Appendix-A

Two-dimensional Partial Differential Equation for slightly compressible fluid

The Diffusivity equation is developed in 1D and 2D using both Cartesian and Radial planes. Combining the law of conservation of mass (in any plane), the basic material balance relation and Darcy's law, will yield the diffusion equation(s) in homogenous and isotropic system.

Mass Balance for Linear Systems (Cartesian Coordinates)

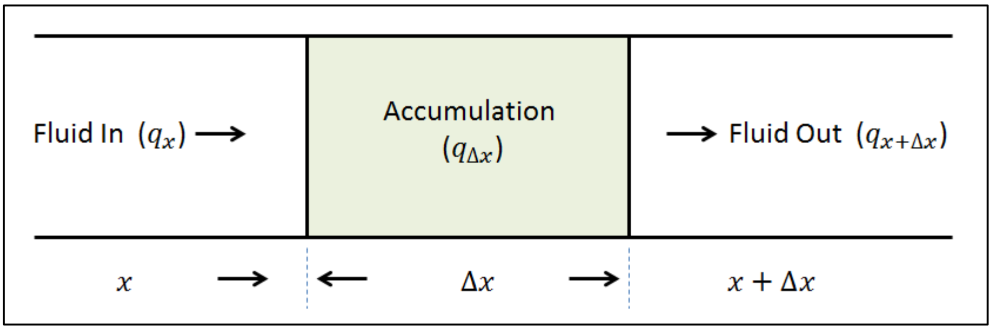

Figure 1: Schematic of the mass flow in a linear system.

While liquid compressibility effects can safely be neglected in most fluid flow situations, however, in oil reservoirs, where large volumes at high pressures are encountered, it is important to account for compressibility effect in the following manner (Houze et al., 2008).

The author has considered the flow of fluid in x-direction through a small area *A* between (x) and $(x + \Delta x)$ and between time (t) and $(t + \Delta t)$, Figure 1, using the following equation:

(Mass-in / Time) – (Mass-out / Time) = (Mass-Accumulation / Time)

Multiply by Δt

Equation 1

$$[(q_x \cdot \rho_x) + (q_y \cdot \rho_y)] \cdot \Delta t - [(q_{x+\Delta x} \cdot \rho_{x+\Delta x}) + (q_{y+\Delta y} \cdot \rho_{y+\Delta y})] \cdot \Delta t = (q_{\Delta x} + q_{\Delta y}) \cdot \rho_{\Delta xy} \cdot \Delta t \qquad (1)$$

One-dimensional Diffusivity Equation

In order to study the liquid compressibility effects, the author has considered the flow of fluid in x-axis only for now, the y-axis will be added later. In this system the fluid flow through a small area *A* between (x) and $(x + \Delta x)$ and between time (t) and $(t + \Delta t)$, hence, the following equation:

(Mass-in / Time) – (Mass-out / Time) = (Mass-Accumulation / Time)

$$(q \cdot \rho)_x - (q \cdot \rho)_{x+\Delta x} = (q \cdot \rho)_{\Delta x}$$

$$-(q_{x+\Delta x} \cdot \rho - q_x \cdot \rho) = (q \cdot \rho)_{\Delta x}$$

The occurrence of pressure transients takes place because the reservoir fluid is slightly compressible and local accumulation or depletion of fluid in the reservoir occurs then:

Multiply by Δt

$$-(q_{x+\Delta x} \cdot \rho - q_x \cdot \rho) \cdot \Delta t = (q \cdot \rho)_{\Delta x} \cdot \Delta t$$

$$-(q_{x+\Delta x} \cdot \rho - q_x \cdot \rho) \cdot \Delta t = \Delta v \cdot \rho_{\Delta x}$$

Where, $\Delta v = A \cdot \Delta x \cdot \varphi$

$$-(q_{x+\Delta x} \cdot \rho - q_x \cdot \rho) \cdot \Delta t = (A \cdot \Delta x) \cdot [(\varphi\rho)_{t+\Delta t} - (\varphi\rho)_t]$$

Divide by $(\Delta x \cdot \Delta t)$ and take limit as $\Delta x \to 0$ & $\Delta t \to 0$:

$$-\lim_{\Delta x \to 0}\left(\frac{q_{x+\Delta x} \cdot \rho - q_x \cdot \rho}{\Delta x}\right) = A \cdot \lim_{\Delta t \to 0}\left(\frac{\varphi\rho}{\Delta t}\right)$$

Equation 2

$$-\frac{\partial q\rho}{\partial x} = A \cdot \frac{\partial \varphi\rho}{\partial t} \qquad (2)$$

Left-hand side term

$\left(-\frac{\partial q\rho}{\partial x}\right)$:

Introduce Darcy's law:

$$q = -\frac{kA}{\mu} \cdot \frac{dp}{dx}$$

Differentiate (w.r.t.) x:

$$\frac{d}{dx} \cdot q = -kA \cdot \frac{d}{dx}\left(\frac{1}{\mu} \cdot \frac{dp}{dx}\right)$$

Then:

$$-\frac{dq\rho}{dx} = kA \cdot \frac{d}{dx}\left(\frac{\rho}{\mu} \cdot \frac{dp}{dx}\right)$$

For: $\frac{d}{dx}\left(\frac{\rho}{\mu} \cdot \frac{dp}{dx}\right) = \frac{\rho}{\mu}\left(\frac{\partial^2 \mathrm{p}}{\partial \mathrm{x}^2}\right) + \frac{1}{\mu}\left(\frac{d\rho}{dx} \cdot \frac{dp}{dx}\right)$

Since: $C_f = -\frac{1}{\rho}\frac{d\rho}{dp} \Rightarrow \frac{d\rho}{dx} = C_f . \rho \left(\frac{dp}{dx}\right)$

Then: $\frac{d}{dx}\left(\frac{\rho}{\mu} \cdot \frac{dp}{dx}\right) = \frac{\rho}{\mu}\left(\frac{\partial^2 \mathrm{p}}{\partial \mathrm{x}^2}\right) + \frac{C_f \cdot \rho}{\mu}\left(\frac{dp}{dx} \cdot \frac{dp}{dx}\right) = \frac{\rho}{\mu}\left(\frac{\partial^2 \mathrm{p}}{\partial \mathrm{x}^2}\right) + \frac{C_f \cdot \rho}{\mu}\left(\frac{dp}{dx}\right)^2$

$$\Rightarrow -\frac{\mathrm{dq\rho}}{\mathrm{dx}} = \mathrm{kA}\left[\frac{\rho}{\mu}\left(\frac{\mathrm{d}^2\mathrm{p}}{\mathrm{dx}^2}\right) + \frac{C_f \cdot \rho}{\mu}\left(\frac{dp}{dx}\right)^2\right]$$

The assumption is now made that the pressure gradient $\left(\frac{dp}{dx}\right)$ with respect to distance is relatively small and hence $\left(\frac{dp}{dx}\right)^2$ can safely be neglected, where, the fluid is slightly compressible. Therefore, it can be considered that:

Equation 3

$$-\frac{dq\rho}{dx} = kA\frac{\rho}{\mu}\left(\frac{d^2p}{dx^2}\right) \tag{3}$$

Right-hand side term

$A \cdot (\frac{\partial \varphi \rho}{\partial t})$:

Use chain rule and differentiate:

$$\frac{\partial \varphi \rho}{\partial t} = \frac{\partial \varphi \rho}{\partial p} \cdot \frac{\partial p}{\partial t} = \left(\varphi \frac{\partial \rho}{\partial p} + \rho \frac{\partial \varphi}{\partial p}\right) \cdot \left(\frac{\partial p}{\partial t}\right)$$

$$\frac{\partial \varphi \rho}{\partial t} = \varphi \rho \left(\frac{1}{\rho} \cdot \frac{\partial \rho}{\partial p} + \frac{1}{\varphi} \cdot \frac{\partial \varphi}{\partial p}\right) \cdot \left(\frac{\partial p}{\partial t}\right) = \varphi \rho \left(c_f + c_r\right) \cdot \left(\frac{\partial p}{\partial t}\right) = \varphi \rho C_t \cdot \left(\frac{\partial p}{\partial t}\right)$$

In the case of single phase liquids the thermodynamic equation of state is adequately represented by the model of a fluid of constant compressibility. The definition of the system compressibility C_t is:

$C_f = -\frac{1}{\rho}\frac{d\rho}{dp}$, fluid compressibility

$C_r = -\frac{1}{\varphi}\frac{d\varphi}{dp}$, rock compressibility

And $C_t = C_f + C_r$

Final equation:

Equation 4

$$A \cdot \frac{d\varphi\rho}{dt} = A \cdot \varphi\rho C_t \cdot \left(\frac{dp}{dt}\right) \quad (4)$$

Substitute Equation 3 and Equation 4 in Equation 2, the one dimensional diffusivity equation (in Cartesian coordinates) is obtained as:

$$\frac{\partial^2 p}{\partial x^2} = \frac{c_t\mu\varphi}{k} \cdot \left(\frac{\partial p}{\partial t}\right)$$

Or

$$\frac{\partial^2 \mathrm{p}}{\partial \mathrm{x}^2} = 1/\eta \cdot \left(\frac{\partial \mathrm{p}}{\partial \mathrm{t}}\right)$$

Two-dimensional Diffusivity Equation

Now the author extends this to include the y-direction parameters as well (based on the above, the fluid compressibility for slightly compressible fluid, can be ignored). Therefore, Equation 1, becomes:

$$[q_x + q_y] \cdot \Delta t - [q_{x+\Delta x} + q_{y+\Delta y}] \cdot \Delta t = (q_{\Delta x} + q_{\Delta y}) \cdot \Delta t$$

Or

$$[q_x - q_{x+\Delta x}] \cdot \Delta t - [q_y - q_{y+\Delta y}] \cdot \Delta t = (q_{\Delta x} + q_{\Delta y}) \cdot \Delta t$$

The compressibility equation: $C = -\frac{1}{v}\frac{\Delta v}{\Delta p} \rightarrow \Delta v = cv\,\Delta p$

$$[q_x - q_{x+\Delta x}] \cdot \Delta t - [q_y - q_{y+\Delta y}] \cdot \Delta t = (q_{\Delta x} + q_{\Delta y}) \cdot \Delta t = c \cdot (A \cdot \Delta x \cdot \varphi) \cdot (p_{t+\Delta t} - p_t) + c \cdot (A \cdot \Delta y \cdot \varphi) \cdot (p_{t+\Delta t} - p_t)$$

$$[q_x - q_{x+\Delta x}] \cdot \Delta t - [q_y - q_{y+\Delta y}] \cdot \Delta t = (c \cdot A \cdot \varphi) \cdot (\Delta x + \Delta y) \cdot (p_{t+\Delta t} - p_t)$$

Divide by$(\Delta x + \Delta y) \cdot \Delta t$ and take limit as $\Delta x \to 0,\ \Delta y \to 0\ \&\ \Delta t \to 0$:

$$-\lim_{\Delta x \to 0}\left(\frac{q_{x+\Delta x} - q_x}{\Delta x}\right) - \lim_{\Delta x \to 0}\left(\frac{q_{y+\Delta y} - q_y}{\Delta y}\right) = \lim_{\Delta t \to 0}\left(\frac{p_{t+\Delta t} - p_t}{\Delta t}\right)(c \cdot A \cdot \varphi)$$

Equation 5

$$-\frac{dq}{dx}-\frac{dq}{dy}=(c\cdot A\cdot\varphi)\frac{dp_1}{dt} \quad (5)$$

Introduce Darcy's law:

$$q_x=-\frac{kA}{\mu}\cdot\frac{dp}{dx}$$

$$q_y=-\frac{kA}{\mu}\cdot\frac{dp}{dy}$$

Differentiate both sides w.r.t. $x\ and\ y$:

$$\frac{d}{dx}\cdot q=-\frac{kA}{\mu}\cdot\left(\frac{d}{dx}\cdot\frac{dp}{dx}\right)$$

$$\boldsymbol{\frac{dq}{dx}=-\frac{kA}{\mu}\cdot\left(\frac{d^2p}{dx^2}\right)}$$

And

$$\frac{d}{dy}\cdot q=-\frac{kA}{\mu}\cdot\left(\frac{d}{dy}\cdot\frac{dp}{dy}\right)$$

$$\boldsymbol{\frac{dq}{dy}=-\frac{kA}{\mu}\cdot\left(\frac{d^2p}{dy^2}\right)}$$

Substitute for$\left(\boldsymbol{\frac{dq}{dx}}\ and\ \boldsymbol{\frac{dq}{dy}}\right)$ in Equation 5; the two-dimensional Partial Differential Equation, for Region-1 is:

$$\frac{\partial^2p_1}{\partial x^2}+\frac{\partial^2p_1}{\partial y^2}=(c_1\mu_1\varphi_1/k_1)\cdot\left(\frac{\partial p_1}{\partial t}\right)$$

The well model solution implements the following assumptions:

- Reservoirs are homogenous and isotropic within each side of the fracture.
- Single phase with slightly compressible fluid (c_f), the reservoir total compressibility is (c_t), constant viscosity (u_f) and formation volume factor (B).
- Fluids properties are independent of pressure.
- Gravity and capillary forces are neglected.

Mass Balance for Radial Systems (Radial Coordinates)

Figure 2, is a schematic of the mass balance in a radial system.

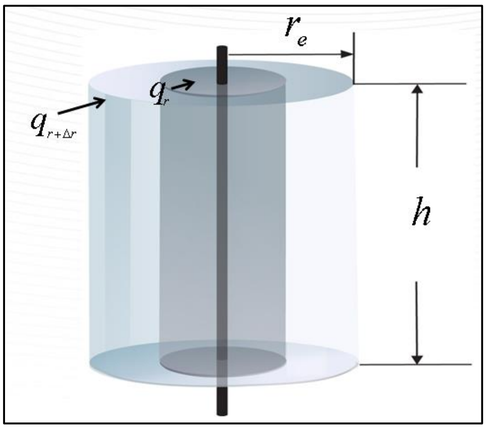

Figure 2: Schematic of the mass flow in a radial system.

$\Delta \boldsymbol{r}$: Finite Difference

q_r: Rate at (r) interface

$q_{\Delta r}$: Accumulation at (Δr)

$q_{r+\Delta r}$: Rate at $(r + \Delta r)$

ρ: Constant

A: $2\pi rh$

V: $(2\pi rh) \cdot (\Delta r \cdot \varphi)$

(Mass-in /Time) – (Mass-out /Time) = (Mass-Accumulation / Time)

$$(q_r \cdot \rho_r) - (q_{r+\Delta r} \cdot \rho_{r+\Delta r}) = q_{\Delta r} \cdot \rho_{\Delta r}$$

Taking into consideration the steps carried-over to handle the liquid compressibility effects for slightly compressible fluid, now directly move forward with passing-over the fluid density and directly multiply by Δt:

$$(q_r \cdot \Delta t) - (q_{r+\Delta r} \cdot \Delta t) = q_{\Delta r} \cdot \Delta t$$

$$(q_r \cdot \Delta t) - (q_{r+\Delta r} \cdot \Delta t) = \Delta v$$

Introduce the compressibility equation to MBE:

Where: $C = -\frac{1}{v}\frac{\Delta v}{\Delta p} \rightarrow \Delta v = cv\, \Delta p$

$\left(q_{r+\Delta r}\text{-}q_r\right) \cdot (\Delta t) = \Delta v = -cv\ \Delta p$

$\left(q_{r+\Delta r}\text{-}q_r\right) \cdot (\Delta t) = c \cdot\ (2\pi rh) \cdot (\Delta r \cdot \varphi) \cdot\ (p_{t+\Delta t} - p_t)$

Divide by $(\Delta r \cdot \Delta t)$ and take limit as $\Delta r \rightarrow 0$ & $\Delta t \rightarrow 0$:

$\lim_{\Delta r \rightarrow 0} \left(\frac{q_{r+\Delta r} - q_r}{\Delta r}\right) = \lim_{\Delta t \rightarrow 0} \left(\frac{p_{t+\Delta t} - p_t}{\Delta t}\right) c \cdot (2\pi rh\varphi)$

Equation 6

$$\frac{dq}{dr} = c \cdot (2\pi rh\varphi) \frac{dp}{dt} \tag{6}$$

Introduce Darcy's law:

$q = \frac{kA}{\mu} \cdot \frac{dp}{dr} = \frac{k(2\pi rh)}{\mu} \cdot \frac{dp}{dr}$

$= \left(\frac{2\pi\ kh}{\mu} r\right) \cdot \left(\frac{dp}{dr}\right)$

Differentiate both sides (w.r.t.) r:

$\frac{d}{dr} \cdot q = \frac{d}{dr} \left[\left(\frac{2\pi\ kh}{\mu} r\right) \cdot \left(\frac{dp}{dr}\right)\right]$

Equation 7

$$\frac{dq}{dr} = \left(\frac{2\pi\ kh}{\mu}\right) \cdot \left(\frac{dp}{dr}\right) + \left(\frac{2\pi\ kh}{\mu} r\right) \cdot \left(\frac{d^2p}{dr^2}\right) \tag{7}$$

Substitute (193) in (192):

$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial p}{\partial r} = \frac{c\mu\varphi}{k} \cdot \left(\frac{\partial p}{\partial t}\right)$

Partial differential equation (Radial coordinates)

Equation 8

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial p}{\partial r} = 1/\eta \cdot \left(\frac{\partial p}{\partial t}\right) \tag{8}$$

Initial and Boundary Conditions

Initial Condition (I.C.)

Initially the reservoir is cylindrical, at uniform initial pressure (P_i), has a thickness of (h), external radius of (r_e) and well radius of (r_w) where rw << r_e, Figure 3.

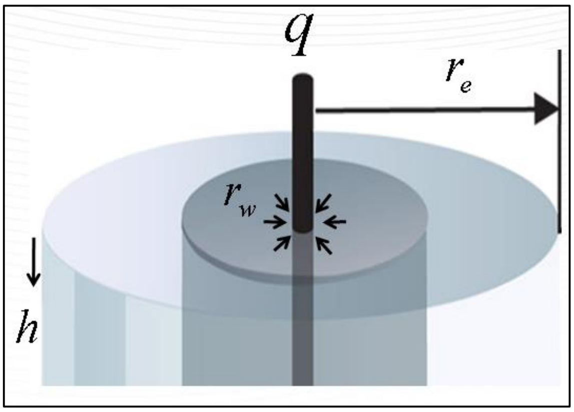

Figure 3: Schematic of a well in cylindrical reservoir.

$P(r, 0^-) = P_i,\ where: r_w \le r \le r_e$

0^-, any time prior to t=0

Boundary Conditions (B.C.)

Inner Boundary Condition (I.B.C.)

At $t = 0$, a constant flow rate (at sand face) is $q_s \Rightarrow\ q\ = q_s\beta$

Applying Darcy's law at sand face:

$$q = \frac{-kA}{\mu} \cdot \frac{dp}{dr}\bigg|_{r \to r_w}$$

$$q_s\beta = q = \frac{-kA}{\mu} \cdot \frac{dp}{dr}\bigg|_{r \to r_w}$$

$$-u|_{r \to r_w} = \frac{q}{A} = \frac{q_s\beta}{A} = \frac{q_s\beta}{2\pi r_w h} = \frac{k}{\mu} \cdot \frac{dp}{dr}\bigg|_{r \to r_w}$$

Re-arranging:

I.B.C.

$$\frac{dp}{dr}\bigg|_{r \to r_w} = \frac{q_s\beta\mu}{2\pi\ kh} \cdot \frac{1}{r_w},\ t \ge 0$$

For general use, the finite wellbore radius B.C.:

$$\lim_{r \to r} \left(r \cdot \frac{dp}{dr} \right) \Big|_{r \to r_w} = \frac{q_s \beta \mu}{2\pi\, kh}, \; t \geq 0$$

Outer Boundary Condition (O.B.C.)

Three B.C.'s and each yield its own unique solution to PDE.

Infinite Reservoir

The well is assumed is situated in a an infinite radius extent ($r \to \infty$**)**

Or a finite reservoir extent provided the pressure disturbance never reaches the outer boundary (r_e). The (O.B.C.) for infinite reservoir is:

$$P(r,t)\,|_{r \to r_\infty} = P_i, \qquad as: r \to \infty, t > 0$$

Closed Boundary Reservoir

The well is assumed to be in a center of a cylindrical reservoir (r_e) with no flow across the exterior boundary. The no flow condition implies zero superficial velocity at the outer boundary (r_e), Figure 4.

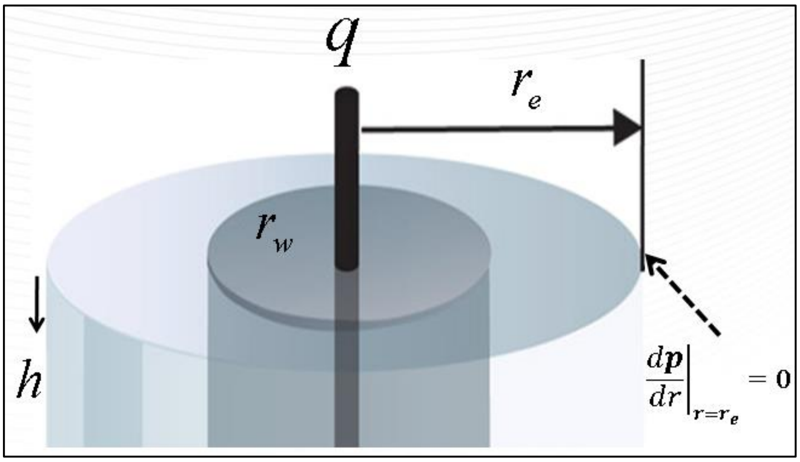

Figure 4: Schematic of a well in a cylindrical reservoir with closed-boundary.

$$\left.\frac{dp}{dr}\right|_{r=r_e} = 0, \text{Must be zero,}$$

$$\frac{dp}{dt} = C,$$

$$\Rightarrow \log t \frac{dp}{dt} = \log t + \; C$$

For $t > 0$

Constant Pressure Outer Boundary

The well is assumed to be in a center of a cylindrical reservoir with constant pressure boundary at (r_e), Figure 5. The O.B.C. for constant pressure support:

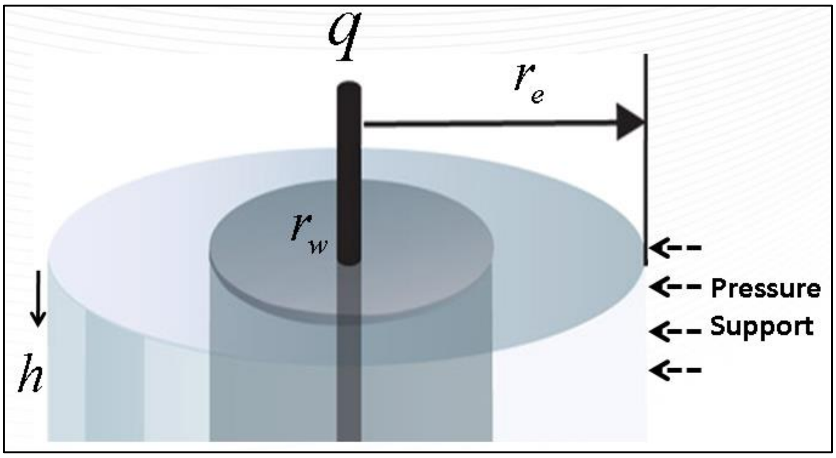

Figure 5: Schematic of a well in a cylindrical reservoir with a constant pressure boundary.

$P(r,t) = P_i$

$as: r \to \infty, \qquad t > 0$

$\frac{dp}{dt} = 0,$

$\Rightarrow \log t \frac{dp}{dt} = 0$