Appendix-C

Deriving the Fracture Equation with the Source Term

(Mass-in / Time) – (Mass-out / Time) = (Mass-Accumulation / Time)

For slightly compressible fluid: (ρ) can be handled as in section **Error! Reference source not found.** and for simplicity will be ignored here, multiply by Δt as previously:

Equation 1

$$\left(\mathrm{q_x} - \mathrm{q_{x+\Delta x}}\right) \cdot (\Delta \mathrm{t}) + (u_1 + u_2) \cdot (h \cdot \Delta x) \cdot \Delta t + \frac{\mathrm{q\beta}}{h} \cdot [\mathrm{H}(x_0 - \mathrm{a}) - \mathrm{H}(x_0 - (\mathrm{a} + \Delta \mathrm{x}))] \cdot (\Delta \mathrm{t}) = q_{\Delta x} \cdot \Delta t \quad (1)$$

Introduce the compressibility equation to MBE:

Where: $C = -\frac{1}{v}\frac{\Delta v}{\Delta p} \rightarrow \Delta v = cv\,\Delta p$

$$\left(\mathrm{q_x} - \mathrm{q_{x+\Delta x}}\right) \cdot (\Delta \mathrm{t}) + (u_1 + u_2) \cdot (h \cdot \Delta x) \cdot \Delta t + \frac{\mathrm{q\beta}}{h} \cdot [\mathrm{H}(x_0 - \mathrm{a}) - \mathrm{H}(x_0 - (\mathrm{a} + \Delta \mathrm{x}))] \cdot (\Delta \mathrm{t}) = c \cdot \left(w_f \cdot h \cdot \Delta x \cdot \varphi\right) \cdot (p_{t+\Delta t} - p_t)$$

Divide by $(\Delta x \cdot \Delta t)$ and take limit as $\Delta x \rightarrow 0,\ \Delta x_0 \rightarrow 0\ \&\ \Delta t \rightarrow 0$:

$$-\lim_{\Delta x \to 0}\left(\frac{q_{x+\Delta x} - q_x}{\Delta x}\right) + (u_1 + u_2) \cdot (h) + \frac{\mathrm{q\beta}}{h} \cdot \lim_{\Delta x \to 0}\left[\frac{\mathrm{H}(x_0 - \mathrm{a}) - \mathrm{H}(x_0 - (\mathrm{a} + \Delta \mathrm{x}))}{\Delta \mathrm{x}}\right] = c \cdot \left(w_f \cdot h \cdot \varphi\right) \lim_{\Delta t \to 0}\left(\frac{p_{t+\Delta t} - p_t}{\Delta t}\right)$$

Then:

Equation 2

$$-\frac{\partial q}{\partial x} + (u_1 + u_2) \cdot (h) + \frac{\mathrm{q\beta}}{h} \cdot \frac{\partial \mathrm{H}(x_0 - \mathrm{a})}{\partial x} = c \cdot \left(w_f \cdot h \cdot \varphi\right) \frac{\partial p}{\partial t} \quad (2)$$

Introduce Darcy's law:

$$q = -\frac{k_f A}{\mu} \cdot \frac{dp}{dx},\ along\ fracture$$

Differentiate both sides w.r.t. (x):

$$\frac{d}{dx} \cdot q = -\frac{k_f A}{\mu} \cdot \left(\frac{d}{dx} \cdot \frac{dp}{dx}\right)$$

Equation 3

$$\frac{dq}{dx} = -\frac{k_f \cdot w_f \cdot h}{\mu} \cdot \left(\frac{d^2 p}{dx^2}\right) \quad (3)$$

Substitute Equation 3 in Equation 2 :

$$\frac{k_f A}{\mu} \cdot \left(\frac{\partial^2 p}{\partial x^2}\right) + (u_1 + u_2) \cdot \mathrm{h} + \mathrm{q\beta} \cdot \frac{\partial \mathrm{H}(x_0 - \mathrm{a})}{\partial x} = c \cdot \left(w_f \cdot h \cdot \varphi\right) \frac{\partial p}{\partial t}$$

Since, $A = w_f \cdot h$, then:

$$\frac{k_f \cdot w_f \cdot h}{\mu} \cdot \left(\frac{\partial^2 p}{\partial x^2}\right) + (u_1 + u_2) \cdot \mathrm{h} + \mathrm{q}\beta \cdot \frac{\partial \mathrm{H}(x_0 - \mathrm{a})}{\partial x} = c \cdot \left(w_f \cdot \mathrm{h} \cdot \varphi\right) \frac{\partial p}{\partial t}$$

Re-arrange:

Two dimensional diffusivity equations for fluid flowing in fracture plane per unit area

$$\frac{\partial^2 p}{\partial x^2} + (u_1 + u_2) \cdot \frac{\mu}{w_f \cdot k_f} + \frac{\mathrm{q}\beta\mu}{w_f \cdot k_f \cdot h} \cdot \frac{\partial \mathrm{H}(x_0 - \mathrm{a})}{\partial x} = \frac{c\mu\varphi}{k_f} \cdot \left(\frac{\partial p}{\partial t}\right)$$

Or:

Equation 4

$$\frac{\partial^2 p}{\partial x^2} + (u_1 + u_2) \cdot \frac{\mu}{w_f \cdot k_f} + \frac{\mathrm{q}\beta\mu}{w_f \cdot k_f \cdot h} \cdot \frac{\partial \mathrm{H}(x_0 - \mathrm{a})}{\partial x} = {}^1\!/\eta_f \cdot \left(\frac{\partial p}{\partial t}\right) \tag{4}$$

Solving for $(\mathrm{u}_1 + \mathrm{u}_2)$ from Darcy's equation, yields:

$$\mathrm{u} = \frac{\mathrm{q}}{\mathrm{A}} = -\frac{\mathrm{k}}{\mu} \cdot \left.\frac{\mathrm{dp}}{\mathrm{dy}}\right|_{\mathrm{y} \to 0}$$

Substitute into Equation 4 and re-arrange will give the diffusivity equation for finite conductivity fracture:

Equation 5

$$\frac{\partial^2 p_f}{\partial x^2} + \frac{1}{k_f \cdot w_f}\left[k_2 \left.\frac{\partial p_2}{\partial y}\right|_{y=0} - k_1 \left.\frac{\partial p_1}{\partial y}\right|_{y=0}\right] + \frac{\mathrm{q}\beta\mu}{w_f \cdot k_f \cdot h} \cdot \frac{\partial \mathrm{H}(x_0 - \mathrm{a})}{\partial x} = 1/\eta_f \cdot \left(\frac{\partial p_f}{\partial t}\right) \tag{5}$$

Where: ${}^1\!/\eta_f = C \cdot \frac{c_t \cdot \varphi \cdot \mu}{k_f}, C = 0.000264$

Convert Variables to Dimensionless Form

Use $\left(\frac{\mathbf{141.2\ q\beta\mu}}{\boldsymbol{kh}} \cdot 2\pi\right)$ as a constant for the well in field units.

Substituting the dimensional by the dimensionless variables will produce the dimensionless Fracture solution with a source term:

Equation 6

$$\frac{\partial^2 p_{Df}}{\partial {x_D}^2} + \frac{1}{F_{CDf}}\left[(k_{D2}) \cdot \left.\frac{\partial p_{D2}}{\partial y_D}\right|_{y_D=0} - (k_{D1}) \cdot \left.\frac{\partial p_{D1}}{\partial y_D}\right|_{y_D=0}\right] + \frac{2\pi}{F_{CDf}} \cdot \delta(x_D - a) = (1/\eta_{Df}) \cdot \frac{\partial p_{Df}}{\partial t_{Df}} \tag{6}$$

Where well is at origin $(a = 0)$

$F_{CDf} = \frac{k_f\, w_f}{k_{rf}\ r_w}$, $k_{rf} = \frac{k_1 + k_2}{2}$, $k_{D3} = \frac{k_1}{k_r}$, $k_{D2} = \frac{k_2}{k_r}$, $k_r = 1.0$

$$\eta_{Df} =\cdot \left(\frac{(\varphi \ c_t \ \mu)_f}{k_f} \cdot \frac{0.000264 \cdot k_{rf}}{(\varphi \ c_t \ \mu)_{rf}}\right) = 0.000264 \cdot \left(\frac{\eta_{rf}}{\eta_f}\right)$$

$$1/\eta_{Df} = \frac{1}{0.000264 \cdot \left(\frac{\eta_{rf}}{\eta_f}\right)}$$

Initial and Boundary Conditions

Initial Boundary condition

$p_{D1} = p_{D2} = p_{Df} = 0 \qquad @ \ t_{Df} = 0$

Boundary condition

$p_{D1} = p_{D2} = p_{Df} \qquad @ \ y_D = 0$

$0 < \mathrm{y} < \infty$

$-\infty < \mathrm{y} < 0$

$-\infty < \mathrm{x} < \infty$