balanced as shown in Fig. 6.12. Suppose the masses of links 1, 2, 4 and 5 are \( m \) and that of link 3 is \( m' \), and the CM of each link is in the middle of the links, a prototype is designed, as shown in Fig. 6.12(c). Rollers and cables are used to achieve zero-free-length springs. The heights of the attachment points \( h_1 = h_2 = h_3 = h_4 = mg/k \) and \( h_2' = m'g/k \).

If the two spherical kinematic chain units of the Bennett 6R double-spherical mechanism have the same parameters as the 3-DOF spherical manipulators in Section 2.3, fewer springs will be required for the static balancing.

**6.3.3 Example 2: Static Balancing of Mechanisms Constructed Using Spherical Chain Units and Other Types of Chain Units**

This section will focus on the mechanisms that are constructed using spherical chain units and other types of chain units. Take the Bennett plano-spherical hybrid linkage as an example (Fig. 6.13), the axes of joints \( R_1, R_2 \) and \( R_3 \) are parallel, and those of \( R_4, R_5 \) and \( R_6 \) intersect at a point. Links 1, 3, 4 and 5 can be easily balanced based on the proposed methods.

The mass of link 2 can be replaced by two point-masses on the joint axes of \( R_2 \) and \( R_3 \), then the mechanism is equivalent to two chain units with payloads, including one mounted on \( R_1 \), and one spherical chain unit composed of \( R_4, R_5 \) and \( R_6 \). \( H_1 \) is a point right above the axis of \( R_1 \), \( H_2 \) and \( H_2' \) are two points right above the intersection of the axes of \( R_4, R_5 \) and \( R_6 \) and \( H_3 \) is a point on the line right above \( R_6 \). Four springs are used, one is attached to \( H_1 \) and to the CM of the augmented link 1 (combining the mass of link 1 and the first point-mass of link 2); the other three are attached to \( H_2 \) and the CM of link 4, \( H_2' \) and the CM of the augmented link 3 (combining the mass of link 3 and the second point-mass of link 2), and \( H_3 \) and the CM of link 5 respectively.
Fig. 6.13 Static balancing of Bennett plano-spherical hybrid linkage: (a) the sketch of the mechanism; (b) the 3D model of the mechanism (the special case when the spring attachment point for balancing link 5 is right above O); (c) the practical design of the system.

Suppose the masses of links 1, 2, 4 and 5 are \( m \) and that of link 3 is \( m/2 \), and the CM of each link is in the middle of the link, a prototype is designed, as shown in Fig. 6.13(c). When distributing the mass of link 2 to link 1 and link 3, the spring attachment point on links 1 and 3 are at the top third of link 1 and top quarter of link 3 respectively. The masses of augmented links 1 and 3 are \( 3m/2 \) and \( m \) respectively. \( h_1 = 3mg/2k \) and \( h_2 = h_2' = h_3 = mg/k \).

A 3-DOF 3-RRS spherical PM, composed of two platforms and three RRS chains [173], is shown in Fig. 6.14. The axes of two R joints in each chain intersect at a point. Different from the spherical mechanism in [117], the axes of the R joints have no common point and the mechanism has no fixed centre of rotation. The two links in each chain are easily...
balanced by attaching the springs to the point right above the point of intersection and to the CM of the links. To balance the upper platform, the mass of the upper platform is replaced by three point-masses located at the three S joints on the platform. The mass and mass moment (about \( O \)) of the upper platform should be equal to those of the three point-masses [Fig. 6.14(a)].

\[
\begin{align*}
\begin{cases}
m_u &= m_{u1} + m_{u2} + m_{u3} \\
m_{u1}g(0, r_1C \varphi_1, r_1S \varphi_1) + m_{u2}g(0, r_2C \varphi_2, r_2S \varphi_2) + m_{u3}g(0, 0, -r_3) = 0
\end{cases}
\end{align*}
\]  

which leads to

\[
\begin{align*}
\begin{cases}
m_{u1}r_1C \varphi_1 + m_{u2}r_2C \varphi_2 &= 0 \\
m_{u1}r_1S \varphi_1 + m_{u2}r_2S \varphi_2 - m_{u3}r_3 &= 0
\end{cases}
\end{align*}
\]  

Since the mechanism is symmetrically distributed,

\[
\begin{align*}
\begin{cases}
r_1 &= r_2 = r_3 \\
\varphi_1 + \varphi_2 &= \pi
\end{cases}
\end{align*}
\]  

Substituting Eq. (6.73) into Eq. (6.71), it is obtained that \( m_{u1} = m_{u2} = m_{u3} = m_u/3 \). By replacing the mass of the upper platform with the three point-masses, the PM is equal to three 2R spherical chain units with payloads. The mass of the first chain unit is balanced by connecting \( H_{11} \) (the point right above the axes of \( R_{11} \)) and the CM of link 11 \( P_{11} \), and \( H_{12} \) (the point right above the intersection of the axes of the two R joints) and the CM of the augmented link 12 \( P_{12}' \) (combining the masses of the upper platform and link 12) using two springs respectively [Fig. 6.14(b)]. Similarly, the other two chain units can also be balanced.

![Fig. 6.14 Static balancing of 3-RRS PM: (a) mass moment substitution of the upper platform; (b) the sketch of the mechanism](image)
6.4 Static Balancing Method of Spatial Manipulators

In this section, the static balancing method of spatial manipulator will be addressed. In Section 6.2, the static balancing method for the planar manipulator has been proposed. Now the method will be extended and applied to spatial manipulators.

6.4.1 Static Balancing of Spatial Manipulators

The spatial manipulators are shown in Fig. 6.15. The spheres represent the positions of the CMs of the links.

Suppose the position vectors of the CM of the \( i \)th link and \( R_i \) [a point that is the intersection of link \((i-1)\) and link \(i\)] in the \(i\)th local coordinate frame are represented by

\[
^i{\mathbf{p}}_i = \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix}^T \quad (6.74a)
\]

\[
^i{\mathbf{p}}_{R_i} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}^T \quad (6.74b)
\]

Let

\[ l_0 = 0, \quad d_1 = 0 \quad (6.75) \]

The position vectors of the CMs of the links (\(P_i\)) and the R joints (\(P_{R_i}\)) expressed in the global coordinate frame are obtained as

\[
\begin{bmatrix} \{P\}^1 \\ \{P\}^2 \end{bmatrix} = \begin{bmatrix} 0 \\ {^1}{\mathbf{p}}^1 \end{bmatrix} = \{\begin{bmatrix} a_1 C\theta_1 - b_1 S\theta_1 \\ -c_1 S\alpha_0 + C\alpha_0 (b_1 C\theta_1 + a_1 S\theta_1) \\ c_1 C\alpha_0 + S\alpha_0 (b_1 C\theta_1 + a_1 S\theta_1) \end{bmatrix}^T \quad (6.76a)
\]

\[
\begin{bmatrix} \{P\}^R_2 \end{bmatrix} = \begin{bmatrix} 0 \\ {^1}{\mathbf{p}}_{R_2} \end{bmatrix} = \{\begin{bmatrix} L_1 C\theta_1 + d_2 S\theta_1 S\alpha_1 \\ P_{R_2y} \\ L_1 S\alpha_0 S\theta_1 + d_2 C\alpha_1 C\alpha_0 - d_2 S\alpha_0 S\alpha_1 C\theta_1 \end{bmatrix}^T \quad (6.76b)
\]

where \(P_{R_2y} = L_1 C\alpha_0 S\theta_1 - d_2 C\alpha_1 S\alpha_0 - d_2 C\alpha_0 S\alpha_1 C\theta_1\)

\[
\begin{bmatrix} \{P\}^2 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} {^2}{\mathbf{p}}^2 = \begin{bmatrix} P_{2x} \\ P_{2y} \\ P_{2z} \end{bmatrix}^T \quad (6.76c)
\]

where

\[
P_{2x} = \{S\theta_1[(c_2 + d_2)S\alpha_1 - C\alpha_1 (b_2 C\theta_2 + a_2 S\theta_2)] + C\theta_1 (L_1 + a_2 C\theta_2 - b_2 S\theta_2)\}
\]

\[
P_{2y} = -S\alpha_0[(c_2 + d_2)C\alpha_1 + S\alpha_1 (b_2 C\theta_2 + a_2 S\theta_2)] + C\alpha_0[-(c_2 + d_2)C\theta_1 S\alpha_1 + C\alpha_1 C\theta_1 (b_2 C\theta_2 + a_2 S\theta_2) + S\theta_1 (L_1 + a_2 C\theta_2 - b_2 S\theta_2)]
\]

\[
P_{2z} = C\alpha_0[(c_2 + d_2)C\alpha_1 + S\alpha_1 (b_2 C\theta_2 + a_2 S\theta_2)] + S\alpha_0[-(c_2 + d_2)C\theta_1 S\alpha_1 + C\alpha_1 C\theta_1 (b_2 C\theta_2 + a_2 S\theta_2) + S\theta_1 (L_1 + a_2 C\theta_2 - b_2 S\theta_2)]
\]
Fig. 6.15 Static balancing of spatial manipulators: (a) manipulator with one link; (b) manipulator with two links; (c) manipulator with three links; (d) the replacement of the springs through vector synthesis
\[ \{ \mathbf{P}_{R3} \} = \frac{1}{T^2} T^{2 \frac{dB}{dt}} T^2 \{ \mathbf{p}_{R3} \} = \{ P_{R3x} \ P_{R3y} \ P_{R3z} \}^T \]  

where

\[
P_{R3x} = C \theta_a (L_1 + L_2 + \alpha_2 + d_3 S_\alpha_2 S_\theta_2) + S \theta_1 [(d_2 + d_3) \alpha_1 + \alpha_2 (d_3 C_\theta_2 S_\alpha_2 + L_2 S_\theta_2)]
\]

\[
P_{R3y} = \alpha_0 S_\alpha_1 (d_3 C_\theta_2 S_\alpha_2 + L_2 S_\theta_2) + \alpha_0 C_\theta_1 (d_3 C_\theta_2 S_\alpha_2 - L_2 S_\theta_2) + \alpha_0 [(d_2 + d_3) \alpha_2] S \alpha_0 + \alpha_0 C_\theta_1 (d_3 C_\theta_2 S_\alpha_2 - L_2 S_\theta_2)
\]

\[
P_{R3z} = C \alpha_0 (d_2 + d_3) \alpha_2 + S \alpha_1 (d_3 C_\theta_2 S_\alpha_2 - L_2 S_\theta_2) + S \alpha_0 [(d_2 + d_3) \alpha_2 + d_3 \alpha_2 S_\alpha_1 + d_3 \alpha_2 S_\theta_2 - L_2 S_\theta_2] + S \alpha_1 (L_1 + L_2 C_\theta_2 + d_3 S_\alpha_2 S_\theta_2)
\]

\[ \{ \mathbf{P}_3 \} = \frac{1}{T^2} T^{2 \frac{dB}{dt}} T^2 \{ \mathbf{p}_3 \} = \{ P_{3x} \ P_{3y} \ P_{3z} \}^T \]  

where

\[
P_{3x} = C \theta_a (L_1 + L_2 + \alpha_2 + d_3 S_\alpha_2) + S \theta_1 [(d_3 + d_2) \alpha_2 - C_\alpha_2 (b_3 C_\theta_3 + a_3 S_\theta_3)] + S_\theta_1 [(d_3 + d_2) \alpha_2] \alpha_2 + \alpha_2 S_\theta_2 (b_3 C_\theta_3 + a_3 S_\theta_3) + S \theta_2 (L_2 + a_3 C_\theta_3 - b_3 S_\theta_3)]
\]

\[
P_{3y} = \alpha_0 C_\theta_1 (d_3 + d_2) \alpha_2 + \alpha_2 S_\theta_2 (b_3 C_\theta_3 + a_3 S_\theta_3) + S \theta_1 (L_1 + (c_3 + d_3) \alpha_2 S_\alpha_1 + (d_3 + d_2) \alpha_2 + \alpha_2 S_\theta_2 (b_3 C_\theta_3 + a_3 S_\theta_3) + S \theta_2 (L_2 + a_3 C_\theta_3 - b_3 S_\theta_3)]
\]

\[
P_{3z} = \alpha_0 \alpha_1 (d_3 + d_2) \alpha_2 + \alpha_2 S_\theta_2 (b_3 C_\theta_3 + a_3 S_\theta_3) + S \alpha_1 (L_1 + (c_3 + d_3) \alpha_2 S_\alpha_1 + (d_3 + d_2) \alpha_2 + \alpha_2 S_\theta_2 (b_3 C_\theta_3 + a_3 S_\theta_3) + S \theta_2 (L_2 + a_3 C_\theta_3 - b_3 S_\theta_3)]
\]

\[
P_{R4x} = C \theta_a (L_1 + C_\theta_2 (L_2 + L_3 C_\theta_3) + S \theta_1 [(d_3 + d_4) \alpha_2 - L_3 C_\alpha_2 S_\theta_3)]
\]

\[
P_{R4y} = \alpha_0 \alpha_1 (d_3 + d_4) \alpha_2 + L_3 C_\alpha_2 S_\theta_2 \theta_3)
\]

\[
P_{R4z} = C \alpha_0 (L_1 + C_\theta_2 (L_2 + L_3 C_\theta_3) + S \theta_1 [(d_3 + d_4) \alpha_2 - L_3 C_\alpha_2 S_\theta_3)]
\]

\[
P_{R4x} = \alpha_0 \alpha_1 (L_2 + L_3 C_\theta_3) S \theta_2 + C_\theta_2 [(d_3 + d_4) \alpha_2 - L_3 C_\alpha_2 S_\theta_3)]
\]

\[
P_{R4y} = \alpha_0 \alpha_1 (L_2 + L_3 C_\theta_3) S \theta_2 + C_\theta_2 [(d_3 + d_4) \alpha_2 - L_3 C_\alpha_2 S_\theta_3)]
\]

\[
P_{R4z} = \alpha_0 \alpha_1 (L_2 + L_3 C_\theta_3) S \theta_2 + C_\theta_2 [(d_3 + d_4) \alpha_2 - L_3 C_\alpha_2 S_\theta_3)]
\]

\[
p^{i-1}T\text{ are provided in Appendix (D). The first link is readily balanced using one spring, the total potential energy of link 1 is computed as:}
\]

\[ V_1 = V_{s1} + V_{m1} = \frac{1}{2} k |\mathbf{P}_1 - \mathbf{H}|^2 + m g P_{1z} = (a_1^2 k^2 + b_1^2 k^2 + c_1^2 k^2 + m_1^2 g^2) / 2k \]
Connecting \( H_{12} \) and one point on the axis of \( R_2 \), and \( H_2 \) and the CM of link 2 using two springs respectively, the second link is then statically balanced. It is noted that attaching the spring to the CM of the link is just an example to illustrate the method, the attachment point on the link can be any points on the line defined by a point on the axis of the \( R \) joint and the CM of the link. The potential energy of link 2 is

\[
V_2 = V_{s2} + V_{m2} = \frac{1}{2} k |P_2 - H_2|^2 + \frac{1}{2} k |P_{R2} - H_{12}|^2 + m_2 g P_{2z} \\
= (a_2 k^2 + b_2 k^2 + c_2 k^2 + d_2 k^2 + 2m_2 g^2)/2k
\]

which is a constant. The third link can be balanced by connecting \( H_{13} \) and one point on the axis of \( R_2 \), \( H_{23} \) and one point on the axis of \( R_3 \), and \( H_3 \) and the CM of link 3 (or any points on the line defined by a point on the axis of the \( R \) joint and the CM of the link) using three springs respectively. The potential energy of link 3 is calculated as

\[
V_3 = V_{s3} + V_{m3} = \frac{1}{2} k |P_3 - H_3|^2 + \frac{1}{2} k |P_{R3} - H_{23}|^2 + \frac{1}{2} k |P_{R2} - H_{13}|^2 + m_3 g P_{3z} \\
= (a_3 k^2 + b_3 k^2 + c_3 k^2 + d_3 k^2 + 3m_3 g^2)/2k
\]

Equations (6.78-6.79) infer that the method proposed above applies to any spatial manipulators, and the four parameters of the link, \( l, d, \alpha \) and \( \theta \) can be with arbitrary values.

The total potential energy of the system is yielded as

\[
V_t = V_1 + V_2 + V_3 = (a_1 k^2 + b_1 k^2 + c_1 k^2 + a_2 k^2 + b_2 k^2 + c_2 k^2 + a_3 k^2 + b_3 k^2 + \sum c_i k^2 + 2L_1 k^2 + L_2 k^2 + 2d_1 k^2 + d_2 k^2 + m_1 g^2 + 2m_2 g^2 + 3m_3 g^2)/2k
\]

It is noted that the spring attachment positions \( H_i \) can be any points right above the axes of the \( R \) joint.

The total number of springs of the \( nR \) serial manipulator using the proposed geometric method is \( n + (n - 1) + (n - 2) + \ldots + 2 + 1 = n(n + 1)/2 \). It is noted that the number of springs can be reduced through vector synthesis. For example, the two springs connected to \( H_2 \) and \( P_2 \), and \( H_{23} \) and \( R_3 \) can be replaced by one spring connected to \( H_2' \) and \( R_3 \), as shown in Fig. 6.15(d). The three springs connected to \( H_1 \) and \( P_1 \), \( H_{12} \) and \( R_2 \) and \( H_{13} \) and \( R_2 \) can be replaced by one spring connected to \( H_1' \) and \( R_2 \).

When a payload is added at the end of the manipulator composed of \( n \) moving links, \( n \) springs are needed to balance the payload. One end of each spring is attached on \( H_i \), which is a point right above \( R_i \), and the other end is to \( R_{i+1} \), as shown in Fig. 6.16.
The total potential energies of the 1-link manipulator, 2-link manipulator and 3-link manipulator with payloads are respectively computed as

\[
V_{1p} = V_{s1p} + V_{m1p} + V_1 = \frac{1}{2}k|\mathbf{P}_{R2} - \mathbf{H}_{1p}|^2 + MgP_{R2x} + V_1 = \left(a_1^2k^2 + b_1^2k^2 + c_1^2k^2 + d_1^2k^2 + l_1^2k^2 + m_1^2g^2 + M^2g^2\right)/2k
\]

\[
V_{2p} = V_{s2p} + V_{m2p} + V_2 = \frac{1}{2}k|\mathbf{P}_{R3} - \mathbf{H}_{2p}|^2 + \frac{1}{2}k|\mathbf{P}_{R2} - \mathbf{H}_{1p}|^2 + MgP_{R3x} + V_2 = \left(2l_2^2k^2 + l_2^2k^2 + a_2^2k^2 + b_2^2k^2 + c_2^2k^2 + 2d_2^2k^2 + d_2^2k^2 + m_2^2g^2 + 2M^2g^2\right)/2k
\]
\[ V_{3p} = V_{s3p} + V_{m3p} + V_3 = \frac{1}{2} k |P_{R4} - H_{3p}|^2 + \frac{1}{2} k |P_{R3} - H_{2p}|^2 + \frac{1}{2} k |P_{R2} - H_{1p}|^2 \\
+ M g P_{R4z} + V_3 = (2L_2^2 k^2 + 2L_3^2 k^2 + 1_{23}^2 k^2 + a_3^2 k^2 + b_3^2 k^2 + c_3^2 k^2 \\
+ 2d_2^2 k^2 + 2d_3^2 k^2 + d_4^2 k^2 + 3m_3^2 g^2 + 3M^2 g^2)/2k \]
(6.83)
which are constants.

### 6.4.2 3D Model of the Statically Balanced Spatial Manipulator

The positions of the spring connecting points on the base can be defined using additional linkages, such as RRRR linkages or RSRS linkages described in [104, 174]. We simply adopt an auxiliary serial mechanism with three translational DOFs for each link. The end-effectors have constant lengths of \( m_i g / k \) (Fig. 6.17).

![Fig. 6.17 3D model of the statically balanced spatial manipulator system](image)

Each end-effector is connected to the R joint of the manipulator using an S joint. Hence, \( H_i \) are always right above the R joints with a distance of \( m_i g / k \). Another method is to assemble one PM with three translation DOFs (such as Delta robot) above each R joint. The masses of the end-effectors of the auxiliary mechanism can be balanced using counterweights.

### 6.4.3 Static Balancing of a Mechanism with Multiple Modes

Figure 6.18 shows a 2-DOF 3-4R PM which has 14 2-DOF operation modes, including four spherical translation modes, six planar motion modes, and four sphere-on-sphere rolling modes [75]. Two approaches are adopted to balance the mechanism. In the first method, the mechanism is equal to three manipulators, including one limb with the upper
platform, which has four R joints, and another two limbs which contains three R joints each. Assume the mass of the link within the limbs is $m$ and that of the platform is noted as $M$. The heights of $H_i$, which are the spring attaching points for the links within the chains, are $h_i = mg/k$ and the heights of $H_{i2}$, which are the spring connecting points for the platform, are $h_{i2} = Mg/k$. In the second method, the mass of the upper platform is replaced by three point-masses located at the three R joints on the platform. Then the mechanism is equal to three manipulators, each with three R joints and one payload at the end-effector. The heights of $H_{i3}$, which are the spring connecting points for the payloads, are $h_{i3} = Mg/3k$. The statically balanced systems using the two methods are respectively shown in Figs. 6.18(a) and (b).

![Fig. 6.18 Static balancing of 3-4R PM with multiple modes [75]: (a) method I; (b) method II](image)

**6.5 Static Balancing Method Using Optimization Tools**

In this section, a novel numerical optimization method will be proposed to derive the static balancing conditions.

**6.5.1 Static Balancing of Planar 1-link Manipulator**

In this section, a 1-link manipulator mounted on an R joint will be balanced to illustrate the optimization method.
The planar 1R manipulator is represented schematically in Fig. 6.19. In the literature (such as [102, 104]), it is stated that the manipulators can only be balanced by connecting the point right above the axis of the R joint and the point on the line defined by the joint and the CM of the link, as shown in Fig. 6.19(a). In this section, it will be shown that the position of spring attachment point has other approximate solutions. These solutions are also valid in the range of permitted errors. Suppose that \( P \) is the CM of the manipulator, \( O \) is a point on the axis of the R joint, the spring attachment points on the manipulator and the base are \( A \) and \( H \) respectively, \( B \) is a point at the same horizontal level as \( O \) and \( H \) is right above \( B \). The lengths of \( OB, OP \) and \( BH \) are noted as \( s, L \) and \( h \) respectively. The position vector of \( A \) in the local coordinate frame is \( \{a \quad b \quad c\}^T \).

The CMs of the positions of \( P \) and \( A \) are calculated as:

\[
P = \{LC\theta \quad 0 \quad LS\theta\}^T \\
A = \{aC\theta - bS\theta \quad -c \quad aS\theta + bC\theta\}^T
\]

Let the spring connecting point \( H \) on the base be

\[
H = \{s \quad 0 \quad h\}^T
\]

The potential energy of the manipulator is expressed as

\[
V = V_s + V_m = \frac{1}{2}k|H - A|^2 + mgP_z
\]

\[
= k[(s - aC\theta + bS\theta)^2 + (bC\theta - h + aS\theta)^2 + c^2]/2 + mgLS\theta
\]

The condition for the static balancing of the manipulator is the total potential energy is a constant. The sum of squared differences between two obtained potential energies when giving different values of variables should be as small as possible. The objective function is set as
\[ V_s = \sum_{i=1}^{9} (V_{i+1} - V_i)^2 = f(a, b, s, h) \] (6.87)

where \( V_i \) is the potential energy when \( \theta = \theta_i \), \( \theta_i \) are random values from 0° to 180°. The objective function is a formula related to \( a, b, s \) and \( h \). It is noted that the variance of the total potential energy is independent of \( c \). The optimization toolbox ‘fmincon’ of MATLAB is adopted to minimize the objective function.

\[
[x, fval] = fmincon(fun,x0,[],[],[],[],lb,ub,nonlcon,options)
\]

where \( x \) starts from \( x0 \) and attempts to find a minimizer \( x \) of the function. \( lb \) and \( ub \) are a set of lower and upper bounds on the design variable. For the 1-link manipulator, the lower and upper bound are set to be (the attachment points should be on the link)

\[
lb = [0, -100, 0] \\
ub = [100, 100, 100, 100] 
\] (6.88)

Let \( k = 0.1 \text{ N/mm}, l = 100 \text{ mm}, m = 0.1 \text{ kg} \) and \( g = 9.8 \text{ N/kg} \). The optimization is regarded as successful if \( fval (V_s) < 10^{-5} \). When giving different initial values in MATLAB, different sets of \( a, b, s \) and \( h \) can be obtained. Four sets of results are listed in Table 6.1 as examples.

<table>
<thead>
<tr>
<th>Initial values</th>
<th>Optimization results</th>
</tr>
</thead>
<tbody>
<tr>
<td>30, 30, 30, 30</td>
<td>3.8138, 27.4887, -34.9777, 4.8528</td>
</tr>
<tr>
<td>50, 50, 50, 50</td>
<td>7.6918, 32.3854, -28.6447, 6.8034</td>
</tr>
<tr>
<td>10, 20, 20, 10</td>
<td>19.9761, 8.3321, -17.4301, 41.7884</td>
</tr>
<tr>
<td>90, 80, 80, 90</td>
<td>7.5853, 38.6085, -24.4396, 4.8016</td>
</tr>
</tbody>
</table>

The results in Table 6.1 show that different from the conclusions in the literature, the spring connecting point on the base can be away from the vertical line passing through the R joints, and that on the manipulator doesn’t have to be on the line defined by the R joint and the CM of the link.

To verify the correctness of the result, one set of \( a, b, s \) and \( h \) (7.6918, 32.3854, -28.6447, and 6.8034) are substituted into the formula of the total potential energy of the manipulator. When giving different values of \( \theta \), the corresponding potential energies are obtained. Several examples are listed in Table 6.2. The plot of the potential energy of the system with respect to \( \theta \) drawn by MATLAB is given in Fig. 6.20.

When \( \theta \) varies from 0° to 180°, the values of the potential energy almost keeps constant. The results verify the proposed optimization method is valid and the obtained sets of results can be the spring attachment points of the statically balanced 1-link manipulator.
### Table 6.2 The values of the potential energy of the 1-link manipulator

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>0</th>
<th>( \pi/20 )</th>
<th>( \pi/10 )</th>
<th>( 3\pi/20 )</th>
<th>( \pi/5 )</th>
<th>( \pi/4 )</th>
<th>( 3\pi/10 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V )</td>
<td>98.7390</td>
<td>98.7390</td>
<td>98.7390</td>
<td>98.7390</td>
<td>98.7390</td>
<td>98.7391</td>
<td>98.7391</td>
</tr>
<tr>
<td>( \theta )</td>
<td>7( \pi/20 )</td>
<td>2( \pi/5 )</td>
<td>9( \pi/20 )</td>
<td>( \pi/2 )</td>
<td>11( \pi/20 )</td>
<td>3( \pi/5 )</td>
<td>13( \pi/20 )</td>
</tr>
<tr>
<td>( V )</td>
<td>98.7390</td>
<td>98.7391</td>
<td>98.7391</td>
<td>98.7391</td>
<td>98.7391</td>
<td>98.7392</td>
<td>98.7392</td>
</tr>
<tr>
<td>( \theta )</td>
<td>7( \pi/10 )</td>
<td>3( \pi/4 )</td>
<td>4( \pi/5 )</td>
<td>17( \pi/20 )</td>
<td>9( \pi/10 )</td>
<td>19( \pi/20 )</td>
<td>( \pi )</td>
</tr>
<tr>
<td>( V )</td>
<td>98.7392</td>
<td>98.7392</td>
<td>98.7392</td>
<td>98.7393</td>
<td>98.7393</td>
<td>98.7393</td>
<td>98.7393</td>
</tr>
</tbody>
</table>

### 6.5.2 Static Balancing of Spherical Manipulators

In this section, the proposed optimization method will be applied to spherical manipulators with links whose weights cannot be neglected.

![Fig. 6.21 Static balancing of 3-link spherical manipulators: (a) the 3-link manipulator; (b) the results of the spring attachment points](image-url)
The 3-link manipulator is shown in Fig. 6.21(a). A coordinate system is fixed to the base with its z-axis pointing vertically upward and with its origin located at the intersection of the axes of the R joints.

Suppose that the position vectors of the CM of the \(i\)th link and the spring connecting point on the link in the \(i\)th local coordinate frame are represented by

\[
{\mathbf{G}}_i = \begin{bmatrix} Z_1 \\ 10 \\ 20 \\ 30 \end{bmatrix}
\]

(6.89a)

\[
{\mathbf{G}}_i = \begin{bmatrix} 7 \\ 8 \\ 9 \\ 1 \end{bmatrix}
\]

(6.89b)

For the sake of conciseness, the joint twist angles of the manipulator are assumed to be equal, which is denoted as \(\alpha\), and \(\alpha = 60^\circ\). It is noted that the parameters of the links can be distinct, the manipulator with identical links is just an example to illustrate the optimization method. The position vectors of the CM of the \(i\)th link and the spring connecting point on the link in the global coordinate frame are calculated as:

\[
\begin{bmatrix} P_1 \\ 1 \end{bmatrix} = ^0_1^T \begin{bmatrix} 1 P_1 \end{bmatrix} = \begin{bmatrix} 10C\theta_1 - 20S\theta_1 \\ 30 \\ -10S\theta_1 - 20C\theta_1 \\ 1 \end{bmatrix}
\]

(6.90a)

\[
\begin{bmatrix} P_2 \\ 1 \end{bmatrix} = ^0_1^T \begin{bmatrix} 2 P_2 \end{bmatrix} = \begin{bmatrix} P_{2x} \\ P_{2y} \\ P_{2z} \\ 1 \end{bmatrix}
\]

(6.90b)

where

\[
P_{2x} = 5[2C\theta_1(C\theta_2 - 2S\theta_2) - S\theta_1(3\sqrt{3} + 2C\theta_2 + S\theta_2)]
\]

\[
P_{2y} = -5(-3 + 2\sqrt{3}C\theta_2 + \sqrt{3}S\theta_2)
\]

\[
P_{2z} = -5[2S\theta_1(C\theta_2 - 2S\theta_2) + C\theta_1(3\sqrt{3} + 2C\theta_2 + S\theta_2)]
\]

\[
\begin{bmatrix} P_3 \\ 1 \end{bmatrix} = ^0_1^T \begin{bmatrix} 3 P_3 \end{bmatrix} = \begin{bmatrix} P_{3x} \\ P_{3y} \\ P_{3z} \\ 1 \end{bmatrix}
\]

(6.90c)

where

\[
P_{3x} = -5[C\theta_1[6\sqrt{3}S\theta_2 - 4C\theta_2(C\theta_3 - 2S\theta_3)] + S\theta_1[3\sqrt{3}(1 + C\theta_2) + (-3 + C\theta_2)(2C\theta_3 + S\theta_3)] + 2S\theta_2[2C(\theta_1 + \theta_3) + S(\theta_1 + \theta_3)]]/2
\]

\[
P_{3y} = 5[3 - 9C\theta_2 - 2\sqrt{3}C\theta_3(1 + C\theta_2 + S\theta_2) - \sqrt{3}(1 + C\theta_2 - 4S\theta_2)S\theta_3]/2
\]

\[
P_{3z} = [5C\theta_1[-3\sqrt{3}(1 + C\theta_2) - 2C\theta_3(-3 + C\theta_2 + S\theta_2) + (3 - C\theta_2 + 4S\theta_2)S\theta_3] + 10S\theta_1[-2C\theta_2(C\theta_3 - 2S\theta_3) + S\theta_2(3\sqrt{3} + 2C\theta_3 + S\theta_3)]]/2
\]

\[
\begin{bmatrix} A_1 \\ 1 \end{bmatrix} = ^0_1^T \begin{bmatrix} 1 A_1 \end{bmatrix} = \begin{bmatrix} a_1C\theta_1 - b_1S\theta_1 \\ c_1 \\ -a_1S\theta_1 - b_1C\theta_1 \\ 1 \end{bmatrix}
\]

(6.90d)

\[
\begin{bmatrix} A_2 \\ 1 \end{bmatrix} = ^0_2^T \begin{bmatrix} 2 A_2 \end{bmatrix} = \begin{bmatrix} A_{2x} \\ A_{2y} \\ A_{2z} \\ 1 \end{bmatrix}
\]

(6.90e)

where

\[
A_{2x} = -S\theta_1(\sqrt{3}c_2 + b_2C\theta_2 + a_2S\theta_2)/2 + C\theta_1(a_2C\theta_2 - b_2S\theta_2)
\]

\[
A_{2y} = [c_2 - \sqrt{3}(b_2C\theta_2 + a_2S\theta_2)]/2
\]
\[ A_{2x} = -C_1(\sqrt{3}c_2 + b_2C\theta_2 + a_2S\theta_2)/2 + S\theta_1(-a_2C\theta_2 + b_2S\theta_2) \]

\[
\begin{bmatrix} A_3 \\ 1 \end{bmatrix} = 1 + 3T^2 \begin{bmatrix} 3A_3 \\ 1 \end{bmatrix}^T
\]

(6.90f)

where

\[
A_{3x} = \{-S\theta_1[\sqrt{3}c_3(1 + C\theta_2) - 3b_3C\theta_3 + b_3C\theta_2C\theta_3 + 2a_3C\theta_3S\theta_2 - 3a_3S\theta_3 + a_3C\theta_2S\theta_3 - 2b_3S\theta_2S\theta_3] + C\theta_1[-2S\theta_2(\sqrt{3}c_3 + b_3C\theta_3 + a_3S\theta_3) + 4C\theta_2(a_3C\theta_3 - b_3S\theta_3)]/4 \}
\]

\[
A_{3y} = \{-c_3 - 3c_3C\theta_2 - \sqrt{3}[C\theta_3(b_3 + b_3C\theta_2 + 2a_3S\theta_2) + (a_3 + a_3C\theta_2 - 2b_3S\theta_2)S\theta_3])}/4 \}
\]

\[
A_{3z} = \{4C\theta_2S\theta_1(-a_3C\theta_3 + b_3S\theta_3) - C\theta_1[\sqrt{3}c_3(1 + C\theta_2) + (-3 + C\theta_2)(b_3C\theta_3 + a_3S\theta_3)] + 2S\theta_2[-a_3C(\theta_1 + \theta_3) + \sqrt{3}c_3S\theta_1 + b_3S(\theta_1 + \theta_3)]}/4 \}
\]

Since the links are identical, let

\[
\begin{align*}
\{a_1 = a_2 = a_3 &= a \\
\{b_1 = b_2 = b_3 &= b \\
\{c_1 = c_2 = c_3 &= c
\end{align*}
\]

(6.91)

The spring connecting point on the base \( H \) is given by

\[
H = \{s \ t \ h\}^T
\]

(6.92)

The total potential energy of the system is calculated as

\[
V = \frac{1}{2}k|A_1 - H_1|^2 + \frac{1}{2}k|A_2 - H_2|^2 + \frac{1}{2}k|A_3 - H_3|^2 + mgP_{1x} + mgP_{2z} + mgP_{3z} = \{|k[6(a^2 + b^2 + c^2 + h^2 + s^2) + 6t^2 + t[c(-7 + 3C\theta_2) + \sqrt{3}(2aS\theta_2 + C\theta_3(b + 2aS\theta_2) + (a - 2bS\theta_2)S\theta_3 + C\theta_2[b(2 + C\theta_3) + aS\theta_3])]| + S\theta_1[4ahk - 40mg + 4bks + 3\sqrt{3}cks - 3ks(bC\theta_3 + aS\theta_3) + C\theta_2[4ahk - 40mg + 2bks + \sqrt{3}cks + (4ahk - 40mg + bks)C\theta_3 + (-4bhk + 80mg + aks)S\theta_3]\} - 2S\theta_2[-10mg(4 + 3\sqrt{3} + 2C\theta_3 + S\theta_3) + k[-as(1 + C\theta_3) + bsS\theta_3 + h[\sqrt{3}c + b(2 + C\theta_3) + aS\theta_3)]]} + C\theta_1[-10gm[8 + 9\sqrt{3} - 6C\theta_3 + 2S\theta_2(1 + C\theta_3 - 2S\theta_3) - 3S\theta_3 + C\theta_2(4 + 3\sqrt{3} + 2C\theta_3 + S\theta_3)] + k[4bh + 3\sqrt{3}ch - 4as - 3h(bC\theta_3 + aS\theta_3) + 2S\theta_2[s[\sqrt{3}c + b(2 + C\theta_3) + aS\theta_3] + h(a + aC\theta_3 - bS\theta_3)] + C\theta_2[h[\sqrt{3}c + b(2 + C\theta_3) + aS\theta_3] - 4s(a + aC\theta_3 - bS\theta_3)]]/4 \}
\]

(6.93)

The objective function is a formula of \( a, b, c, s, t \) and \( h \).

\[
Vs = \Sigma_{i=1}^9(V_{i+1} - V_i)^2 = f(a, b, c, s, t, h)
\]

(6.94)

When giving different initial values, the optimization results of the parameters are obtained, which are given in Table 6.3. From the results, it can be seen that \( s \) and \( t \) are always approximately equal to zero, which means the spring attachment point on the base
should always be right above the intersection of the R joints. Besides, \( \{a \ b \ c\}^T \) and the position vector of the CM of each link in the local coordinate frame are always proportional, i.e., the spring attachment point on the link is on the line defined by the intersection of the R joints and the CM of the link [Fig. 6.21(b)].

Table 6.3 The optimization results of the 3-link spherical manipulator with three springs

<table>
<thead>
<tr>
<th>Initial values</th>
<th>Optimization results</th>
</tr>
</thead>
<tbody>
<tr>
<td>30, 30, 30, 30, 30, 30</td>
<td>1.6775, 3.3652, 5.0331, 0.0383, 0.0502, 58.3455</td>
</tr>
<tr>
<td>10, 20, 30, 40, 50, 60</td>
<td>2.1011, 4.3442, 6.3128, 0.3315, 0.4289, 45.9785</td>
</tr>
<tr>
<td>90, 80, 70, 60, 50, 40</td>
<td>23.2670, 46.8468, 69.8214, 0.0061, 0.0080, 4.2000</td>
</tr>
<tr>
<td>50, 40, 30, 30, 40, 50</td>
<td>7.2258, 14.4949, 21.6802, 0.0088, 0.0115, 13.5453</td>
</tr>
</tbody>
</table>

Now statically balanced 3-link manipulator with two springs will be designed. The two springs are attached to links 1 and 3 respectively [Fig. 6.22(a)].

![Fig. 6.22 Static balancing of spherical manipulators with two springs: (a) 3-link manipulator; (b) 4-link manipulator](image)

The spring connecting points on the base \( H_1 \) and \( H_2 \) are given by

\[
H_1 = \begin{bmatrix} 0 \\ 0 \\ h_1 \end{bmatrix}^T \quad (6.95)
\]

\[
H_2 = \begin{bmatrix} 0 \\ 0 \\ h_2 \end{bmatrix}^T \quad (6.96)
\]

The potential energy of the 3-link manipulator with two springs is obtained as:

\[
V = \frac{1}{2} k |A_1 - H_1|^2 + \frac{1}{2} k |A_3 - H_2|^2 + mgP_{1z} + mgP_{2z} + mgP_{3z} = 2(a_1^2 + a_3^2 + b_1^2 + b_3^2 + c_1^2 + c_3^2 + h_1^2 + h_2^2)k + C\theta_1[k(4b_1h_1 + \sqrt{3}c_3h_2 + \sqrt{3}c_3h_2C\theta_2 + h_2C\theta_3[b_3(-3 + C\theta_2) + 2a_3S\theta_2] + h_2[a_3(-3 + C\theta_2) - 2b_3S\theta_2]S\theta_3] - 10mg[b + 9\sqrt{3} - 6C\theta_3 + 2S\theta_2(1 + C\theta_3 - 2S\theta_3 - 3S\theta_3 + C\theta_2(4 + 3\sqrt{3} + 2C\theta_3 + S\theta_3))] -
\]
\[ 2S\theta_1 \{10mg[2 + 2C\theta_2(1 + C\theta_3 - 2S\theta_3) - S\theta_2(4 + 3\sqrt{3} + 2C\theta_3 + S\theta_3)] + k[-2a_1h_1 + h_2S\theta_2(\sqrt{3}c_3 + b_3C\theta_3 + a_3S\theta_3) + 2h_2C\theta_2(-a_3C\theta_3 + b_3S\theta_3)]} / 4 \]  

The objective function is in terms of \( a_1, b_1, a_3, b_3, c_3, h_1 \) and \( h_2 \).

\[ V_s = \sum_{i=1}^{9} (V_{i+1} - V_i)^2 = f(a_1, b_1, a_3, b_3, c_3, h_1, h_2) \]  

(6.98)

### Table 6.4 The optimization results of the 3-link spherical manipulator with two springs

<table>
<thead>
<tr>
<th>Initial values</th>
<th>Optimization results</th>
</tr>
</thead>
<tbody>
<tr>
<td>30, 30, 30, 30, 30, 30</td>
<td>0.0024, 8.0239, 0.0045, 24.7639, 65.6994, 43.9671, 7.9186</td>
</tr>
<tr>
<td>50, 50, 50, 50, 50, 50</td>
<td>0.0017, 5.7170, 0.0045, 24.9049, 66.0734, 61.7083, 7.8737</td>
</tr>
<tr>
<td>10, 20, 30, 40, 30, 20, 10</td>
<td>0.0032, 40.2335, 0.0007, 15.1062, 40.0956, 8.7653, 12.9765</td>
</tr>
<tr>
<td>90, 80, 70, 60, 70, 80, 90</td>
<td>0.0013, 19.7935, 0.0012, 28.8591, 76.6015, 17.8165, 6.7924</td>
</tr>
</tbody>
</table>

### Table 6.5 The values of the potential energy of the 3-link spherical manipulator with two springs

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( 0, 0, 0 )</th>
<th>( \pi/8, \pi/7, \pi/6 )</th>
<th>( \pi/3, \pi/4, \pi/5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V )</td>
<td>349.4992</td>
<td>349.5019</td>
<td>349.5039</td>
</tr>
<tr>
<td>( \theta )</td>
<td>( \pi/2, \pi/2, \pi/2 )</td>
<td>( \pi, \pi, \pi )</td>
<td>( \pi/10, \pi/5, \pi/10 )</td>
</tr>
<tr>
<td>( V )</td>
<td>349.5079</td>
<td>349.4656</td>
<td>349.5024</td>
</tr>
<tr>
<td>( \theta )</td>
<td>( \pi/5, 2\pi/5, \pi/5 )</td>
<td>( 3\pi/10, 3\pi/5, 3\pi/10 )</td>
<td>( 2\pi/5, 4\pi/5, 2\pi/5 )</td>
</tr>
<tr>
<td>( V )</td>
<td>349.5038</td>
<td>349.5077</td>
<td>349.5134</td>
</tr>
<tr>
<td>( \theta )</td>
<td>( \pi/2, \pi, \pi/2 )</td>
<td>( 3\pi/5, 6\pi/5, 3\pi/5 )</td>
<td>( 7\pi/10, 7\pi/5, 7\pi/10 )</td>
</tr>
<tr>
<td>( V )</td>
<td>349.5128</td>
<td>349.5011</td>
<td>349.4846</td>
</tr>
<tr>
<td>( \theta )</td>
<td>( 4\pi/5, 8\pi/5, 4\pi/5 )</td>
<td>( 9\pi/10, 9\pi/5, 9\pi/10 )</td>
<td>( \pi, 2\pi, \pi )</td>
</tr>
<tr>
<td>( V )</td>
<td>349.4751</td>
<td>349.4754</td>
<td>349.4768</td>
</tr>
</tbody>
</table>

When giving different initial values, seven parameters in Table 6.4 are obtained. The results show that \( a_1 \) and \( a_3 \) should be zero, namely, the spring attachment point for each link should be on the plane defined by the two R joints of the link. Substituting the first set of parameters (0.0024, 8.0239, 0.0045, 24.7639, 65.6994, 43.9671, and 7.9186) into Eq. (6.97), the potential energies are calculated, as shown in Table 6.5.

Let \( \theta_2 = 2\theta_1 \), the surface of potential energies of the system with respect to \( \theta_1 \) and \( \theta_3 \) is drawn in Fig. 6.23. Table 6.5 and Figure 6.23 show that the potential energy almost keeps constant and is approximately equal to 349.5 N.mm.
Then the statically balanced 4-link manipulator with two springs will be designed using the optimization method. The two springs are attached to links 2 and 4 respectively [Fig. 6.22(b)]. The position vectors of the CM of the 4th link, and the spring connecting point on link 4 and the total potential energy of the system are calculated as

\[
\{P_4\} = T^{1/2}T_3^{1/2}T_4^{1/2}\{P_4\} = \{P_{4x} \ P_{4y} \ P_{4z} \ 1\}^T
\] (6.99a)

where

\[
P_{4x} = \{5S\theta_1 \{-6\sqrt{3}[1 - 3C\theta_3 + C\theta_2(1 + C\theta_3) - 2S\theta_2S\theta_3] + 4C\theta_4[3 + C\theta_3(3 - 2S\theta_2) + (3 + 2S\theta_2)S\theta_3 - C\theta_2(-3 + C\theta_3 + S\theta_3)] + 6(1 + C\theta_2) - 2C\theta_3(-3 + C\theta_2 - 8S\theta_2) + 4(-6 + 2C\theta_2 + S\theta_2)S\theta_3]S\theta_4]/2 - 10C\theta_1[S\theta_2[\sqrt{3}(1 + C\theta_3) + 2C\theta_4(-3 + C\theta_3 + S\theta_3) + (-3 + C\theta_3 - 4S\theta_3)S\theta_4] + C\theta_2[-4C\theta_3(C\theta_4 - 2S\theta_4) + 2S\theta_3(3\sqrt{3} + 2C\theta_4 + S\theta_4)]]/4
\]

\[
P_{4y} = 5\{(1 + 3C\theta_2)(-3 + 2\sqrt{3}C\theta_4 + \sqrt{3}S\theta_4) + C\theta_3[9S\theta_2 - \sqrt{3}[C\theta_4 + C(\theta_2 + \theta_4) - 2[S\theta_4 + S(\theta_2 + \theta_4)]]]/4
\]

\[
P_{4z} = \{5C\theta_1 \{-6\sqrt{3}[1 - 3C\theta_3 + C\theta_2(1 + C\theta_3) - 2S\theta_2S\theta_3] + 4C\theta_4[2S\theta_2(-C\theta_3 + S\theta_3) - C\theta_2(-3 + C\theta_3 + S\theta_3) + 3(1 + C\theta_3 + S\theta_3)] + 6(1 + C\theta_2) - 2C\theta_3(-3 + C\theta_2 - 8S\theta_2) + 4(-6 + 2C\theta_2 + S\theta_2)S\theta_3]S\theta_4]/2 - 10S\theta_4[3\sqrt{3}(1 + C\theta_3) - 2C\theta_4(-3 + C\theta_3 + S\theta_3) + (3 - C\theta_3 + 4S\theta_3)S\theta_4] + C\theta_2[4C\theta_3(C\theta_4 - 2S\theta_4) - 2S\theta_3(3\sqrt{3} + 2C\theta_4 + S\theta_4)]]/4
\]

\[
\{A_4\} = T^{1/2}T_3^{1/2}T_4^{1/2}\{A_4\} = \{A_{4x} \ A_{4y} \ A_{4z} \ 1\}^T
\] (6.99b)

where
\[ A_{4x} = \{S\theta_1(-2\sqrt{3}c_4[1 - 3C\theta_1 + C\theta_2(1 + C\theta_3) - 2S\theta_2S\theta_3] + 2C\theta_4(3b_4(1 + C\theta_2) - C\theta_3[b_4(-3 + C\theta_2) + 4a_4S\theta_2]) + 2(3a_4 - a_4C\theta_2 + b_4S\theta_2)S\theta_3) + [6a_4 + 6a_4C\theta_2 + 2C\theta_3(3a_4 - a_4C\theta_2 + 4b_4S\theta_2) + 4[b_4(-3 + C\theta_2) + a_4S\theta_2]S\theta_3]S\theta_4]/2 + C\theta_1(-2\theta_2\sqrt{3}c_4(1 + C\theta_3) + (3\theta_3)(b_4C\theta_4 + a_4S\theta_4)) - 4C\theta_2(\sqrt{3}c_4S\theta_3 + C\theta_3(2a_4C\theta_4 + 4b_4S\theta_4)) - 4S\theta_3[b_4C(\theta_2 + \theta_4) + a_4S(\theta_2 + \theta_4))]/8\]

\[ A_{4y} = \{c_4[2 - 6(1 + C\theta_2)C\theta_3 + 12S\theta_2S\theta_3] - 2\sqrt{3}C\theta_4[b_4 + C\theta_3(b_4 + 4b_4C\theta_2 + 4a_4S\theta_2) + 2(a_4 + a_4C\theta_2 - b_4S\theta_2)S\theta_3] + \sqrt{3}(-2\theta_2[1 + (1 + C\theta_2)C\theta_3] + 8b_4C\theta_3S\theta_2 + 4(b_4 + 4b_4C\theta_2 - a_4S\theta_2)S\theta_3]S\theta_4 + 6C\theta_2[-c_4 + \sqrt{3}(b_4C\theta_4 + a_4S\theta_4))]/16\]

\[ A_{4z} = \{(C(\theta_1 - \theta_2/2) - 3C(\theta_1 + \theta_2/2))\sqrt{3}c_4 - 3b_4C\theta_4 - 3a_4S\theta_4) + C\theta_3[3(\theta_1(-3 + C\theta_2)] + 2\theta_2(\sqrt{3}c_4 + 4b_4C\theta_4 + a_4S\theta_4)] + C\theta_2(-4a_4C\theta_4 + 4b_4S\theta_2)] + 2\sqrt{3}c_4(2C\theta_2S\theta_1 + C\theta_1S\theta_2) + b_4(3S(\theta_1 - \theta_4) + S(\theta_1 - \theta_2) - 3S(\theta_1 + \theta_4) + 3S(\theta_1 + \theta_2 + \theta_4))\}]/8\]

\[ ^{i-1}_1T \text{ are given in Appendix (C). The total potential energy of the 4-link manipulator with two springs are yielded as:} \]

\[ V = \frac{1}{2}k|A_2 - H_1|^2 + \frac{1}{2}k|A_4 - H_2|^2 + mgP_{1z} + mgP_{2x} + mgP_{3z} + mgP_{4x} = \]

\[ \{4(a_2 + a_2^2 + b_2^2 + b_2 + c_2^2 + c_2^2 + h_2^2 + b_2^2)k + C\theta_1(\sqrt{3}(c_4h_1 + 4c_2h_2)k - 10(16 + 21\sqrt{3})gm - 3b_4h_1kC\theta_4 + 60gmC\theta_4 + 4a_2h_2kS\theta_2 - 40gmS\theta_2 - 3a_4h_1kS\theta_4 + 30gmS\theta_4 + C\theta_3(-3\sqrt{3}c_4h_1k + 30(4 + 3\sqrt{3})gm - 40gmS\theta_2 + C\theta_4[-3b_4h_1k + 60gm + 4(a_4h_1k - 10gm)S\theta_2] + [-3a_4h_1k + 30gm + (-4b_4h_1k + 80gm)S\theta_2]S\theta_4) + C\theta_2[\sqrt{3}c_4h_1k + 4b_4h_2k - 10(8 + 9\sqrt{3})gm + \sqrt{3}c_4h_1kC\theta_3 - 40gmC\theta_3 - 30\sqrt{3}gmC\theta_3 - 20gmS\theta_3 + C\theta_4[(b_4h_1k - 20gm)(-3\theta_3 + 3\theta_2) + 2(a_4h_1k - 10gm)S\theta_3] + [(a_4h_1k - 10gm)(-3\theta_4) - 2(b_4h_1k - 20gm)S\theta_3]S\theta_4) + 2S\theta_3[C\theta_4(-3a_4h_1k + 30gm - b_4h_1kS\theta_2 + 20gmS\theta_2) - h_1k(\sqrt{3}c_4S\theta_2 - 3b_4S\theta_4 + a_4S\theta_2) + 10gm[3 - 6S\theta_4 + S\theta_2(4 + 3\sqrt{3} + S\theta_4)]\} - 2S\theta_1[10gm[4 + 4C\theta_2[1 + C\theta_3(1 + C\theta_4 + 2S\theta_4)] - 2C\theta_2S\theta_3(4 + 3\sqrt{3} + 2C\theta_4 + S\theta_4) + S\theta_2(-8 - 9\sqrt{3} + 6C\theta_4 - 2S\theta_3(1 + C\theta_4 - 2S\theta_4) + 3S\theta_4 - C\theta_3(4 + 3\sqrt{3} + 2C\theta_4 + S\theta_4))] + k[S\theta_2[\sqrt{3}c_4h_1 + 4b_2h_2 + \sqrt{3}c_4h_1C\theta_3 + h_1C\theta_4[b_4(-3 + C\theta_3) + 2a_4S\theta_3] + h_1(a_4(-3 + C\theta_3) - 2b_4S\theta_3]S\theta_4) + 2C\theta_2[-2a_2h_2 + h_1S\theta_3(\sqrt{3}c_4 + b_4C\theta_4 + a_4S\theta_4) + 2h_1C\theta_3(-a_4C\theta_4 + b_4S\theta_4))]/8\]
The objective function is in terms of $a_2, b_2, c_2, a_4, b_4, c_4, h_1$ and $h_2$.

$$V_s = \sum_{i=1}^{9} (V_i - V_i)^2 = f(a_2, b_2, c_2, a_4, b_4, c_4, h_1, h_2)$$

Table 6.6 The optimization results of the 4-link spherical manipulator with two springs

<table>
<thead>
<tr>
<th>Initial values</th>
<th>Optimization results</th>
</tr>
</thead>
<tbody>
<tr>
<td>30, 30, 30, 30, 30, 30, 30, 30</td>
<td>0.0036, 17.1379, 29.6707, 0.0010, 11.4028, 30.2685, 20.5805, 17.1901</td>
</tr>
<tr>
<td>10, 10, 10, 10, 10, 10, 10, 10</td>
<td>0.0015, 6.8826, 11.9157, 0.0006, 7.0988, 18.8437, 51.2464, 27.6121</td>
</tr>
<tr>
<td>10, 20, 30, 40, 40, 30, 20, 10</td>
<td>0.0025, 11.9307, 20.6555, 0.0021, 24.9150, 66.1365, 29.5629, 7.8673</td>
</tr>
<tr>
<td>10, 20, 30, 40, 40, 30, 20, 10</td>
<td>0.0013, 24.3884, 42.2374, 0.0001, 4.7615, 12.6402, 14.4591, 41.1640</td>
</tr>
</tbody>
</table>

The parameters when giving different initial values are listed in Table 6.6. Similar to the 3-link manipulator with two springs, $a_2$ and $a_4$ should be zero, namely, the spring attachment point for each link should be on the plane defined by the two R joints of the link. The potential energy yielded using the first set of obtained parameters (0.0036, 17.1379, 29.6707, 0.0010, 11.4028, 30.2685, 20.5805, and 17.1901) are given in Table 6.7. Let $\theta_2 = 2\theta_1$ and $\theta_4 = 2\theta_3$, the total potential energy of the manipulator associated with $\theta_1$ and $\theta_3$ is depicted in Fig. 6.24. Table 6.7 and Figure 6.24 show that the potential energy almost keeps constant and is approximately equal to 146.9 N.mm.

Table 6.7 The values of the potential energy of the 4-link spherical manipulator with two springs

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\pi/2, \pi/2, \pi/2, \pi/2$</th>
<th>$\pi/2, \pi, \pi, \pi$</th>
<th>$\pi/10, \pi/5, \pi/10, \pi/5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V$</td>
<td>146.9587</td>
<td>146.9807</td>
<td>146.9617</td>
</tr>
<tr>
<td>$\theta$</td>
<td>$\pi/5, 2\pi/5, \pi/5, 2\pi/5$</td>
<td>$3\pi/10, 3\pi/5, 3\pi/10, 3\pi/5$</td>
<td>$2\pi/5, 4\pi/5, 2\pi/5, 4\pi/5$</td>
</tr>
<tr>
<td>$V$</td>
<td>146.9581</td>
<td>146.9546</td>
<td>146.9533</td>
</tr>
<tr>
<td>$\theta$</td>
<td>$\pi/2, \pi, \pi/2, \pi$</td>
<td>$3\pi/5, 6\pi/5, 3\pi/5, 6\pi/5$</td>
<td>$7\pi/10, 7\pi/5, 7\pi/10, 7\pi/5$</td>
</tr>
<tr>
<td>$V$</td>
<td>146.9576</td>
<td>146.9701</td>
<td>146.9818</td>
</tr>
<tr>
<td>$\theta$</td>
<td>$4\pi/5, 8\pi/5, 4\pi/5, 8\pi/5$</td>
<td>$9\pi/10, 9\pi/5, 9\pi/10, 9\pi/5$</td>
<td>$\pi, 2\pi, \pi, 2\pi$</td>
</tr>
<tr>
<td>$V$</td>
<td>146.9825</td>
<td>146.9761</td>
<td>146.9712</td>
</tr>
</tbody>
</table>
Now statically balanced spatial manipulators will be designed using the proposed optimization method. A global coordinate is attached to the ground, with the \( z \)-axis pointing in the direction opposite to the gravitational acceleration vector. Let \( \theta_1 \), \( \theta_2 \), and \( \theta_3 \) be the joint variables associated with the three revolute joints respectively and the link lengths are represented as \( l \). It is obtained above that the \( i^{th} \) link of the \( n \)-link manipulator, in which the masses of the links cannot be neglected, can be balanced using \( i \) springs. In this section, each link will be balanced using only one spring. One end of each spring is
attached to $H_i$, which is a point right above $R_i$, and the other end is attached to $A_i$, as shown in Fig. 6.25.

Suppose the position vectors of the CM, the spring attachment point of the $i^{th}$ link, and $R_i$ in the $i^{th}$ local coordinate frame are represented by

$$i\mathbf{p}_i = \begin{bmatrix} 10 & 20 & 30 \end{bmatrix}^T$$  \hspace{1cm} (6.102a)

$$i\mathbf{a}_i = \begin{bmatrix} a_n & b_n & c_n \end{bmatrix}^T$$  \hspace{1cm} (6.102b)

$$i\mathbf{b}_i = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$$  \hspace{1cm} (6.102c)

It is noted that the parameters of the links can be distinct, the manipulator with identical links is just an example to illustrate the optimization method. The spring connecting point on the base $H_i$, which is a point right above the $B_i$, is given by

$$H_i = \{B_{ix} \ B_{iy} \ h + B_{iz}\}^T$$  \hspace{1cm} (6.103)

The heights of $H_i$ are all set to be $h$ to facilitate the design of the system. One can also give different values of $h_i$. Let

$$l_0 = 0, l_1 = l_2 = l$$  \hspace{1cm} (6.104a)

$$\alpha_0 = 30^\circ, \alpha_1 = 60^\circ, \alpha_2 = 90^\circ$$  \hspace{1cm} (6.104b)

$$d_1 = d_2 = d_3 = 0$$  \hspace{1cm} (6.104c)

The position vectors of the CMs of the links ($P_i$), the spring attachment points on the links and the R joints ($B_i$) expressed in the global coordinate frame are obtained as

$$\{\mathbf{P}_1\} = {^0}_1T_{^1_1}\{\mathbf{P}_1\}$$

$$= \begin{bmatrix} 10C\theta_1 - 20S\theta_1 & 10\sqrt{3}C\theta_1 - 15 + 5\sqrt{3}a_1S\theta_1 & 15\sqrt{3} + 10C\theta_1 + 5S\theta_1 & 1 \end{bmatrix}^T$$  \hspace{1cm} (6.105a)

$$\{\mathbf{P}_2\} = {^0}_1T_{^2_2}\{\mathbf{P}_2\} = \begin{bmatrix} A_{2x} & A_{2y} & A_{2z} & 1 \end{bmatrix}^T$$  \hspace{1cm} (6.105b)

where

$$P_{2x} = C\theta_1(l + 10C\theta_2 - 20S\theta_2) + 5S\theta_1(3\sqrt{3} - 2C\theta_2 - S\theta_2)$$

$$P_{2y} = [-15 - 45C\theta_1 + \sqrt{3}lS\theta_1 + 10\sqrt{3}C\theta_2(-1 + C\theta_1 + S\theta_1) + 5\sqrt{3}(-1 + C\theta_1 - 4S\theta_1)S\theta_2]/2$$

$$P_{2z} = [10C\theta_2(3 + S\theta_1) + S\theta_1(l - 20S\theta_2) + 15(\sqrt{3} + S\theta_2) + 5C\theta_1(-3\sqrt{3} + 2C\theta_2 + S\theta_2)]/2$$

$$\{\mathbf{P}_3\} = {^0}_1T_{^3_3}\{\mathbf{P}_3\} = \begin{bmatrix} A_{3x} & A_{3y} & A_{3z} & 1 \end{bmatrix}^T$$  \hspace{1cm} (6.105c)

where

$$P_{2x} = C\theta_1[l + 30S\theta_2 + C\theta_2(l + 10C\theta_3 - 20S\theta_3)] + S\theta_1[30C\theta_2 - S\theta_2(l + 10C\theta_3 - 20S\theta_3) + 10\sqrt{3}(2C\theta_3 + S\theta_3)]/2$$
\[ P_{2y} = \{2\sqrt{3}lS\theta_1 + \sqrt{3}l(-1 + C\theta_1)S\theta_2 + 2\sqrt{3}C\theta_2[15 - 15C\theta_1 + S\theta_1(l + 10C\theta_3 - 20S\theta_3)] - 10(2C\theta_3 + S\theta_3) - 30C\theta_1(2C\theta_3 + S\theta_3) + 10\sqrt{3}S\theta_2[(-1 + C\theta_1)C\theta_3 + 6S\theta_1 - 2(-1 + C\theta_1)S\theta_3)]/4 \]
\[ P_{2z} = \{2lS\theta_1 + C\theta_2[-90 - 30C\theta_1 + 2S\theta_1(l + 10C\theta_3 - 20S\theta_3)] - 10\sqrt{3}(-1 + C\theta_1)(2C\theta_3 + S\theta_3) + S\theta_2[3l + lC\theta_1 + 10(3 + C\theta_1)C\theta_3 + 60S\theta_1 - 20(3 + C\theta_1)S\theta_3)]/4 \]

where

\[ \{A_1\} = {}^0T_1^T \begin{bmatrix} 1 \\ A_1x \\ A_1y \\ A_1z \end{bmatrix} \]

\[ \{A_2\} = {}^0T_1^T \begin{bmatrix} 2A_2 \\ A_2x \\ A_2y \\ A_2z \end{bmatrix} \]

\[ \{A_3\} = {}^0T_2^T \begin{bmatrix} 3A_3 \\ A_3x \\ A_3y \\ A_3z \end{bmatrix} \]

where

\[ A_{2x} = lC\theta_1 + a_2(C\theta_1C\theta_2 - S\theta_1S\theta_2)/2 - b_2(C\theta_1S\theta_2 + C\theta_2S\theta_1)/2 + (\sqrt{3}c_2S\theta_1)/2 \]
\[ A_{2y} = -c_2 + C\theta_1[-3c_2 + \sqrt{3}(b_2C\theta_2 + a_2S\theta_2)] - \sqrt{3}[b_2C\theta_2 + a_2S\theta_2 - 2S\theta_1(l + a_2C\theta_2 - b_2S\theta_2)]/4 \]
\[ A_{2z} = [-\sqrt{3}c_2(-1 + C\theta_1) + 3b_2C\theta_2 + b_2C\theta_1C\theta_2 + 2lS\theta_1 + 2a_2C\theta_2S\theta_1 + 3a_2S\theta_2 + a_2C\theta_1S\theta_2 - 2b_2S\theta_1S\theta_2]/4 \]

\[ \{B_1\} = {}^0T_1^T \begin{bmatrix} 1 \\ B_1x \\ B_1y \\ B_1z \end{bmatrix} \]

\[ \{B_2\} = {}^0T_1^T \begin{bmatrix} 2B_2 \\ B_2x \\ B_2y \\ B_2z \end{bmatrix} \]

\[ \{B_3\} = {}^0T_2^T \begin{bmatrix} 3B_3 \\ B_3x \\ B_3y \\ B_3z \end{bmatrix} \]
where

\[
B_{3x} = lC\theta_1 + l(C\theta_1 C\theta_2 - S\theta_1 S\theta_2/2)
\]

\[
B_{3y} = l(\sqrt{3}C\theta_1 S\theta_2/4 - \sqrt{3}S\theta_2/4 + \sqrt{3}C\theta_2 S\theta_1/2) + \sqrt{3}lS\theta_1/2
\]

\[
B_{3z} = l(3S\theta_2/4 + C\theta_1 S\theta_2/4 + C\theta_2 S\theta_1/2) + lS\theta_1/2
\]

\(i\frac{dT}{d\theta_1}\) are provided in Appendix (D). The total potential energy of the system is yielded as

\[
V = V_s + V_m = \frac{1}{2}k|A_1 - H_1|^2 + \frac{1}{2}k|A_2 - H_2|^2 + \frac{1}{2}k|A_3 - H_3|^2 + mgP_{1x} + mgP_{2x} + mgP_{3x} = \{mg\{4(5 + l)S\theta_1 + 2C\theta_2[-15 + S\theta_1(10 + l + 10C\theta_3 - 20S\theta_3)] + S\theta_2(30 + 3l + 30C\theta_3 + 20S\theta_1 - 60S\theta_3) + 10\sqrt{3}(9 + 2C\theta_3 + S\theta_3)\} + C\theta_1\{gm[-10C\theta_2 + S\theta_2(10 + l + 10C\theta_3 - 20S\theta_3) - 10(-4 + 3\sqrt{3} + 2\sqrt{3}C\theta_3 + \sqrt{3}S\theta_3)] + hk[-2b_1 + (-b_2 + c_3)C\theta_2 + C\theta_3(\sqrt{3}b_3 - a_3S\theta_2) + \sqrt{3}(c_2 + a_3S\theta_3) + S\theta_2(-a_2 + b_3S\theta_3)]\} + k\{2a_2^2 + 2a_2^2 + 2a_3^2 + 2b_2^2 + 2b_2^2 + 2c_1^2 + 2c_2^2 + 2c_2^2 - 2\sqrt{3}c_1h - \sqrt{3}c_2h + 6h^2 + h\{-2a_1S\theta_1 - \sqrt{3}(b_3C\theta_3 + a_3S\theta_3) + S\theta_2[-3a_2 - 3a_3C\theta_3 + 2(b_2 - c_3)S\theta_1 + 3b_3S\theta_3] + C\theta_2[-3b_2 + 3c_3 - 2S\theta_1(a_2 + a_3C\theta_3 - b_3S\theta_3)]\}\}/4
\]

(6.106)

The objective function is in terms of \(a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3\) and \(h\).

\[
V_s = \sum_{i=1}^{9}(V_{i+1} - V_i)^2 = f(a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3, h)
\]

(6.107)

<table>
<thead>
<tr>
<th>Table 6.8 The optimization results of the 3-link spatial manipulator</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Initial values</strong></td>
</tr>
<tr>
<td>30, 30, 30, 30, 30, 30, 30, 30, 30, 30</td>
</tr>
<tr>
<td>50, 50, 50, 50, 50, 50, 50, 50, 50, 50</td>
</tr>
<tr>
<td>10, 20, 30, 10, 20, 30, 10, 20, 30, 10</td>
</tr>
<tr>
<td>10, 20, 30, 40, 50, 50, 40, 30, 20, 10</td>
</tr>
</tbody>
</table>

When giving different initial values, ten parameters in Table 6.8 are obtained. Substituting the first set of parameters (76.5880, 34.6767, 45.3864, 40.1175, 27.9223, 42.5598, 3.6471, 7.2941, 31.5694, and 26.8711) into Eq. (6.106), the potential energy is calculated, as shown in Table 6.9. Let \(\theta_2 = 2\theta_1\), the plot of the potential energy of the system with respect to \(\theta_1\) and \(\theta_3\) is given in Fig. 6.26. The obtained potential energy in
Table 6.9 and Figure 6.26 show that the potential energy varies in a small range, which can be neglected.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\pi, \pi, \pi$</th>
<th>$\pi/2, \pi/2, \pi/2$</th>
<th>$\pi/3, \pi/4, \pi/5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V$</td>
<td>710.9435</td>
<td>710.9431</td>
<td>710.9431</td>
</tr>
<tr>
<td>$\theta$</td>
<td>$\pi/9, \pi/8, \pi/7$</td>
<td>0, 0, 0</td>
<td>$\pi/10, \pi/5, \pi/10$</td>
</tr>
<tr>
<td>$V$</td>
<td>710.9434</td>
<td>710.9436</td>
<td>710.9433</td>
</tr>
<tr>
<td>$\theta$</td>
<td>$\pi/5, 2\pi/5, \pi/5$</td>
<td>$3\pi/10, 3\pi/5, 3\pi/10$</td>
<td>$2\pi/5, 4\pi/5, 2\pi/5$</td>
</tr>
<tr>
<td>$V$</td>
<td>710.9431</td>
<td>710.9431</td>
<td>710.9431</td>
</tr>
<tr>
<td>$\theta$</td>
<td>$\pi/2, \pi, \pi/2$</td>
<td>$3\pi/5, 6\pi/5, 3\pi/5$</td>
<td>$7\pi/10, 7\pi/5, 7\pi/10$</td>
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<tr>
<td>$V$</td>
<td>710.9432</td>
<td>710.9433</td>
<td>710.9434</td>
</tr>
<tr>
<td>$\theta$</td>
<td>$4\pi/5, 8\pi/5, 4\pi/5$</td>
<td>$9\pi/10, 9\pi/5, 9\pi/10$</td>
<td>$\pi, 2\pi, \pi$</td>
</tr>
<tr>
<td>$V$</td>
<td>710.9435</td>
<td>710.9436</td>
<td>710.9437</td>
</tr>
</tbody>
</table>

Fig. 6.26 The potential energy of the 3-link spatial manipulator

6.6 Statically Balanced Deployable Mechanisms with Multiple Modes

In this section, deployable mechanisms with multiple modes will be balanced based on the static balancing methods proposed in the previous sections. These mechanisms will be statically balanced using springs with and without auxiliary links.

6.6.1 Static Balancing of Single-loop 8R Linkages

First, statically balanced deployable 8R linkages will be addressed.

a) Statically Balanced 8R Linkage using only springs.
The static balancing method of spatial manipulators in Section 6.4 will be adopted to balance the 8R linkage. Link 8 is fixed on the ground, and $R_8$ is perpendicular to the ground plane. Hence, link 7 always moves on the ground, and has no need to balance (Fig. 6.27). The linkage is then equal to two manipulators, including the one composed of links 1, 2 and 3, and the one with links 6, 5 and 4. Each manipulator can be balanced using the proposed static balancing method of spatial manipulators in Section 6.4. Suppose that $H_1$, $H_2$, $H_3$, $H_4$, $H_5$ and $H_6$ are points right above joint $R_1$, $R_2$, $R_3$, $R_5$, $R_6$ and $R_7$ respectively. All the links are assumed to be the same, with the mass of $m$, and the heights of the spring connecting points on the base are $h = mg/k$. It is noted that $R_7$, the first R joint of the second manipulator moves on the ground, which has no influence on the balancing condition of the system, since the height of the CM of the manipulator is constant during the translational motion on the plane perpendicular to the direction of gravity. Link 1 is balanced by connecting $H_1$ and $P_1$ using one spring; link 2 is balanced by connecting $H_2$ and $P_2$, and $H_1$ and $R_2$; Link 3 is statically balanced by connecting $H_3$ and $P_3$, $H_1$ and $R_2$, $H_2$ and $R_3$. Similarly, link 4 can be balanced using three springs, link 5 is balanced using two springs and link 6 is balanced using one spring. The resulting 8R linkage with multiple modes is fully compensated for gravity in any modes.

![Fig. 6.27 Statically balanced 8R linkage with multiple modes](image)

### b) Statically Balanced 8R Linkage using Auxiliary Links

Now the static balancing method using auxiliary parallelograms will be designed. Special auxiliary parallelograms are adopted to identify the CM of the spatial manipulator, and then the CM is mounted on an S joint. The method is similar to the principal vectors proposed by Fischer [125] but is easier to achieve. Since the CMs of the manipulators are
fixed, the mechanisms are statically balanced. A two-link manipulator is shown in Fig. 6.28(a), in which the R joints of the auxiliary parallelogram are parallel with the second R joint of the manipulator. The method applies to manipulators with arbitrary links, as long as the links and the identical auxiliary links that are used as counterweights are arranged symmetrically about the centre of the parallelogram. The two additional links in the middle are used to define the CM of the augmented manipulator.

In the 3-link manipulator [Fig. 6.28(b)], the CM of the first two links are identified using the method above. Then an additional auxiliary parallelogram is used to connect the CM of the two-link manipulator and the CM of the third link using two S joints. Suppose the weight of the two-link manipulator is \( n \) times of the weight of the third link, the position of the intersection of the two links in the middle to define the CM is set at \( n \) out of \( (n+1) \) point.

Fig. 6.28 Statically balanced manipulators using auxiliary parallelograms: (a) 2-link manipulator; (b) 3-link manipulator; (c) 4-link manipulator; (d) Bricard linkage; (e) 8R linkage

Based on the 2-link manipulator and the 3-link manipulator modular, countless statically balanced mechanisms can be constructed. For instance, when connecting two 2-link manipulators, and adding an additional parallelogram to connect the two CMs of the two 2-link manipulators, a statically balanced 4-DOF manipulator is then obtained.
[Fig. 6.28(c)]. When adopting the additional parallelogram to connect two balanced 3-link manipulators, a statically balanced Bricard linkage is constructed [Fig. 6.28(d)]. Similarly, a statically balanced 8R linkage is designed, as shown in Fig. 6.28(e). It is noted the mechanisms using the auxiliary parallelogram may have the limited range of motions due to the interference of the parallelograms.

c) Statically Balanced 8R Linkage Using Springs and Auxiliary Links I

This section uses springs and auxiliary links to balance the manipulators. The 2-link manipulator with identical links and perpendicular joint axes is taken as an example to illustrate the approach. To let the local position of the CM of the manipulator be fixed, the auxiliary links with identical parameters of the links of the manipulator are adopted as counterweights, as shown in Fig. 6.29(a). As a result, the CM of the manipulator is located at the second R joint of the manipulator \( P \). The manipulator is then equivalent to a payload mounted on an R joint. One end of the spring is attached on \( H \), which is a point right above the first R joint, the other end is fixed on \( P \). It can be readily proved the manipulator is statically balanced.

Suppose the mass of the link of the manipulator is \( m \), the point of attachment of the springs on the base is

\[
H = \begin{pmatrix} 0 & 0 & 4mg/k \end{pmatrix}^T
\]  \hspace{1cm} (6.108)

The position of the CM of the manipulator \( P \) and the potential energy are yielded as

\[
P = \begin{pmatrix} lC\theta & 0 & lS\theta \end{pmatrix}^T
\]  \hspace{1cm} (6.109)

\[
V = V_s + V_m = \frac{1}{2}k|\begin{vmatrix} \begin{pmatrix} P - H \end{pmatrix} \end{vmatrix}^2 + mgP_z = \left( t^2k^2 + 16m^2g^2 \right)/2k
\]  \hspace{1cm} (6.110)

Eq. (6.109) verifies the system is statically balanced. By connecting the 2-link manipulators above, deployable single-loop linkages are obtained. The R joints used to connect the 2-link manipulators all intersect at a point. As described in Section 4, the DOF of the constrained linkages are all one during the deploying process, if \( R_i \) \( (i = 2, 4, 6...) \) are constrained on the same plane.
Fig. 6.29 Statically balanced mechanisms using auxiliary links and springs: (a) 2-DOF manipulator; (b) Bricard linkage; (c) extension of Bricard linkage

The static balancing of Bricard linkage is then discussed. The position vectors of $R_i$ $(i = 2, 4, 6)$ are calculated as

\[ P_2 = \{lC\theta \ 0 \ lS\theta\}^T \quad (6.111a) \]
\[ P_4 = \{l(C\theta - 1) \ l(C\theta + 1) \ \frac{2C\theta + 1}{(C\theta + 1)^2} \ lS\theta\}^T \quad (6.111b) \]
\[ P_6 = \{-l \ 0 \ 0\}^T \quad (6.111c) \]

Links 1 and 2 (of the first 2-link manipulator) are already balanced. The potential energy is calculated as

\[ V_2 = V_{s2} + V_{m2} = \frac{1}{2}k|P_2 - H|^2 + mgP_{2z} = (l^2k^2 + 16m^2g^2)/2k \quad (6.112) \]

Links 5 and 6 (of the third 2-link manipulator) always lie on the ground and do not need to balance. Now it will be verified that links 3 and 4 (of the second manipulator) are balanced, by connecting $H$ and $P_4$ using one spring. The potential energy is computed as

\[ V_4 = V_{s4} + V_{m4} = \frac{1}{2}k|P_4 - H|^2 + mgP_{4z} = (3l^2k^2 + 16m^2g^2)/2k \quad (6.113) \]

Eq. (6.113) infers that the second 2-link manipulator is also balanced. Eqs. (6.112-6.113) imply that the static balancing method is valid by connecting $R_2$ ($R_4$), and the point right above $R_1$. The extension of the Bricard linkage in Fig. 6.29(c) is also balanced.
Fig. 6.30 Statically balanced 8R linkage and 12R linkage using auxiliary links and springs: (a) 8R linkage; (b) 12R linkage

Now the static balancing of 8R linkage will be addressed. When connecting four statically balanced 2-link manipulators, a balanced 8R linkage is obtained, as shown in Fig. 6.30(a). Three springs are used to connect $H$ and $P_2$, $P_4$ and $P_6$ respectively. The position vectors of $R_i$ ($i = 2, 4, 6, 8$) in the global frame were obtained in Section 4 as

\[
\begin{align*}
P_2 &= \begin{bmatrix} l \cos \theta & 0 & l \sin \theta \end{bmatrix}^T \\
P_4 &= \begin{bmatrix} l(2C\theta - 1) & 2l(C\theta + 1)\sqrt{(C\theta/(C\theta + 1))^2} & 2l \sin \theta \end{bmatrix}^T \\
P_6 &= \begin{bmatrix} l(C\theta - 2) & 2l(C\theta + 1)\sqrt{(C\theta/(C\theta + 1))^2} & l \sin \theta \end{bmatrix}^T \\
P_8 &= \begin{bmatrix} -l & 0 & 0 \end{bmatrix}^T
\end{align*}
\]

The potential energies are given by

\[
\begin{align*}
V_2 &= V_{s2} + V_{m2} = \frac{1}{2} k \|P_2 - H\|^2 + mgP_{2z} = (l^2 k^2 + 16m^2 g^2)/2k \\
V_4 &= V_{s4} + V_{m4} = \frac{1}{2} k \|P_4 - H\|^2 + mgP_{4z} = (5l^2 k^2 + 16m^2 g^2)/2k \\
V_6 &= V_{s6} + V_{m6} = \frac{1}{2} k \|P_6 - H\|^2 + mgP_{6z} = (5l^2 k^2 + 16m^2 g^2)/2k \\
V_8 &= V_8 = (13l^2 k^2 + 16m^2 g^2)/2k
\end{align*}
\]

Eqs. (6.115-6.117) imply that the 8R linkage is also balanced using the static balancing method. It is noted that the method is only valid when the 8R linkage is in the constrained configuration. Similarly, all the deployable single-loop linkages in the constrained configuration, such as the 12R linkage in Fig. 6.30(b), can be balanced. In a 12R linkage,

\[
\varphi = \arccos[(2 - C\theta)/(C\theta + 1)]
\]

The potential energies of the system are yielded as

\[
\begin{align*}
V_2 &= (l^2 k^2 + 16m^2 g^2)/2k \\
V_4 &= V_{10} = (7l^2 k^2 + 16m^2 g^2)/2k \\
V_6 &= V_8 = (13l^2 k^2 + 16m^2 g^2)/2k
\end{align*}
\]
which are also constants.

\textit{d) Statically Balanced 8R Linkage Using Springs and Auxiliary Links II}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{static_balanced_linkage.png}
\caption{Statically balanced manipulators using auxiliary parallelograms and springs: (a) 2-DOF manipulator; (b) 3-DOF manipulator; (c) 4-DOF manipulator; (d) Bricard linkage; (e) 8R linkage}
\end{figure}

We connect the manipulator with auxiliary links in b) to a link, which mounted on an S joint, using another S joint. The manipulator can be balanced by connecting the point right above the S joint on the base and the CM of the manipulator using one spring. The statically balanced 2-link, 3-link and 4-link manipulators, Bricard linkage and 8R linkage are provided in Fig. 6.31.

\textbf{6.6.2 Static Balancing of DPMs}

The DPMs proposed in Chapters 4 and 5 will be balanced in this section.

\textit{a) Statically Balanced DPMs using a Special 1-DOF Mechanism}
DPMs can be developed into the statically balanced mechanisms using a special 1-DOF mechanism. The first mechanism discussed is the prism mechanism based on Bricard linkages using S joints. Since the mechanism is always plane-symmetric about the mirror plane and axisymmetric about the line defined by the intersection of $R_2$, $R_4$ and $R_6$ and that of $R_2'$, $R_4'$ and $R_6'$, the CM of the mechanism is always at the centre of the triangle defined by the three interval S joints. A 1-DOF triangle mechanism composed of prismatic joints is attached to the S joints to identify the CM of the mechanism, as shown in Fig. 6.32(a). When mounting the CM of the mechanism on the base through an S joint, the mechanism becomes statically balanced, and can keep its postures during the deploying process. Similarly, the prism based on Bricard linkages using RRR chains can also be balanced. For the deployable prism mechanism based on 8R linkages using S joints or half the number of RRR chains, a parallelogram is added, instead of the triangle mechanism. The CM of the DPM and that of the parallelogram are coincident. Mounting the CM of the mechanism on an S joint, then the statically balanced deployable
mechanism based on 8R linkages with multiple modes is obtained. It is noted that the method is not valid when the DPMs are in the rotation modes, since the mechanisms in these modes are not symmetric about the mirror plane.

*b*) **Statically Balanced DPMs using springs**

In this section, only springs will be used to balance the DPMs proposed in Chapters 4 and 5. Since the prism mechanisms are always symmetric about the mirror plane, the mechanisms are cut in half to facilitate balancing. We start from the DPM based on 8R linkages using S joints, as shown in Fig. 6.33(a). All the S joints keep on the plane (the plane is fixed on the ground) and have 3-DOF rotations. Hence, each link can be balanced by connecting the CM of the link and the point right above the S joint using one spring, as shown in Fig. 6.33(b).

Then the statically balanced DPM based on 8R linkages using RRR chains [Fig. 6.34(a)] will be addressed. When cutting in half, the DPM is equivalent to the linkage composed of eight 2-link 2R manipulators in Fig. 6.34(b). The first link in each manipulator has a 1-DOF rotation on the plane and can be balanced by connecting the CM of the first link and the point right above the first R joint using one spring. As obtained in Section 6.4, the second link of the manipulator can be balanced using two springs which connect the CM of the second link and the point right above the second R joint, and the second R joint and the point right above the first R joint respectively.

![Fig. 6.33 Statically balanced DPM based on 8R linkages using S joints](image)
When connecting two 8R linkages using four RRR chains, the linkage constrained on the mirror plane is equivalent to the linkage comprised of four 3-link 3R manipulators [Fig. 6.35(b)]. The first link can be balanced by connecting the CM of the first link and the point right above the first R joint using one spring. The second link of the manipulator can be balanced using two springs which connect the CM of the second link and the point right above the second R joint, and the second R joint and the point right above the first R joint respectively. When connecting the CM of the third link and the point right above the third R joint, the third R joint and the point right above the second R joint, and the second R joint and the point right above the first R joint respectively, the third link is then statically balanced.
This method applies to any prism mechanisms in any modes, except for the rotation mode of the mechanisms with half the number of the RRR chains.

6.7 Summary

This chapter is to develop the deployable mechanisms proposed in Chapters 3, 4 and 5 into the statically balanced mechanisms. First, the static balancing method of the planar 4R parallelogram, planar manipulators, spherical mechanisms and spatial mechanisms have been proposed, with almost no calculation. In addition, a numerical optimization approach has also been addressed to obtain the balancing conditions of the mechanisms. The sum of squared differences of the potential energies of the system is set as the objective function to obtain the spring attachment points. Then the proposed static balancing methods have been applied to the deployable mechanisms, including the ones with multiple modes. Statically balanced DPMs using only springs, auxiliary links and the combination of auxiliary links and springs have also been designed.
CHAPTER 7 – CONCLUSIONS

7.1 General Conclusions

This thesis has addressed the type synthesis and static balancing of a class of deployable mechanisms with multiple modes. First, the existing single-loop linkages, the deployable/foldable mechanisms and mechanisms with multiple modes have been reviewed. The statically balanced mechanisms using springs, counterweights and other approaches have also been presented.

Chapter 2 has discussed the theoretical tools used in this thesis, including DOF analysis, kinematic analysis and optimization tools, and the fundamentals of static balancing.

In Chapter 3, the construction method for foldable single-loop 8R linkages with multiple modes has been proposed. Planar 4R linkages and spherical 4R linkages were used to build spatial 8R linkages. The links are thick panels and the joints are offset to allow folding. There are two types of 8R linkages involved, including the one constructed using two spherical 4R linkages and the one connected using one planar 4R linkage and one spherical 4R linkage. The mechanisms have multiple modes, including an 8R linkage mode, a 6R linkage mode and 4R linkage modes. Besides, the mechanisms can be spread onto a plane and be folded into two or four layers.

Using a spatial triad, single-loop linkages have been connected in Chapters 4 and 5. Then the single-loop linkages were inserted to faces of polyhedrons to obtain DPMs. Two types of DPMs were designed, including the ones constructed using S joints and the ones using RRR chains. The DPMs have only 1-DOF when deployed. Several mechanisms have multiple modes and can switch among different modes through transition positions. Prototypes have been fabricated, using rigid or flexible joints. The ones with rigid joints have precise movement and the ones with compliant joints have a larger deploying ratio and can recover to the initial state after deploying.

In Chapter 6, static balancing methods for planar mechanisms, spherical mechanisms and spatial mechanisms have been addressed. All the mechanisms in this thesis are composed of links whose weights cannot be disregarded. Both an algebraic method and a geometric method have been adopted to design statically balanced 4R parallelogram. The 4R parallelogram can be balanced using three, two or one springs. The spherical manipulators and spatial manipulators are balanced using a geometric method readily, with almost no calculation. A novel numerical optimization method has also been
proposed, by setting the sum of squared differences of potential energies as the objective function. Using the proposed static balancing method, the deployable 8R linkage and DPMs have been developed into the ones that are statically balanced.

7.2 Main Contributions

The main contributions of this thesis are:

1) A novel construction method for deployable mechanisms using single-loop linkages has been presented. DPMs with multiple modes are obtained using this method. The mechanisms have simple structures and few DOFs. It has been revealed that by connecting single-loop linkages using half the number of the RRR chains, the mechanisms obtain an additional rotation mode.

2) A novel construction method for foldable single-loop 8R linkages with multiple modes is proposed. Instead of connecting consecutively as in the literature, the compositional 4R linkages are staggered to obtain the 8R linkages.

3) The general method for the static balancing of the spherical and spatial manipulators has been proposed. Using this method, any manipulators, including the ones with multiple modes, can be readily balanced with almost no calculation.

4) The 4R parallelogram linkage has been balanced using only one external spring. The concept of virtual rotation centre is first introduced in the design of the statically balanced system using a geometric method.

5) A novel numerical optimization method is proposed, which is suitable for the statically balanced systems using springs. More possible solutions can be found using this method, than the current methods in the literature.

7.3 Future Work

In the future, the work can be developed in the following aspects. First, deployable mechanisms with multiple modes have been designed in this thesis. However, there is still a lack of detailed DOF analysis and kinematic analysis for these mechanisms. In the future, the set of equations for each mechanism will be derived and then solved to obtain all the possible motion modes of the mechanisms. The application of these mechanisms also needs explorations. They may apply in the fields of entertainment, education and aerospace. Prototypes with better quality will be built and a control method will be used to actuate the mechanisms.
Secondly, the static balancing conditions for planar, spherical and spatial manipulators have been obtained using a geometric method, with almost no calculation. However, the number of springs can be reduced through mass moment substitution and vector synthesis. The interference between springs and links should be avoided.

Thirdly, the statically balanced system designed in this thesis all have constant payloads. In the cases that the payloads of the manipulators change, for example, picking up or dropping off objects, the system should be adjusted. Therefore, the devices with variable payloads deserve further explorations. In addition, springs and counterweights will be combined to balance the mechanisms. Constant force springs, gears and cams may also be adopted.

Finally, the static balancing methods have only been theoretically verified by calculating the total potential energy of the system. It would be better to design and fabricate practical prototypes to verify the feasibility and sensibility of the statically balanced mechanisms. Rollers and pulleys will be used to achieve the zero-free-length springs.
APPENDIX THE TRANSFER MATRIXES OF THE MECHANISMS

A The Transfer Matrixes of the Bricard Linkage.

The transfer matrixes of the Bricard linkage are

\[
\begin{align*}
^6T = ^3T = ^4T &= \begin{bmatrix}
C\theta & -S\theta & 0 & l \\
0 & 0 & 1 & 0 \\
-S\theta & -C\theta & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \\
^1T = ^3T = ^5T &= \begin{bmatrix}
-C\theta/(C\theta + 1) & -\sqrt{1 - C^2\theta/(C\theta + 1)^2} & 0 & l \\
0 & 0 & -1 & 0 \\
\sqrt{1 - C^2\theta/(C\theta + 1)^2} & -C\theta/(C\theta + 1) & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\end{align*}
\] (A1)

B The Transfer Matrixes of the 8R Linkage

The transfer matrixes of the 8R linkage are

\[
\begin{align*}
^8T = ^3T = ^4T = ^9T &= \begin{bmatrix}
C\phi & -S\phi & 0 & l \\
0 & 0 & -1 & 0 \\
S\phi & C\phi & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \\
^1T = ^5T &= \begin{bmatrix}
C\phi_2 & -S\phi_2 & 0 & l \\
0 & 0 & -1 & 0 \\
S\phi_2 & C\phi_2 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \\
^3T = ^7T &= \begin{bmatrix}
C\phi_2 & -S\phi_2 & 0 & l \\
0 & 0 & -1 & 0 \\
S\phi_2 & C\phi_2 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\end{align*}
\] (B1, B2, B3)

C The Transfer Matrixes of the Spherical Manipulator

The transfer matrixes of the spherical manipulator are

\[
^0T = \begin{bmatrix}
C\theta_1 & -S\theta_1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-S\theta_1 & -C\theta_1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\] (C1)
D The Transfer Matrixes of the Spatial Manipulator

The transfer matrixes of the spatial manipulator are

\[ 1^T = \begin{bmatrix} C\theta_2 & -S\theta_2 & 0 & 0 \\ C\alpha_1 S\theta_2 & C\alpha_1 C\theta_2 & S\alpha_1 & 0 \\ -S\alpha_1 S\theta_2 & -S\alpha_1 C\theta_2 & C\alpha_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (C2) \]

\[ 2^T = \begin{bmatrix} C\theta_3 & -S\theta_3 & 0 & 0 \\ C\alpha_2 S\theta_3 & C\alpha_2 C\theta_3 & S\alpha_2 & 0 \\ -S\alpha_2 S\theta_3 & -S\alpha_2 C\theta_3 & C\alpha_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (C3) \]

\[ 3^T = \begin{bmatrix} C\theta_4 & -S\theta_4 & 0 & 0 \\ C\alpha_3 S\theta_4 & C\alpha_3 C\theta_4 & S\alpha_3 & 0 \\ -S\alpha_3 S\theta_4 & -S\alpha_3 C\theta_4 & C\alpha_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (C4) \]
REFERENCES


