Morita equivalence of semigroups

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Abstract

Morita equivalence is a general way of classifying structures by means of their actions that is weaker than isomorphism but at the same time useful.

It arose first in the study of unital rings in the 1950's [35] but has since been extended to many other kinds of strucures, including classes of non-unital rings. It was first applied to semigroup theory in the 1970's in the work of Banaschewski [5] and Knauer [19] who independently determined when two monoids were Morita equivalent. However they were unable to extend their definition to arbitrary semigroups since Banaschewski showed that Morita equivalence reduced to isomorphism.

It was not until the 1990's that Talwar [40, 41] was able to find a good definition of Morita equivalence for a class of semigroups that included all monoids but also all regular semigroups: the class of semigroups with local units. Such a semigroup is one in which each element has a left and a right idempotent identity. Talwar's work was not developed further until the twenty-first century when a variety of mathematicians including Funk, Laan, Lawson, Márki and Steinberg started to develop the Morita theory of semigroups in detail [9, 20, 25, 39]. Our thesis takes as its starting point Lawson's reinterpretation of Talwar's work.

The thesis consists of three chapters. An essential ingredient in Morita theory is the notion of an equivalence of categories. For this reason, Chapter 1 of this thesis reviews all the categorical definitions needed. In Chapter 2, we describe in detail the work of Banaschewski and Knauer on the Morita theory of monoids. These two chapters contain no new work. We begin Chapter 3 by explaining why the obvious way of defining the Morita equivalence of two semigroups does not work. We then describe Lawson's approach to Talwar's work. This provides the foundation for our thesis. Our new contributions to the theory are contained in Sections 3.2, 3.3 and 3.4 and are based on Rees matrix semigroups.

Talwar showed that the classical Rees matrix theorem for completely simple semigroups could be regarded as a Morita theorem: a semigroup is Morita equivalent to a group if and only if it is completely simple if and only if it is isomorphic to a Rees matrix semigroup over a group. This raises the question of determining what role Rees matrix semigroups play in the Morita theory of semigroups with local units. We investigate three different problems based on this idea:

Section 3.2 In this section, we try to prove an exact generalisation of the Rees theorem. We are interested in the case where S is Morita equivalent to T if and only if S is *isomorphic* to some kind of Rees matrix semigroup over T.

Section 3.3 In this section, we prove that S is Morita equivalent to T if and only if S is a locally isomorphic image of a special kind of Rees matrix semigroup over T. This result was first proved by Laan and Márki [20] but we give a new proof that generalizes the classical proof of the Rees theorem.

Section 3.4 Finally, we solve the following problem: given an inverse semigroup S find all inverse semigroups T which are Morita equivalent to S. Our solution uses special kinds of Rees matrix semigroups over S. In this section, we also describe those semigroups which are Morita equivalent to semigroups with commuting idempotents. This builds on early work by Khan and Lawson [17, 18].

Dedication

In the memory of my father and my brother, in the memory of all innocent children who had lost their lives in Syria just because one person and people around him want to keep the power for themselves, in the memory of all those people who had paid their lives while they were seeking their freedom I dedicate my research.

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Contents

Chapter 1

Categories

Category theory is that part of mathematics that studies structures and the relationships between them using morphisms. It provides a language in which to do algebra. The goal of this chapter is to give all the definitions and results on categories that will be needed in the rest of the thesis. All of the material in this section is standard, and we make no claim to originality. We have used as references [3, 10, 16, 27, 34, 38], and most of the proofs are adapted from [16] unless otherwise stated. The most important definition in this chapter is the definition of an equivalence of categories. This is the basis of the definition of Morita equivalence.

1.1 Definitions

A *class* A is a collection of elements such that we can determine for any element x if it is an element of the class A or not. If x is an element of A then we write $x \in A$, and if x is not an element of A then we write $x \notin A$. A *set* A is a class such that there is another class B such that $A \in B$. So a set is a particular case of class. If a class is not a set then it is called a *proper class*. A set is called a *small class* and a proper class is called a *large class*. Roughly speaking, sets are 'small' and classes are 'large'.

A *(directed) graph* consists of a set V called *vertices*, and a set E called *arrows* and two operations as follows: the *domain operation,* which gives for each arrow f a vertex a such that $a = \text{dom} f$, and the *codomain operation,* which gives for each arrow f a vertex b such that $b = \text{cod} f$. We say that f is an arrow *starting at* its domain a and *ending at* its codomain b. We write

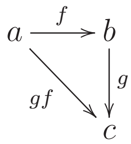
$f\colon a \to b$ or $a \xrightarrow{f} b$.

There are two different ways to define categories. In the first definition categories have both objects and arrows. In the second, categories have just arrows. The key point is that in categories it is not the particular nature of the objects and arrows which is important, but the way the arrows behave.

For our first definition, a *category* is defined to be a graph satisfying (C1) to (C4) below.

(C1) For each object a there is an arrow $1_a\colon a \to a$, called the *identity arrow*.

(C2) To each pair of arrows f and g with dom $g =$ cod f there is an arrow $gf\colon \mathrm{dom} f \to \mathrm{cod} g$, called the *composition of f and g*, which can be pictured as follows



(C3) For objects and arrows as follows

$$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{k} d$$

we always have the equality

$$k(gf) = (kg)f.$$

This is called *associativity*.

(C4) For each arrow $f\colon a \to b$ composition with the identity arrows 1_a and 1_b gives $f1_a = f$ and $1_b f = f$.

Our second definition of a category is the same as the first except that objects are ignored and replaced by identity arrows.

In each definition there is something which tell us when pairs of arrows can be multiplied together. In the first definition, the graph tell us when pairs of arrows can be multiplied together because we know the domain and the codomain of each arrow, so if the domain of the arrow g for example is equal to the codomain of the arrow f then we can multiply g by f otherwise they cannot be multiplied in this direction. That is, if

$$f : a \to b \quad \text{and} \quad g : b \to c$$

then since dom$g = b =$ cod$f$ the mutiplication gf exists as an arrow in this category. Its domain is the domain of f and its codomain is the codomain of g, so we can write it as follows

$$gf : a \to c.$$

In the second definition, the identity elements tell us when arrows can be multiplied together by knowing that the right identity of the first arrow is equal to the left identity of the second one, otherwise they cannot be multiplied together. This means that if f and g are two arrows in a category C and if 1_a is an identity arrow in C satisfing

$$g1_a = g \quad \text{and} \quad 1_a f = f$$

then gf exists in C, so g and f can be multiplied.

A *subcategory* S of a category C is a collection of some of the objects in C with some of the arrows in C between those objects with the identity of each object in S and such that the composition of each two arrows in S should be in S. This means that S itself is a category. A subcategory S is a *full subcategory* of C if all arrows between the S objects in C are in S.

For any two objects a and b in a category C, we define the *hom-class* of a and b to be

$$\hom(a,b) = \hom_C(a,b) = \{f \in C\colon f \ \text{ is an arrow } \ f : a \to b\}.$$

The hom-class of two objects a and b in a category C consists of all arrows in this category with domain a and codomain b. We usually assume that this is a set, and so call it a *hom-set*. If we want to define the hom-set depending on arrows, then the hom-set of two identities e and e' in a category C is

$$\hom(e,e') = \{f \in C\colon fe = f \quad \text{and} \quad e'f = f\}.$$

We say that a category is *large* if the objects and arrows form a class. We say that a category is *small* if the objects and arrows form a set.

Let C be a category. Then we say that C is *strongly connected* if for each pair of identities $e, e' \in C$, $\hom(e,e') \neq \emptyset$.

Let $f\colon a \to b$ be an arrow in a category C. Then if there is an arrow $g\colon b \to a$ such as $gf = 1_a$ and $fg = 1_b$ then g is unique because if $h\colon b \to a$ is another arrow such that $hf = 1_a$ and $fh = 1_b$ then

$$h = h1_b = h(fg) = (hf)g = 1_a g = g.$$

This unique arrow is called the *inverse* of f and is denoted by f^{-1}. We say that f is *invertible* if it has an inverse; it is also said to be an *isomorphism.* Two objects a and b are *isomorphic* in the category C if and only if there is an invertible arrow $f : a \to b$. If every arrow in a category C is invertible then the category is called a *groupoid.*

1.2 Examples of categories

In this section, we shall describe some examples of categories which will be important in this thesis.

Example 1.2.1 A monoid M is a category with one object which is the identity, and the other elements in it are the arrows. The composition is defined in it as the multiplication in the monoid. Since M is a monoid the identity law is satisfied and the associativity is satisfied. Conversely, a category with a single object can be regarded as a monoid.

Example 1.2.2 Continuing the previous example, we see that a group can be viewed as a groupoid with one object, and every groupoid with one object can be regarded as a group.

Example 1.2.3 A relation $\leq$ on a set X is called a *preorder* if and only if it is reflexive, which means $x \leq x$ for each $x \in X$, and transitive, which means for each $x, y, z \in X$ such that $x \leq y$ and $y \leq z$ then $x \leq z$. If $(X, \leq)$ is a preordered set, let C be the set of all ordered pairs (x, y) where $x \leq y$. Define a partial product $(x, y)(y, z) = (x, z)$. Then it is easy to check that C is a category in which every hom-set contains at most one arrow. Conversely, let C be a small category having the property that every hom-set contains at most one arrow. Let X be the set of identities of C. For $x, y \in X$, define $x \leq y$ if $\hom(y, x) \neq \emptyset$. Then $(X, \leq)$ is a preordered set.

These examples suggest that categories can be viewed as generalizations of both monoids and preorders. We now give some examples of categories of structures.

Example 1.2.4 We start with sets and the functions between them denoted by **Set**. We show that this is a category. If we consider sets as objects and functions as arrows then an arrow between two sets has a domain and a codomain which are objects in **Set** so **Set** is a graph. For each object A in **Set** there is an identity function $1_A\colon A \to A$ such that $1_A(x) = x$ for each $x \in A$. Also for any two functions $f : A \to B$ and $g : B \to C$ in **Set** their composition $g \circ f : A \to C$ is a function as well. The identity function 1_B satisfies $1_B \circ f = f$ because

$$(1_B \circ f)(x) = 1_B(f(x)) = f(x) \text{ for each } x \in A.$$

Similarly, $g \circ 1_B = g$. Hence the identity law is satisfied in **Set**. Let A, B, C and D be sets and let f, g and h be functions between them as follows

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D\ .$$

Then $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are both functions from A to D and they are equal because they have the same domain and codomain and for each $x \in A$ we have

$$(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x))) = (h \circ g)(f(x)) = ((h \circ g) \circ f)(x).$$

Therefore **Set** is a category. If we ignore the sets in this category and just think about the arrows then these sets will be replaced by the identity arrows and then we can determine whether two arrows can be composed or not from the left and right identities. So when we study sets and functions between them as a category we are just interested in the functions and we do not know anything about the sets themselves; we do not care about them, we just care about the arrows between them.

Example 1.2.5 We now study all monoids and all homomorphisms between them denoted by **Mon**. Since every homomorphism has a monoid as a domain and a monoid as a codomain then **Mon** is a graph. On the other hand, for each monoid M there is a function $1_M\colon M \to M$ such that $1_M(x) = x$ for each $x \in M$, and it is a homomorphism because

$$1_M(xy) = xy = 1_M(x)1_M(y).$$

It follows that the identity condition is satisfied. Also, if M, N and S are three monoids and $f : M \to N$ and $g : N \to S$ are two homomorphisms, then the composition $g \circ f : M \to S$ is a homomorphism because for each $x, y \in M$ we have

$$(g \circ f)(xy) = g(f(xy)) = g(f(x)f(y)) = g(f(x))g(f(y)) = (g \circ f)(x)(g \circ f)(y).$$

It also satisfies associativity because homomorphisms are functions. The identity law holds because homomorphisms are functions. Hence **Mon** is a category and it is a subcategory of **Set**.

Example 1.2.6 We next study the category of left A-acts where A is a monoid. Let A be a monoid with identity 1. Then a set M with a function $A \times M \to M$, denoted by $(a, m) \mapsto a \cdot m = am$, is called a *left A-act* if and only if this function satisfies

$$1m = m \text{ and } (ab)m = a(bm)$$

for all $m \in M$ and $a, b \in A$. Let N and M be two left $A-$acts. Then a function $f\colon N \to M$ is called a *left A-homomorphism* if and only if $f(ax) = af(x)$ for each $a \in A$ and $x \in N$. Define A-**Act** to consist of the left A-acts and all left A-homomorphisms between them. This is easily seen to be a category. In particular, observe that if we have two left A-homomorphisms

$$f\colon M_1 \to M_2 \text{ and } g\colon M_2 \to M_3$$

then their composition $g \circ f\colon M_1 \to M_3$ is a function and for all $a \in A$ and $x \in M_1$ we have that

$$(g \circ f)(ax) = g(f(ax)) = g(af(x)) = ag(f(x)) = a(g \circ f)(x).$$

Therefore $g \circ f$ is a left A-homomorphism.

Example 1.2.7 Let A be a monoid with identity 1. We may define the category of *right A-acts*, denoted by **Act**-A. Here set M with a function $M \times A \to M$, denoted by $(m, a) \mapsto m \cdot a = ma$, is called a *right A-act* if and only if this function satisfies

$$m1 = m \text{ and } m(ab) = (ma)b$$

for all $m \in M$ and $a, b \in A$.

Example 1.2.8 Let A and B be two monoids. Then a set M is an (A, B)-*biact* if and only if the following two conditions are satisfied

(B1) M is a left A-act and a right B-act.

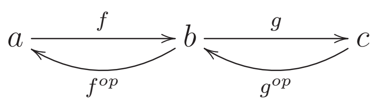
(B2) For all $a \in A, b \in B$ and $m \in M$

$$(am)b = a(mb).$$

Let M and N be two (A, B)-biacts. Then $f: M \to N$ is an (A, B)-*homomorphism* if and only if f is a left A-homomorphism and a right B-homomorphism. There is then a category A-**Act**-B whose objects are (A, B)-biacts and whose arrows are the (A, B)-homomorphisms.

Example 1.2.9 Let R be a unital ring, that is, a ring with an identity. Let M be an abelian group. We say that M is a *left R-module* if R acts on the set M on the left via its multiplicative monoid and $r(m+n) = rm + rn$ and $(r+s)m = rm+sm$. If M and N are two left R-modules a function $f: M \to N$ is a *left R-homomorphism* if it is a homomorphism of abelian groups and $f(rm) = rf(m)$. We may form a category R-**mod** whose objects are left R-modules and whose arrows are left R-homomorphisms. The definition of modules generalizes the definition of vector spaces. A good introduction to modules can be found in [4]. We shall meet this category again in Section 2.1 where we shall explain how it motivates this thesis.

Define the *dual* or *opposite category* $(C^{op}, *)$ from a given category $(C, \circ)$ as follows: C and C^{op} have the same objects but for each arrow $f : a \to b$ in C we form an arrow $f^{op} : b \to a$ in C^{op} and these arrows are the only arrows in C^{op}: therefore C^{op} consists of all objects in C and the opposite arrows f^{op} for each $f \in C$. The composition $f^{op} * g^{op}$ is defined in C^{op} if and only if the composition $g \circ f$ is defined in C and :



$$f^{op} * g^{op} = (g \circ f)^{op}.$$

The goal of category theory is to define things in terms of arrows. The notion of a dual category leads to the idea of each definition being associated with a dual definition obtained by 'reversing the arrows'. We shall see many examples of this. If a definition is made then its dual definition is usually prefixed with 'co-'.

Example 1.2.10 Let $(A, \cdot)$ be a monoid and let $(A^{op}, \circ)$ be the dual monoid with the same elements in A but the binary oberation on it is defined by $a \circ b = b \cdot a$ for each a and b in A. Let M be a right A-act. We prove that M is a left A^{op}-act. To prove this, define the map $* : A \times M \to M$ by $a * x = xa$ for each $a \in A$ and $x \in M$. Then

$$e * x = xe = x$$

and

$$a * (b * x) = (b * x)a = (xb)a = x(b \cdot a) = x(a \circ b) = (a \circ b) * x$$

for each $a, b \in A$ and $x \in M$. Therefore M is a left A^{op}-act. We can also conclude that, if M is a left A-act, then M is a right A^{op}-act.

1.3 Properties of arrows

The theme of this section is to show that some important mathematical properties can be described in purely categorical terms. An arrow $m: a \to b$ in a category C is said to be a *monomorphism, monic* or *left cancellable* when for each pair of arrows $f_1, f_2: d \to a$ the equality $mf_1 = mf_2$ implies that $f_1 = f_2$. The motivation for this definition comes from the following result.

Proposition 1.3.1 *In the category* **Set***, the function f is monic if and only if it is injective.*

Proof Let $f: B \to C$ be injective. Then if $g, h: A \to B$ are two arrows in C such that $f \circ g = f \circ h$ then $(f \circ g)(x) = (f \circ h)(x)$ for each $x \in A$. Therefore $f(g(x)) = f(h(x))$. But f is injective, and so $g(x) = h(x)$ for each $x \in A$. Hence $g = h$ and f is monic.

Now suppose that $f: B \to C$ is monic. Suppose $x, y \in B$ are such that $f(x) = f(y)$. Let $\{a\}$ be any one-element set and define two arrows $g, h : \{a\} \to B$ by $g(a) = x$ and $h(a) = y$. Then

$$f \circ g, f \circ h: \{a\} \to C,$$

and

$$(f \circ g)(a) = f(g(a)) = f(x) = f(y) = f(h(a)) = (f \circ h)(a).$$

Therefore $f \circ g = f \circ h$. But f is monic, and so $g = h$. Hence

$$x = g(a) = h(a) = y,$$

and so f is injective. ■

The above result is not true in all categories, although it is true that injections are always monics but the following standard example shows that monics are not always injections.

Example 1.3.2 An abelian group G is a *divisible* if and only if for any $g \in G$ and for any $0 \neq n \in \mathbb{Z}$ there is $h \in G$ such that $nh = g$. For example $\mathbb{Q}$ is a divisible abelian group, but $\mathbb{Z}$ is not. We prove that in the category of divisible abelian groups monics do not have to be injectives. Define $f : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ by $f(x) = \mathbb{Z} + x$. We prove first that f is monic in this category. Let A be a divisible abelian group and let $g, h : A \to \mathbb{Q}$ be two arrows in this category such that $fg = fh$. If we suppose that $g \neq h$ then there is $a \in A$ such that $g(a) \neq h(a)$. But $fg = fh$ therefore $fg(a) = fh(a)$ and so $\mathbb{Z}+g(a) = \mathbb{Z}+h(a)$ and so $g(a)-h(a) \in \mathbb{Z}$. Let $g(a)-h(a) = n$. Then $n \neq 0$ and so $2n \neq 0$. But A is divisible, therefore there is $b \in A$ such that $2nb = a$. Hence $g(2nb) - h(2nb) = n$ and so $2n(g(b) - h(b)) = n$ therefore $g(b) - h(b) = 1/2$. But $b \in A$ so $fg(b) = fh(b)$ and so $\mathbb{Z} + g(b) = \mathbb{Z} + h(b)$ therefore $g(b) - h(b) \in \mathbb{Z}$ which means that $1/2 \in \mathbb{Z}$, contradiction. We have proved that $g = h$ and f is monic. However, f is not injective because

$$f(1) = \mathbb{Z} + 1 = \mathbb{Z} = \mathbb{Z} + 2 = f(2)$$

but $1 \neq 2$ in $\mathbb{Q}$.

An arrow $h : a \to b$ is said to be an *epimorphism, epic* or *right cancellable* when for each pair of arrows $g_1, g_2 : b \to c$ the equality $g_1 h = g_2 h$ implies that $g_1 = g_2$.

Lemma 1.3.3 *In the catgeory* **Set**, *f is epic if and only if it is surjective.*

Proof Let $f : A \to B$ be surjective, and suppose $g \circ f = h \circ f$. We shall prove that $g = h$. Let $b \in B$. Then since f is surjective, there is $a \in A$ such that $b = f(a)$. Thus

$$g(b) = g(f(a)) = (g \circ f)(a) = (h \circ f)(a) = h(f(a)) = h(b).$$

It follows that $g = h$ and so f is epic.

Let $f\colon A \to B$ be an epic arrow. Suppose that f is not surjective. Then there is $y \in B$ such that $y \notin f(A)$. Let $C = \{c_1, c_2\}$ be any two-element set and define $g_1, g_2\colon B \to C \in$ **Set** by $g_1(b) = c_1$ for each $b \in B$ and $g_2(b) = c_1$ for each $b \neq y \in B$ and $g_2(y) = c_2$. Then $g_1 \neq g_2$. On the other hand, $g_1 \circ f, g_2 \circ f : A \to C$ satisfy

$$(g_1 \circ f)(a) = g_1(f(a)) = c_1 = g_2(f(a)) = (g_2 \circ f)(a)$$

for all $a \in A$. It follows that $g_1 \circ f = g_2 \circ f$. But f is epic and so $g_1 = g_2$, contradiction. We have proved that f is surjective, as required. ■

The above result is not true in all categories, as the following standard example shows.

Example 1.3.4 We work in the category **Mon** of monoids. Let $i\colon \mathbb{N} \to \mathbb{Z}$ be the inclusion monoid homomorphism. We prove that i is epic. Let M be a monoid and let $g, h\colon \mathbb{Z} \to M$ be two homomorphisms such that $gi = hi$. Suppose that $g \neq h$. Then there is $n \in \mathbb{Z}$ such that $g(n) \neq h(n)$ and so $g(-n) \neq h(-n)$ but n or $-n$ belongs to $\mathbb{N}$. Suppose that $n \in \mathbb{N}$. Then

$$g(n) = g(i(n)) = (gi)(n) = (hi)(n) = h(i(n)) = h(n).$$

This is a contradiction. We have proved that i is epic. However, it is clear that i is not surjective.

An arrow $r\colon b \to a$ is a *right inverse* of another arrow $f\colon a \to b$ if and only if $fr = 1_b$. A right inverse is called a *section* of f. It is not necesarily unique.

An arrow $l\colon b \to a$ is a *left inverse* of an arrow $f\colon a \to b$ if and only if $lf = 1_a$. A left inverse of f is also called a *retraction* of f. It is also not necessarily unique.

Lemma 1.3.5 *Let $f\colon a \to b$ and $g\colon b \to a$ be two arrows such that $gf = 1_a$. Then g is epic and f is monic.*

Proof If $g_1, g_2\colon a \to c$ satisfy $g_1 g = g_2 g$ then

$$(g_1 g)f = (g_2 g)f.$$

Therefore

$$g_1(gf) = g_2(gf).$$

Hence

$$g_1 1_a = g_2 1_a.$$

Thus

$$g_1 = g_2.$$

It follows that g is epic.

For each pair of arrows $f_1, f_2 : d \to a$ such that $ff_1 = ff_2$ then

$$g(ff_1) = g(ff_2).$$

Therefore

$$(gf)f_1 = (gf)f_2.$$

Hence

$$1_a f_1 = 1_a f_2$$

and so

$$f_1 = f_2.$$

It follows that f is monic. ■

Lemma 1.3.5 told us that if an arrow f has a right inverse then f is epic, and if f has a left inverse then f is monic. Also in **Set** we have the following.

Lemma 1.3.6 *In the category* **Set** *if f is epic then f has a right inverse.*

Proof Let $f: A \to B$ be an epic arrow in **Set**. Then by Lemma 1.3.3 f is surjective. Define $g: B \to A$ as follows: for each $b \in B$ choose $a \in A$ such that $f(a) = b$ and define $g(b) = a$. Then $fg : B \to B$ satisfies for each $b \in B$

$$(fg)(b) = f(g(b)) = f(a) = b.$$

Thus $fg = 1_B$ and f has a right inverse. ■

Lemma 1.3.7 *Let* $a \underset{h}{\overset{g}{\rightleftarrows}} b$ *be such that* $gh = 1_a$. *Then* $f = hg: b \to b$ *is an idempotent.*

Proof The composition $ff: b \to b$ is defined and satisfies

$$f^2 = ff = (hg)(hg) = h(gh)g = h1_a g = hg = f.$$

Therefore f is an idempotent. ∎

An idempotent f is said to be *split* when there are arrows h and g such that $f = hg$ and $gh = 1$, where 1 is an identity. This definition will play an important role in Chapter 3.

Two objects a and b in a category C is said to be *isomorphic* if there is an arrow $f : a \to b$ with a left and right inverse $f^{-1} : b \to a$ and then f is called an *isomorphism*.

1.4 Functors

We can think of categories as structures: as generalizations of monoids. Functors will then play the role of 'homomorphisms between categories'. Let B and C be categories. Then a *functor* $F: B \to C$ consists of two functions: the object function F which takes each object b in B to an object $F(b)$ in C, and the arrow function which takes each arrow $f: b \to b'$ in B to an arrow $F(f): F(b) \to F(b')$ in C, in such a way that $F(1_b) = 1_{Fb}$ and $F(fg) = F(f)F(g)$ when the composition fg is defined in B. We can also define functors on the arrows-only definition of a category and then it really does look like a generalization of a monoid homomorphism. In Sections 2.2.2 and 2.2.3, we shall describe some important examples of functors: hom-functors and tensor functors.

Functors can be composed. If $F: A \to B$ and $G: B \to C$ are functors then $GF: A \to C$ is a functor. For each category C there is an identity functor $1_C: C \to C$. Let $F: A \to B$ be a functor. Then F is an *isomorphism* if there is a functor $G: B \to A$ such that $GF = 1_A$ and $FG = 1_B$. We say that the two categories A and B are *isomorphic* if there is an isomorphism between them.

Lemma 1.4.1 *Let $F: A \to B$ be a functor. Then F is an isomorphism if and only if F is bijective.*

Proof Suppose that F is an isomorphism. Then there is a functor $G : B \to A$ such that $GF = 1_A$ and $FG = 1_B$. But F and G are functions and so F is bijective.

Conversely, suppose F is bijective. Then there is a function $G : B \to A$ such that $GF = 1_A$ and $FG = 1_B$. Now we have to prove that G is a functor. Let $1_b : b \to b$ be any identity in the category B. Then since F is surjective there is $a \in A$ such that $b = F(a)$, and so the identity $1_a : a \to a$ satisfies that $F(1_a) = 1_{F(a)} = 1_b$. It follows that $G(1_b) = G(F(1_a)) = GF(1_a) = 1_A(1_a) = 1_a$. Also, since F is bijective for each two composable arrows $h : x \to y, k : y \to z \in B$ there is $f : a \to b, g : c \to d \in A$ such that $F(f) = h$ and $F(g) = k$. Thus $F(a) = x$ and $F(b) = y$ and $F(c) = y$ and $F(d) = z$ which means that $F(b) = F(c)$ but F is bijection so $b = c$ and so f and g are composable. Then

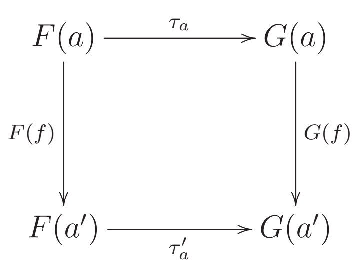
$$G(kh) = G(F(g)F(f)) = G(F(gf)) = GF(gf) = 1_A(gf) = gf$$
$$= 1_A(g)1_A(f) = GF(g)GF(f) = G(k)G(h).$$

It follows that G is a functor. We have proved that F is an isomorphism. ■

From our experience in algebra we would expect that isomomorphisms between categories would be important. However, this is not the case. The more important notion is that of an equivalence of categories. We shall explain this idea in Section 1.6, but first we need a way of comparing functors.

1.5 Natural transformations

In the last section, we defined functors as the natural notion of a homomorphism between categories. We now define the natural notion of a homomorphism between functors. Let $A \underset{G}{\overset{F}{\rightrightarrows}} B$ be parallel functors between categories A and B. We define an arrow $\tau : F \Rightarrow G$ between the functors, called a *natural transformation*, as follows: for each object a in A, there is an arrow $\tau_a : F(a) \to G(a)$ in B such that for each arrow $f : a \to a'$ in A the following diagram commutes in B



If each τ_a is an isomorphism for all objects a then the natural transformation τ is called a *natural isomorphism.*

We can compose natural transformations. Let $\tau: F \Rightarrow G$ and $\sigma: G \Rightarrow H$ be natural transformations. We define the natural transformation $\sigma\tau: F \Rightarrow H$ by $(\sigma\tau)_a = \sigma_a \tau_a$.

1.6 Equivalence of categories

We needed the definitions of category, functor and natural transformation to make the definition of this section which is the most important one in this chapter. We need it even to define what we mean by Morita equivalence.

Two categories A and B are *equivalent* if and only if there are functors

$$F : A \to B \quad \text{and} \quad G : B \to A$$

such that there are natural isomorphisms

$$\tau : GF \Rightarrow 1_A \quad \text{and} \quad \eta : FG \Rightarrow 1_B.$$

Isomorphic categories are equivalent, but the converse is not true and we will give an example later to show that. The notion of equivalent categories allows us to compare categories which are essentially the same but of very different sizes; we shall also see this in our later example.

To show that two categories are equivalent looks complicated but can be simplified. To do this we shall need the following three definitions.

A functor $T: B \to C$ is *full* when it satisfies the following condition: for each pair of objects b, b' in B and for each arrow $g: T(b) \to T(b')$ in C there is an arrow $f: b \to b'$ in B such that $g = T(f)$. The composition of full functors is a full functor.

A functor $T: B \to C$ is *faithful* if it satisfies the following condition: for each pair of objects b, b' in B and for each pair of arrows $f, f': b \to b'$ satisfying $T(f) = T(f'): T(b) \to T(b')$ then $f = f'$. The composition of faithful functors is a faithful functor.

A functor $T: A \to B$ is said to be *essentially surjective* if and only if each object in B is isomorphic to the image under T of an object of A.

A functor which is full, faithful and essentially surjective is called a *weak equivalence.*

Lemma 1.6.1 *Let $F: A \to B$ be a full and faithful functor. Suppose that $f: a \to b$ is an arrow in A such that $F(f)$ is an isomorphism. Then f is an isomorphism.*

Proof By assumption, there is an element $\bar{f}: F(b) \to F(a)$ such that $F(f)\bar{f}$ and $\bar{f}F(f)$ are identities. Because F is full and faithful, there exists a unique arrow $f': b \to a$ such that $F(f') = \bar{f}$. But $F(ff')$ and $F(f'f)$ are both identities and so from the fact that F is full and faithful, it follows that ff' and $f'f$ are identities. Thus f is invertible with inverse f'. ■

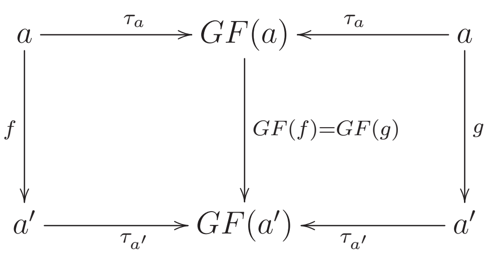
We can now state the following extremely useful theorem. The proof uses the Axiom of Choice.

Theorem 1.6.2 *A pair of categories are equivalent if and only if there is a weak equivalence between them.*

Proof Suppose first that A and B are equivalent categories. Then there are two functors $F: A \to B$ and $G: B \to A$ such that there are natural isomorphisms

$$\tau: 1_A \Rightarrow GF \text{ and } \sigma: 1_B \Rightarrow FG.$$

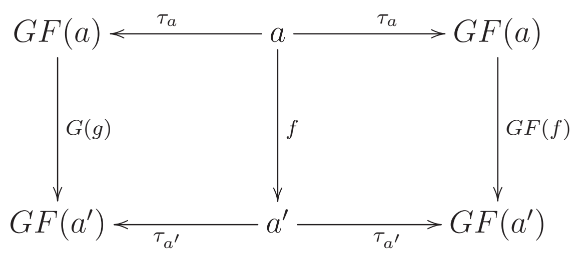
We prove first that F is faithful. Let $f, g: a \to a'$ be two arrows in A such that $F(f) = F(g)$. Then since τ is a natural isomorphism between 1_A and GF the following diagram will be commutative



Therefore $\tau_{a'}f = \tau_{a'}g$. But τ is natural isomorphism and so $\tau_{a'}$ is an isomorphism. It follows that it is left cancellable and so $f = g$. By symmetry we deduce that G is also faithful.

To show that F is essentially surjective, observe that for each object $b \in B$ we have $FG(b) \cong 1_B(b) = b$. Define $a = G(b)$ then a is an object in A such that b is isomorphic to $F(a)$.

It remains to show that F is full. Let a and a' be two objects in A and let $g\colon F(a) \to F(a')$ be an arrow in B. Define $f = \tau_{a'}^{-1}G(g)\tau_a$. Then the following diagram commutes



It follows that $G(g)\tau_a = GF(f)\tau_a$. But τ_a is an isomorphism so it is right cancellable. Thus $G(g) = GF(f)$. But G is faithful and so $g = F(f)$.

To prove the converse, suppose that there is a functor $F\colon A \to B$ which is full, faithful and essentially surjective. Thus for each object $b \in B$, we may choose an object $a \in A$ such that b is isomorphic to $F(a)$. Denote this object by $G(b)$ Then for each object $b \in B$ there is an object $G(b) \in A$ such that $FG(b)$ is isomorphic to b. Choose an isomorphism $x_b\colon FG(b) \to b$. Let $g\colon b \to b'$ be an arrow in B. There are isomorphisms $x_b\colon FG(b) \to b$ and $x_{b'}\colon FG(b') \to b'$. If we take the composition $x_{b'}^{-1}gx_b\colon FG(b) \to FG(b')$ it will be an arrow in B. But F is full and faithful. Thus there is a unique arrow, denoted by $G(g)\colon G(b) \to G(b')$, such that $FG(g) = x_{b'}^{-1}gx_b$.

We shall prove first that $G\colon B \to A$ is a functor. We have that

$$FG(1_b) = x_b^{-1}1_b x_b = 1_{FG(b)} = F(1_{G(b)}).$$

But F is a faithful. Thus $G(1_b) = 1_{G(b)}$. Also, if $g\colon b \to b'$ and $h\colon b' \to b''$ are two arrows in B, then $hg\colon b \to b''$ is an arrow in B and

$$FG(hg) = x_{b''}^{-1}hgx_b = x_{b''}^{-1}hx_{b'}x_{b'}^{-1}gx_b = FG(h)FG(g) = F(G(h)G(g)).$$

But F is faithful. Therefore $G(hg) = G(h)G(g)$. We have proved that G is a functor.

Now we have to prove that there are natural isomorphisms $FG \cong 1_B$ and $GF \cong 1_A$.

Since $FG(g) = x_{b'}^{-1}gx_b$ we have that $x_{b'}FG(g) = gx_b$ so the following diagram commutes

$$\begin{array}{ccc} FG(b) & \xrightarrow{x_b} & b \\ {\scriptstyle FG(g)}\downarrow & & \downarrow{\scriptstyle g} \\ FG(b') & \xrightarrow[x_{b'}]{} & b' \end{array}$$

Each x_b is an isomorphism, and so $FG \cong 1_B$.

For each $a \in A$, $F(a)$ is an object in B so there is an isomorphism $x_{F(a)}\colon FGF(a) \to F(a)$. Then $x^{-1}_{F(a)}\colon F(a) \to FGF(a)$. But F is full and faithful, and so there is a unique morphism $\tau_a\colon a \to GF(a)$ such that $F(\tau_a) = x^{-1}_{F(a)}$. Since $x_{F(a)}$ is an isomorphism, so $x^{-1}_{F(a)}$ is an isomorphism. By Lemma 1.6.1, τ_a is an isomorphism because F is full and faithful. For each arrow $f\colon a \to a'$ in the category A the arrow $F(f)\colon F(a) \to F(a')$ is an arrow in the category B, but as we prove $FG \cong 1_B$ and so the diagram below commutes

$$\begin{array}{ccc} F(a) & \xrightarrow{F(\tau_a)} & FGF(a) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle FGF(f)} \\ F(a') & \xrightarrow[F(\tau_{a'})]{} & FGF(a') \end{array}$$

and since F is faithful the following diagram commutes

$$\begin{array}{ccc} a & \xrightarrow{\tau_a} & GF(a) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle GF(f)} \\ a' & \xrightarrow[\tau_{a'}]{} & GF(a') \end{array}$$

Since τ_a is an isomorphism for each object $a \in A$, we have that $GF \cong 1_A$.

Hence A and B are equivalent categories. ■

We shall now give an explicit example of a pair of categories which are equivalent but not isomorphic because they have completely different sizes.

Example 1.6.3 Let G be a group, and let I be any non-empty set. Then the set

$$I \times G \times I = \{(i, g, j) \colon i, j \in I, g \in G\}$$

becomes a category when we define

$$(i, g, j)(k, h, l) = (i, gh, l) \text{ if and only if } j = k.$$

The identities of this category are the elements of the form $(i, 1, i)$ and we have $\text{dom}(i, g, j) = (j, 1, j)$ and $\text{cod}(i, g, j) = (i, 1, i)$. It is easy to show that $I \times G \times I$ is a category. It is also a groupoid because for each arrow (i, g, j) then (j, g^{-1}, i) is an arrow satisfying

$$(i, g, j)(j, g^{-1}, i) = (i, 1, i)$$

and

$$(j, g^{-1}, i)(i, g, j) = (j, 1, j)$$

and so each arrow in $I \times G \times I$ has an inverse. It is also strongly connected. Conversely, it is easy to show that every strongly connected groupoid is isomorphic to a groupoid of the form $I \times G \times I$.

We now show that the group G and the category $I \times G \times I$ are equivalent. Define a functor as follows. Choose and fix $i \in I$. Then

$$F \colon G \to I \times G \times I \text{ is defined by } F(g) = (i, g, i).$$

Then F is full, faithful and essentially surjective. Thus G and $I \times G \times I$ are equivalent by Theorem 1.6.2. Observe that we said nothing about the cardinality of I. Thus I can be chosen so that the sets G and $I \times G \times I$ do not have the same cardinality; it therefore follows that under this assumption they cannot be isomorphic.

1.7 Limits and colimits

In this section, we shall describe some important constructions that may be carried out in some categories.

A *terminal object* t in a category C is an object in this category such that for each object a in C there is exactly one arrow $a \to t$.

Lemma 1.7.1 *If t_1, t_2 are terminal objects in a category C. Then t_1 and t_2 are isomorphic.*

Proof Since t_1 is a terminal object there is exactly one arrow $f: t_2 \to t_1$, and since t_2 is a terminal object there is exactly one arrow $g: t_1 \to t_2$. The compositions $fg: t_1 \to t_1$ and $gf: t_2 \to t_2$ both exist and have to be unique and so must equal their respective identities. ∎

A dual concept to that of a terminal object in a category C is an *initial object* i. This is an object in this category such that for each object a in C there is a unique arrow $i \to a$. Using a similar argument to that in Lemma 1.7.1, we can prove that any two initial objects are isomorphic.

Example 1.7.2 In the category **Set**, any set consisting of one element is a terminal object, and the empty set is an initial object.

In the category **Mon**, the monoid with one element, the identity, is both a terminal and an initial object.

To motivate the next definition we begin by looking at a construction in sets from a categorical point of view. The *product* of two sets A and B is the set

$$A \times B = \{(x, y): x \in A, y \in B\}$$

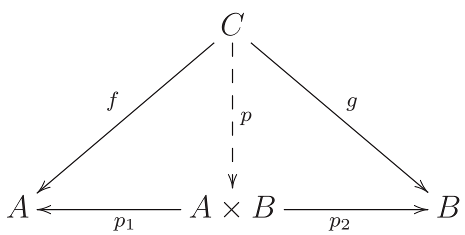
with two special maps

$$p_1: A \times B \to A \text{ defined by } p_1((x, y)) = x$$

and

$$p_2: A \times B \to B \text{ defined by } p_2((x, y)) = y.$$

Lemma 1.7.3 *Let A and B and p_1, p_2 be as above. Let C be another set and $f: C \to A$ and $g: C \to B$ be two maps. Then $p: C \to A \times B$ which is defined by $p(c) = (f(c), g(c))$ for each $c \in C$ is the unique map which makes the following diagram commute*



Proof For each $c \in C$

$$(p_1p)(c) = p_1(p(c)) = p_1(f(c), g(c)) = f(c)$$

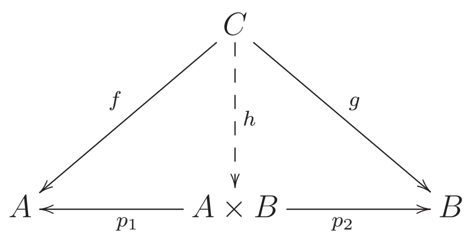
and

$$(p_2p)(c) = p_2(p(c)) = p_2(f(c), g(c)) = g(c)$$

therefore

$$p_1p = f \quad \text{and} \quad p_2p = g$$

and the diagram commutes. Now let $h : C \to A \times B$ be a map which makes the following diagram commute



Then

$$p_1h = f \quad \text{and} \quad p_2h = g.$$

Since h is a map from C to $A \times B$ then for each $c \in C$ there is $(a', b') \in A \times B$ such that $h(c) = (a', b')$, and so

$$f(c) = p_1h(c) = p_1(a', b') = a'$$

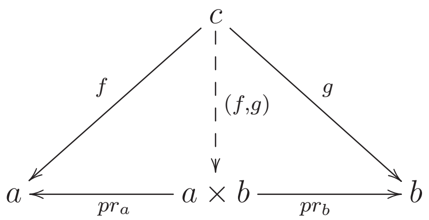
and

$$g(c) = p_2h(c) = p_2(a', b') = b'$$

it follows that $h(c) = (f(c), g(c)) = p(c)$ for each $c \in C$ which means that $h = p$ and p is unique. ■

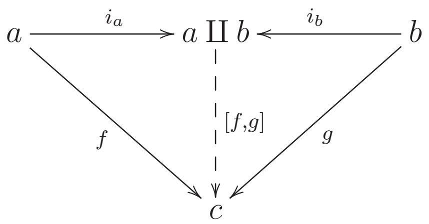
The above lemma shows that products of sets can be charactized in terms of arrows only without reference to elements. In a category C, we define the *product* of two objects a and b to be an object $a \times b$ together with a pair of arrows $(pr_a\colon a \times b \to a, pr_b\colon a \times b \to b)$ such that for any other pair of arrows $(f\colon c \to a, g\colon c \to b)$ in C there is a unique arrow denoted $(f, g)\colon c \to a \times b$

making the following diagram commute



The arrow (f, g) is called the *product* of the two arrows f and g.

The *coproduct* is the dual notion of product. It is defined as follows. Let a and b be two objects in a category C. Then their *coproduct* is an object $a \amalg b$ in C with a pair of arrows $(i_a : a \to a \amalg b, i_b : b \to a \amalg b)$ in C such that for any other pair of arrows $(f : a \to c, g : b \to c)$ in C there is a unique arrow denoted $[f, g] : a \amalg b \to c$ in C making the following diagram commute



We now describe the coproduct in **Set**.

Lemma 1.7.4 *Let A and B be two sets in* **Set**. *Then their coproduct is expressed as their* disjoint union

$$A \amalg B = A \times \{1\} \cup B \times \{2\}$$

together with the functions

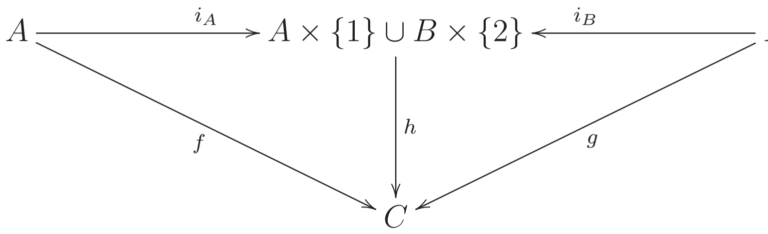
$$i_A : A \to A \amalg B \quad \textit{such that} \quad i_A(a) = (a, 1) \quad \textit{for each} \quad a \in A$$

and

$$i_B : B \to A \amalg B \quad \textit{such that} \quad i_B(b) = (b, 2) \quad \textit{for each} \quad b \in B.$$

Proof Let C be a set and let $f\colon A \to C, g\colon B \to C$ be two functions. Define $h\colon A \times \{1\} \cup B \times \{2\} \to C$ by $h(x, 1) = f(x)$ and $h(x, 2) = g(x)$. Then h

makes the following diagram commute



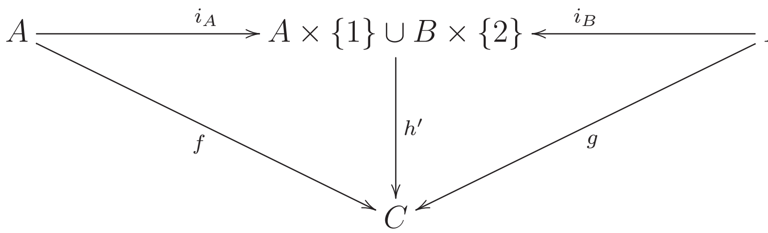
because

$$hi_A(a) = h(i_A(a)) = h(a, 1) = f(a)$$

and

$$hi_B(b) = h(i_B(b)) = h(b, 2) = g(b)$$

for each $a \in A$ and $b \in B$. Suppose that $h'\colon A \times \{1\} \cup B \times \{2\} \to C$ is another arrow making the following diagram commute



Let $(x, i) \in A \times \{1\} \cup B \times \{2\}$. Then $(x, i) = (a, 1)$ or $(x, i) = (b, 2)$ such that $a \in A$ and $b \in B$. If $(x, i) = (a, 1)$ then

$$h'(x, i) = h'(a, 1) = h'(i_A(a)) = h'i_A(a) = f(a)$$

and if $(x, i) = (b, 2)$ then

$$h'(x, i) = h'(b, 2) = h'(i_B(b)) = h'i_B(b) = g(b)$$

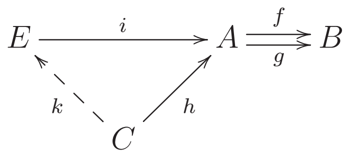
and so, $h' = h$. Hence h is unique. ∎

Example 1.7.5 To understand the above definitions better we look at a concrete example. Let $(X, \leq)$ be a preordered set regarded as a category. Then there is a unique arrow $a \to b$ if and only if $a \leq b$. This category has a terminal element if and only if it has a maximum element, and an initial element if and only if it has a minimum element. A pair of elements x, y has a product if and only if their greatest lower bound $x \wedge y$ exists, and they have a coproduct if and only if the least upper bound $x \vee y$ exists.

Our next definitions look at the case where we have more than one arrow between two objects. Again we motivate the general definition by considering the case of the category of sets first. Let f and g be parallel functions $f, g\colon A \rightrightarrows B$ in **Set**. Define

$$E = \{x \in A \colon f(x) = g(x)\}.$$

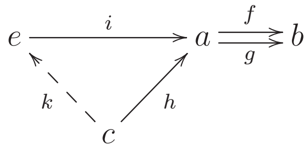
Then the inclusion function $i\colon E \to A$ has the property that $fi = gi$ and is called the *equalizer* of f and g. If $h\colon C \to A$ is another function which satisfies that $fh = gh$, then there is a unique function $k\colon C \to E$ defined by $k(c) = h(c)$ for each $c \in C$ making the following diagram commute



We now define equalizers in general categories. An arrow $i\colon e \to a$ in C is an *equalizer* of a pair of parallel arrows $f, g\colon a \to b$ in C if and only if:

1. $fi = gi$.

2. If $h\colon c \to a$ is another arrow in C satisfying $fh = gh$, then there is a unique arrow $k : c \to e$ in C making the following diagram commute

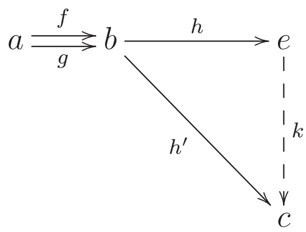


The dual notion to that of the equalizer is coequalizer. The *coequalizer* of two parallel arrows $f, g\colon a \to b$ in any category C is an arrow $h\colon b \to e$ such that

1. $hf = hg$.

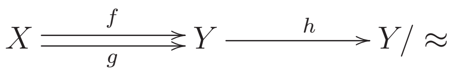
2. If $h'\colon b \to c$ satisfies the same property $h'f = h'g$ in C, then there is a

unique arrow $k\colon e \to c$ making the following diagram commute

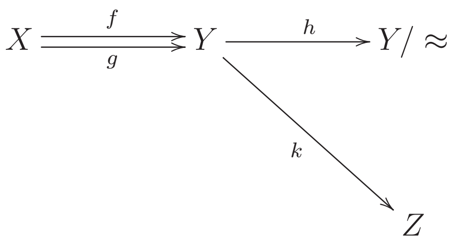


Lemma 1.7.6 *Coequalizers exist in the category* **Set***.*

Proof Let $f, g : X \to Y$ be two functions between sets X and Y. Define the relation $\sim$ on Y by $y_1 \sim y_2$ in Y if and only if $y_1 = f(x)$ and $y_2 = g(x)$ for some $x \in X$. Let $\approx$ be the equivalence relation on Y generated by $\sim$; this is described in Lemma 2.2.6 and what follows. Let $h : Y \to Y/\approx$ be the associated natural map taking y to the $\approx$-equivalent class containing y, denoted by $[y]$. We have the following diagram of functions



Let $x \in X$. Then $f(x) \sim g(x)$ and so since $\sim \subseteq \approx$ we have that $[f(x)] = [g(x)]$. Thus $(hf)(x) = (hg)(x)$ for all $x \in X$. Now let $k : Y \to Z$ be any function such that $kf = kg$. We therefore have the following diagram of functions



The kernel of k is an equivalence relation on Y. Suppose that $y_1 \sim y_2$ in Y. Then $y_1 = f(x)$ and $y_2 = g(x)$ for some $x \in X$. But

$$k(y_1) = (kf)(x)$$

$$k(y_2) = (kg)(x)$$

and $(kf)(x) = (kg)(x)$ and so $k(y_1) = k(y_2)$. Thus $(y_1, y_2) \in \ker(k)$. It follows that $\sim \subseteq \ker(k)$. However $\approx$ is the smallest equivalence relation on

Y containing $\sim$ and so $\approx \subseteq \ker(k)$. Define $p : Y/\approx \to Z$ by $p([y]) = k(y)$. We have to show that this function is well-defined. Suppose that $[y] = [y']$. Then $y \approx y'$. But from what we have just proved, $k(y) = k(y')$, as required. Thus p makes the diagram commute. Suppose $p' : Y/\approx \to Z$ is another function making the diagram commute. Let $[y] \in Y/\approx$. Then $[y] = h(y)$, by definition. Thus

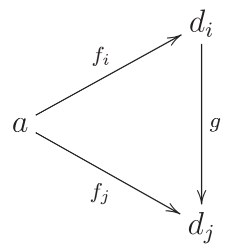
$$p'([y]) = p'(h(y)) = k(y) = p(h(y)) = p([y]).$$

It follows that $p' = p$, and p is the unique function making the diagram commute. ■

We now make the general definitions of which the above are special cases.

A *diagram* D in a category C is a collection of objects d_i and arrows between them $g\colon d_i \to d_j$. We can think of D as a directed graph in the category.

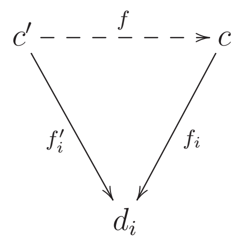
A *cone* for a diagram D is an object a with a set of arrows from a to each object $d_i \in D$ making all triangles below commute



A *limit* for a diagram D, denoted by

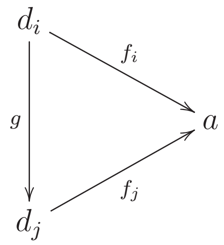
$$\varprojlim D$$

is a cone $\{f_i : c \to d_i\}$ for this diagram which satisfies the following: for each cone $\{f'_i : c' \to d_i\}$ there is a unique arrow $f : c' \to c$ making the following triangles commute for each object $d_i \in D$



This means that the limit for a diagram D can be considered as a terminal object of all cones for D. Thus limits are massive generalizations of greatest lower bounds or meets.

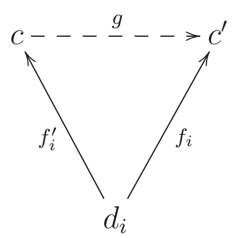
A *cocone* is the dual of the cone, in other words the cocone for the diagram D is an object a with a set of arrows from each object $d_i \in D$ to a making all triangles below commute



A *colimit* for a diagram D, denoted by

$$\varinjlim D$$

is a cocone $\{f_i : d_i \to c\}$ for this diagram which satisfies the following: for each cocone $\{f_i' : d_i \to c'\}$ there is a unique arrow $g : c \to c'$ making the following triangles commute for each object $d_i \in D$



This means that the colimit for a diagram D can be considered as an initial object of all cocones for D. Thus colimits are massive generalizations of least upper bounds or joins.

In Chapter 2, we shall deal exclusively with colimits.

Examples 1.7.7

1. Let D be the empty diagram. We shall show that the limit for D is a terminal object. A cone for D is just any object a in the category. There are no arrows from a to D. A limit for D is another object t

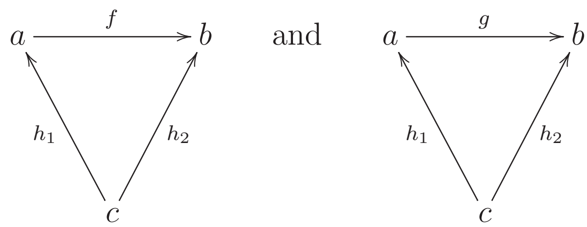
with no arrows to D such that there is a unique arrow from a to t. This unique arrow from a to t makes the diagram commute because it is the only arrow in this diagram. It follows that the terminal object is a limit for D. Using a similar argument, the initial object is a colimit for the empty diagram.

2. Take the diagram D in C consisting of two objects a and b. Then any other object c with two arrows $f : c \to a$ and $g : c \to b$ is a cone and a limit cone for them is the product of a and b in C. A colimit for this diagram is a coproduct of a and b.

3. Take the diagram D in C which consists of two objects with two arrows f and g between them as follows

$$a \overset{f}{\underset{g}{\rightrightarrows}} b$$

Then a cone for D is an object c in C with two arrows $h_1 : c \to a$ and $h_2 : c \to b$ making the following triangles



commute. Which means that $h_2 = fh_1 = gh_1$. Therefore we can say that the cone for D in this case is an arrow $h_1 : c \to a$ making the diagram below commute

$$c \xrightarrow{h_1} a \overset{f}{\underset{g}{\rightrightarrows}} b\ .$$

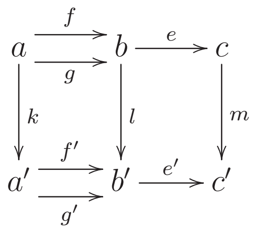
The limit is then the equalizer of f and g. Coequalizers are the colimits of the same diagram.

We shall now prove two well-known results we shall need in Chapter 2.

Lemma 1.7.8 *Let $e\colon b \to c$ be the coequalizer of $f, g\colon a \to b$. Let $h\colon d \to a$ be an epimorphism. Then e is the coequalizer of $fh, gh\colon d \to b$.*

Proof Clearly $efh = egh$. Suppose that $e' : b \to c'$ is such that $e'fh = e'gh$. The arrow h is an epimorphism and so right cancellable. Thus $e'f = e'g$. By assumption there is a unique arrow $k : c \to c'$ such that $e' = ke$. Hence the result. ■

Lemma 1.7.9 *Let $f, g : a \to b$ have the coequalizer $e : b \to c$. Let $f', g' : a' \to b'$ have the coequalizer $e' : b' \to c'$. Let $k : a \to a'$, $l : b \to b'$ and $m : c \to c'$ be arrows such that we have the following diagram*



We assume that $me = e'l$, $lf = f'k$ and $lg = g'k$. Then if k and l are isomorphisms then m is an isomorphism.

Proof Observe that because e is an epimorphism, the arrow m is the unique arrow making the rightmost square of the above diagram commute. Observe that

$$el^{-1}f' = el^{-1}g'.$$

Thus by the definition of a coequalizer, there exists a unique arrow $m' : c' \to c$ such that $m'e' = el^{-1}$. Observe that

$$m'me = m'(me) = m'e'l = (m'e')l = (el^{-1})l = e.$$

But e is an epimorphism and so right cancellable. It follows that $m'm$ is an identity. By symmetry, mm' is an identity. We have proved that m is an isomorphism, as required. ■

1.8 Adjoints

Let $F : C \to D$ and $G : D \to C$ be functors. For each pair of objects $c \in C$ and $d \in D$, we shall compare the set

$$\hom_D(F(c), d)$$

with the set

$$\hom_C(c, G(d)).$$

We shall suppose that there is a bijection

$$\phi_{c,d}: \hom_D(F(c), d) \to \hom_C(c, G(d))$$

which is natural in both c and d. We need to explain what the phrase 'natural in c and d means'. Suppose that α: $d \to d'$ then we require that

$$G(\alpha)_* \phi_{c,d} = \phi_{c,d'} \alpha_*,$$

where the lower star means 'multiply on the left'. Suppose that β: $c' \to c$ then we require that

$$\beta^* \phi_{c,d} = \phi_{c',d} F(\beta)^*$$

where upper star means 'multiply on the right'. If this occurs we say that F is *left adjoint to* G, and that G is *right adjoint* to F.

Adjointness is the most general relationship linking two functors together. We shall need the following three results about adjoints.

For the following, see Section 4 of Chapter IV of [27].

Lemma 1.8.1 *If the functors $F: C \to D$ and $G: D \to C$ determine an equivalence of categories then they are adjoint to each other.*

For the following see Corollary 1 of Section 1 of Chapter IV of [27].

Lemma 1.8.2 *Let $F, F': C \to D$ and $G: D \to C$ be functors such that F and F' are left adjoint to G. Then F and F' are naturally isomomorphic. The dual result also holds.*

Before stating the final result we shall need, we begin with a motivating example.

Example 1.8.3 Let P and Q be partially ordered sets and let $f: P \to Q$ be an order-preserving function. Thus P and Q are special kinds of categories, by Example 1.2.3, and f is actually a functor between them. Suppose that in P the elements x and x' have a least upper bound $x \vee x'$. Since $x, x' \leq x \vee x'$ we have that $f(x), f(x') \leq f(x \vee x')$. We cannot deduce that $f(x) \vee f(x')$ exists in Q, and even if it did we can only then deduce that $f(x) \vee f(x') \leq f(x \vee x')$; we cannot say, in general, that they are equal.

Suppose now that there is an order-preserving function $g\colon Q \to P$ such that for all $x \in P$ and $y \in Q$ we have that

$$f(x) \leq y \Leftrightarrow x \leq g(y).$$

In other words, f is left adjoint to g. As before suppose that $x \vee x'$ exists in P. We now use the properties of adjoints. Then $f(x), f(x') \leq f(x \vee x')$. Let $f(x), f(x') \leq y$. Then $x, x' \leq g(y)$. Thus $x \vee x' \leq g(y)$. But this implies that $f(x \vee x') \leq y$. We have proved that the least upper bound of $f(x)$ and $f(x')$ is $f(x \vee x')$. Hence $f(x) \vee f(x') = f(x \vee x')$.

The above example is generalized in the following result. We say that a functor *preserves* limits if the image of a (co)limit under F is the (co)limit of the image of the digram under F.

Lemma 1.8.4 *Let $F\colon C \to D$ and $G\colon D \to C$ be functors such that F is left adjoint to G. Then F preserves colimits and G preserves limits.*

1.9 The category of left A-acts

This category was defined in Example 1.2.6 and will be studied in detail in Section 2.2. We shall now describe some of the important properties of this category that we will use there.

Isomorphisms of left A-acts are simply bijective left A-homomorphisms.

A *subact* of a left A-act M is a subset $N \subseteq M$ such that $AN \subseteq N$.

We defined epimorphisms in arbitrary categories in Section 1.3 and we saw that in the category of sets epimorphisms are surjections this is not true in all categories; for example, in the category of monoids. However we do have the following result proved in [5].

Lemma 1.9.1 *Epimorphisms in A-**Act** are precisely the surjections.*

We shall use this result without further comment throughout Section 2.2.

It can be easily proved that all coequalizers are epimorphisms: see Proposition II.2.18(2) of [16]. However, in general not all epimorphisms are coequalizers. Those that are, are said to be *regular*. In many categories, regular epimorphisms are closer to the idea of what we think of as a quotient. There is a good discussion of this in Chapter 7 of [3]. The following is proved as Theorem II.2.44 of [16].

Lemma 1.9.2 *Every epimorphism in* A-**Act** *is a coequalizer of a pair of left* A*-homomorphisms.*

A *coproduct* in A-**Act** for M and N is the disjoint union

$$M \amalg N = M \times \{1\} \cup N \times \{2\}$$

where we define $a(x, i) = (ax, i)$ for $i = 1, 2$. It can be proved that this really is the coproduct in the category of left A-acts. There is no initial object in the category A-**Act** so all coproducts exist *except* over the empty set.

Lemma 1.9.3 *In* A-**Act** *all coproducts of non-empty sets of objects exist.*

Chapter 2

Morita equivalence of monoids

Classifying structures up to isomorphism is usually an unreachable ideal, so we need a way of classifying structures which is weaker than isomorphism but is still useful. Morita equivalence of structures is just such a way. In this chapter, we shall briefly describe the Morita theory of unital rings and then show in detail how this theory was adapted to monoids. None of the results in this chapter is new. It is based on the papers by Banaschewski [5] and Knauer [19] and the book [16].

2.1 The Morita theory of rings

Morita theory was introduced by Morita in 1958 [35]. In the first instance, the Morita theory of rings was defined for *unital* rings, meaning those rings having an identity. Let R and S be two such rings. In Example 1.2.9, we described the categories of left R-modules and left S-modules. We say that R and S are *Morita equivalent* if these categories are equivalent. This definition is the reason why we had to introduce equivalences of categories in Chapter 1. Although the definition is phrased in terms of left modules it turns out to be self-dual: see Proposition 18.32 of [21].

The detailed theory of Morita equivalence is described in [4]. We do not describe it here because we shall describe the analogous theory for monoids in the next section. Instead, we give an algebraic characterization of Morita equivalence.

If R is a ring then $M_n(R)$ is the set of all $n \times n$ matrices with entries from R. An idempotent e in a ring R is said to be *full* if $R = ReR$.

The following is proved as Proposition 18.33 of [21].

Theorem 2.1.1 (Algebraic characterization) *Let R and S be unital rings. Then R and S are Morita equivalent if and only if S is isomorphic to $eM_n(R)e$ for some finite n and full idempotent e in $M_n(R)$.*

The Morita equivalence of unital rings is an important way of classifying them. A special case is the Artin-Wedderburn theorem. This can be interpreted as saying that a ring is semisimple if and only if it is Morita equivalent to a finite direct product of division rings.

2.2 The Morita theory of monoids

Let A and B be monoids. We say that they are *Morita equivalent* if the category A-**Act** is equivalent to the category B-**Act**.

The obvious question is how can we algebraically determine when A and B are Morita equivalent? This is answered in Theorem 2.2.36.

2.2.1 Cyclic left A-acts

If M and N are two left A-acts we denote the set of all left A-homomorphisms between them either by $\hom_A(M, N)$ or by $\hom(M, N)$.

The set X is called a *generating set* of a left A-act M if and only if $M = AX$. If X contains exactly one element then M is said to be *cyclic*. The set X of generating elements of the left A-act M is said to be a *basis* of M if every element $m \in M$ can be uniquely written as $m = ax$ such that $a \in A$ and $x \in X$. This means that if $m = a_1x_1 = a_2x_2$, then $a_1 = a_2$ and $x_1 = x_2$. If a left A-act M has a basis then it is called a *free act*. In particular, A is a free act with basis $\{1\}$. We describe free left A-acts in Proposition 2.2.20

Let A be a cyclic left A-act of the form Ae where e is an idempotent. Such acts will play an important role in what follows. Here are two useful results about such left A-acts.

Lemma 2.2.1 *Let i and j be idempotents. Then the set of left A-homomorphisms from Ai to Aj is in bijective correspondence with the triples (i, x, j) where $ixj = x$ for some $x \in A$.*

Proof Let $f\colon Ai \to Aj$ be a left A-homomorphism. Put $f(i) = x$. Then $xj = x$ since $x \in Aj$ and $ix = if(i) = f(i) = x$. Thus $ixj = x$. Conversely, let $x \in A$ such that $ixj = x$. Define $f\colon Ai \to Aj$ by $f(a) = ax$. This is a left A-homomorphism. It is clear that this defines a bijection, as claimed. ■

In the following result, it is important to write the arguments of left A-homomorphisms on the left rather than on the right. Thus the value of f at x is written $(x)f$ instead of $f(x)$.

Lemma 2.2.2 *Let A be a monoid and $e^2 = e \in A$. Then the monoid $\hom_A(Ae, Ae)$ is isomorphic to the monoid eAe.*

Proof Define $F\colon \hom(Ae, Ae) \to eAe$ by $F(f) = (e)f$. Since $(e)f \in Ae$ and $(e)f = (ee)f = e(e)f$ we have that $(e)f \in eAe$. Thus F is a function from $\hom(Ae, Ae)$ to eAe. We show that it is a monoid homomorphism. First, $F(1_{Ae}) = (e)1_{Ae} = e$ the identity of eAe. Next, let $f, g \in \hom(Ae, Ae)$. Then $(e)f \in eAe$ and so there is $a \in A$ such that $(e)f = eae$. Thus

$$F(fg) = (e)(fg) = ((e)f)g = (eae)g = (eae)((e)g) = (e)f(e)g = F(f)F(g).$$

It follows that F is a monoid homomorphism from $\hom(Ae, Ae)$ to eAe.

We show now that F is a bijection. Let f_1 and f_2 be such that $(e)f_1 = (e)f_2$. Then for each $ae \in Ae$ where $a \in A$, we have that $(ae)f_1 = a(e)f_1 = a(e)f_2 = (ae)f_2$. Thus $f_1 = f_2$ and F is injective. Finally, let $eae \in eAe$. Define $f \in \hom(Ae, Ae)$ by $(be)f = beae$. Then f is a left A-homomorphism and $(e)f = eae$, as required. ■

The following sections do the following

- Sections 2.2.2, 2.2.3, and 2.2.4 show how to construct functors between A-**Act** and B-**Act**.
- Sections 2.2.5, 2.2.6 and 2.2.7 describe special kinds of left A-acts.
- Section 2.2.8 determines when two monoids are Morita equivalent.

2.2.2 Hom functors

We defined the Morita equivalence of A and B in terms of an equivalence between the category of left A-acts and the category of left B-acts. Therefore we will be interested in defining functors from A-**Act** to B-**Act**. In this section, and the next, we shall describe two different ways of doing this.

Let A be a monoid and P a left A-act. Define

$$\hom(P, -)\colon A\text{-}\mathbf{Act} \to \mathbf{Set}$$

as follows: for each left A-act X, we have that $\hom(P, X)$ is the set of all left A-homomorphisms from P to X; for each left A-homomorphism $f\colon X \to Y \in A$-**Act** define

$$\hom(P, f)\colon \hom(P, X) \to \hom(P, Y)$$

by $\hom(P, f)(g) = fg$ for each $g \in \hom(P, X)$. We often write $\hom(P, f)(g) = f_*(g) = fg$.

Lemma 2.2.3 *The function* $\hom(P, -)$ *is a functor from* A-**Act** *to* **Set**.

Proof Let X be a left A-act and and let $1_X\colon X \to X$ be the identity left A-homomorphism. Then $(1_X)_*$ is just the identity function on $\hom(P, X)$. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two left A-homomorphisms. Then f_*g_* is defined and is equal to $(fg)_*$. We have proved that $\hom(P, -)$ is a functor.■

Functors such as this are called *hom functors*.

We would like to define a functor from A-**Act** to B-**Act** and the above lemma only tells us how to construct a functor from A-**Act** to the category of sets. We can do this by ensuring that P has extra structure.

Theorem 2.2.4 *Let* P *be an* (A, B)*-biact. Then*

$$\hom_A(P, -)\colon A\text{-}\mathbf{Act} \to B\text{-}\mathbf{Act}$$

is a functor.

Proof We show first that for each left A-act X, the set $\hom(P, X)$ is a left B-act. If $f \in \hom(P, X)$ and $b \in B$ then define $b \cdot f$ by $(b \cdot f)(p) = f(pb)$ for any $p \in P$. Observe that

$$((bb') \cdot f)(p) = f(p(bb')) = f((pb)b') = (b' \cdot f)(pb) = (b \cdot (b' \cdot f))(p).$$

Let $h\colon X \to Y$ be a left A-homomorphism. Then $\hom(P, h)\colon \hom(P, X) \to \hom(P, Y)$. Let $g \in \hom(P, X)$. We show that $h_*(b \cdot g) = b \cdot h_*(g)$. By definition

$$(h_*(b \cdot g))(p) = (h(b \cdot g))(p) = h((b \cdot g)(p)) = h(g(pb)) = (hg)(pb)$$

$$= (b \cdot (hg))(p) = (b \cdot h_*(g))(p)$$

Thus $h_*(b \cdot g) = b \cdot h_*(g)$.

It is now easy to show that this defines a functor. ■

2.2.3 Tensor functors

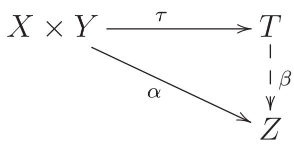
We now describe a second technique for constructing functors. To do this we need a new construction. Most of the material from this section is based on Section 8.1 of [15].

Let X be a right A-act, Y a left A-act and Z a set. Then a map

$$\alpha: X \times Y \to Z$$

is said to be *A-balanced* if $\alpha(xa, y) = \alpha(x, ay)$ for each $x \in X$, $y \in Y$ and $a \in A$.

Let X be a right A-act and let Y be a left A-act. Then a set T together with an A-balanced map $\tau : X \times Y \to T$ is called a *tensor product* of X and Y if for each set Z and for each balanced map $\alpha: X \times Y \to Z$ there is a unique map $\beta: T \to Z$ making the following diagram commute



The following is a standard uniqueness argument in category theory.

Lemma 2.2.5 *Let (T, τ) and (T', τ') be two tensor products of X and Y. Then T and T' are isomorphic in the category of sets.*

We now show that tensor products exist. To do this, we begin by recalling some standard definitions and results.

Let X be a set and let ρ be an equivalence on X. For each $x \in X$, we put

$$\rho(x) = \{y \in X : x \rho y\},$$

called an *equivalence class*. The set

$$X/\rho = \{\rho(x) : x \in X\}$$

of equivalence classes forms a partition of X called the *quotient set* of X. Let X be a set. Then

$$1_X = \{(x, x) \in X \times X : x \in X\}$$

is the equality relation on X. Let R be any relation on a set X. Then R^{-1} is the *opposite relation* on X defined by $(x, y) \in R^{-1}$ if and only if $(y, x) \in R$. The relation $R \circ R$ is defined by $(x, z) \in R \circ R$ if and only if there is $y \in X$ such that $(x, y), (y, z) \in R$. We sometimes write R^2 instead of $R \circ R$, and more generally, R^n for $R \circ R \circ \cdots \circ R$ composed n times.

Lemma 2.2.6 *Let R be a relation on a set X. Then the intersection of any set of equivalence relations containing R is again an equivalence relation.*

Let R be a relation on a set X. Then the intersection of all equivalence relations on X containing R, which is an equivalence relation by the above lemma, is called the equivalence relation *generated by* R, and is denoted by R^e. We now describe R^e more explicitly.

Let S be a reflexive relation on a set X. This means that $1_X \subseteq S$. Then $S = S \circ 1_X \subseteq S \circ S$. Thus $S \subseteq S^2$. It follows that

$$S \subseteq S^2 \subseteq S^3 \subseteq \cdots.$$

The relation

$$S^\infty = \bigcup_{n=1}^{\infty} S^n$$

is called the *transitive closure of the relation S*.

Lemma 2.2.7 *For every reflexive relation S on a set X, the relation S^∞ is the smallest transitive relation on X containing S.*

We now have the following important result.

Proposition 2.2.8 *Let R be any relation on a set X. Then*

$$R^e = (R \cup R^{-1} \cup 1_X)^\infty.$$

Proof Put $E = (R \cup R^{-1} \cup 1_X)^\infty$. From Lemma 2.2.7 the relation E is a transitive relation containing R. Since

$$1_X \subseteq R \cup R^{-1} \cup 1_X \subseteq E,$$

the relation E is also reflexive. The relation $S = R \cup R^{-1} \cup 1_X$ is symmetric because $S^{-1} = (R \cup R^{-1} \cup 1_X)^{-1} = R^{-1} \cup R \cup 1_X = S$. It follows that for each $n \geq 1$

$$S^n = (S^{-1})^n = (S^n)^{-1}.$$

Thus S^n is symmetric. If $(x, y) \in E$, then there is $n \geq 1$ such that $(x, y) \in S^n$ and so $(y, x) \in S^n$. Thus $(y, x) \in E$. It follows that $E = S^\infty$ is symmetric. We have proved that E is an equivalence relation containing R.

Suppose σ is an equivalence relation containing R. Then $1_X \subseteq \sigma$ and $R^{-1} \subseteq \sigma^{-1} = \sigma$. Thus

$$S = R \cup R^{-1} \cup 1_X \subseteq \sigma.$$

But

$$S \circ S \subseteq \sigma \circ \sigma = \sigma,$$

and so $S^n \subseteq \sigma$ for all $n \geq 1$. It follows that $E = S^\infty \subseteq \sigma$. We have proved that $E = (R \cup R^{-1} \cup 1_X)^\infty$ is the smallest equivalence relation on X containing R. Thus

$$R^e = (R \cup R^{-1} \cup 1_X)^\infty$$

as required. ■

We can write Proposition 2.2.8 in more explicit way.

Proposition 2.2.9 *Let R be a relation on a set X, and let R^e be the smallest equivalence on X containing R. Then $(x, y) \in R^e$ if and only if either $x = y$ or for some $n \geq 2$, there is a sequence*

$$x = z_1 \to z_2 \to \cdots \to z_n = y$$

in which, for each $i = 1, 2, \ldots, n-1$, either $(z_i, z_{i+1}) \in R$ or $(z_{i+1}, z_i) \in R$.

We may now show that tensor products exist.

Theorem 2.2.10 *Let X be a right A-set and B a left A-set. Let σ be the equivalence relation on $X \times Y$ generated by*

$$\Sigma = \{((x, ay), (xa, y)) \colon a \in A, x \in X, y \in Y\}.$$

Denote by $x \otimes y$ the equivalence class $\sigma(x, y)$, put $X \otimes Y = (X \times Y)/\sigma$ and define $\tau \colon X \times Y \to X \otimes Y$ by $\tau(x, y) = x \otimes y$. Then $(X \otimes Y, \tau)$ is the tensor product of X and Y.

From the definition of σ, we see that

$$xa \otimes y = x \otimes ay$$

for all $x \in X$, $a \in A$ and $y \in Y$. The following is now an immediate deduction from the theory developed above.

Proposition 2.2.11 *The two elements $x \otimes y$ and $x' \otimes y'$ are equal in $X \otimes Y$ if and only if either $(x, y) = (x', y')$ or for some $n \geq 2$ there is a sequence*

$$(x, y) = (x_1, y_1) \to (x_2, y_2) \to \cdots \to (x_n, y_n) = (x', y')$$

in which for each $i = 1, \ldots, n-1$, either $((x_i, y_i), (x_{i+1}, y_{i+1})) \in \Sigma$ or $((x_{i+1}, y_{i+1}), (x_i, y_i)) \in \Sigma$.

So tensor products are just sets. To get more, we need to assume more about the acts we start with.

Proposition 2.2.12 *Let P be an (A, B)-biact and Q be a (B, C)-biact. Define $a(p \otimes q) = ap \otimes q$ and $(p \otimes q)c = p \otimes qc$ for each $p \otimes q \in P \otimes Q$ and $a \in A$ and $c \in C$. Then $P \otimes Q$ is an (A, C)-biact.*

Proof By symmetry, it is enough to prove that $P \otimes Q$ is a left A-act. To do this, it is enough to show that the definition of the action is well-defined. That is if $p \otimes q = p' \otimes q'$ then $ap \otimes q = ap' \otimes q'$. But this follows from Proposition 2.2.11. ■

The following result will be important in showing that the Morita theory of semigroups with local units is a generalization of the Morita theory of monoids.

Proposition 2.2.13 *Let X be a left A-act. Define the map $\mu_X : A \otimes X \to X$ by $a \otimes x \mapsto ax$. Then μ_X is a well-defined left A-isomorphism.*

Proof Observe that the function from $A \times X$ to X given by $(a, x) \mapsto ax$ is balanced. Thus μ_X is a well-defined map. It is clearly a left A-homomorphism and it is surjective because if $x \in X$ then $1 \cdot x = x$ and so $\mu_X(1 \otimes x) = x$. It remains to show that it is injective. Suppose that $ax = a'x'$. Then

$$a \otimes x = 1 \otimes ax = 1 \otimes a'x' = a' \otimes x',$$

as required. ■

To finish off this section, we shall show that tensor products can be used to define functors.

Let X and X' be right A-acts and let Y and Y' be left A-acts. Let $f : X \to X'$ be a right A-homomorphism and let $g : Y \to Y'$ be a left A-homomorphism. Define a map from $X \times Y$ to $X' \otimes Y'$ by $(x, y) \mapsto f(x) \otimes g(y)$. This is a balanced map and so there is a unique function from $X \otimes Y$ to $X' \otimes Y'$ that maps $x \otimes y$ to $f(x) \otimes g(x)$. We denote this map by $f \otimes g$.

Lemma 2.2.14 *Let Q be a right A-act. Define*

$$Q \otimes - : A\text{-}\mathbf{Act} \to \mathbf{Set}$$

as follows: it takes each left A-act P to the set $Q \otimes P$ and each left A-homomorphism $f : X \to Y$ to the function $1_Q \otimes f : Q \otimes X \to Q \otimes Y$. Then it is a functor.

The most interesting version of the above construction is the following.

Theorem 2.2.15 *Let P be an (A, B)-biact. Then*

$$P \otimes - : B\text{-}\mathbf{Act} \to A\text{-}\mathbf{Act}$$

is a functor.

2.2.4 Adjointness of hom and tensor functors

Let P be an (A, B)-biact. Then by Theorems 2.2.4 and 2.2.15, we have defined two functors going in opposite directions

$$\hom_A(P, -) : A\text{-}\mathbf{Act} \to B\text{-}\mathbf{Act}$$

and

$$P \otimes - : B\text{-}\mathbf{Act} \to A\text{-}\mathbf{Act}.$$

We now explain the connection between these two functors. This is where the notion of an adjoint, defined in Chapter 1, is used.

Let X be a left B-act and let Y be a left A-act. We have that $P \otimes X$ is a left A-act and so we can look at the set $\hom_A(P \otimes X, Y)$ of all left A-homomorphisms from $P \otimes X$ to Y. Now $\hom_A(P, Y)$ is a left B-act and so we can look at the set $\hom_B(X, \hom_A(P, Y))$ of all left B-homomorphisms from X to $\hom_A(P, Y)$.

Let $f: P \otimes X \to Y$ be a left A-homomorphism. Thus $f(p \otimes x) = y$. Fix x. Then for each $p \in P$ we have a map $g(x)$ defined by $g(x)(p) = f(p \otimes x)$. Thus $g: X \to \hom_A(P, Y)$. We need to check that g is a left B-homomorphism. That is, we need to check that $bg(x) = g(bx)$ for all $b \in B$ and $x \in X$. That is, $g(x)(pb) = g(bx)(p)$ for all $p \in P$. But $g(x)(pb) = f(pb \otimes x)$ and $g(bx)(p) = f(p \otimes bx)$. But these two elements are equal by the definition of the tensor product. We have therefore defined a function

$$\phi_{X,Y}: \hom_A(P \otimes X, Y) \to \hom_B(X, \hom_A(P, Y)).$$

This function is then the basis for proving the following, together with Lemma 1.8.4.

Theorem 2.2.16 *Let P be an (A, B)-biact. Then $P \otimes -$ is left adjoint to* $\hom_A(P, -)$. *It follows that $P \otimes -$ preserves colimits.*

2.2.5 Indecomposables

In the next three sections, we shall describe some important kinds of left A-acts: indecomposables, projectives and generators. The main outcome will be a complete description, in Theorem 2.2.33, of the indecomposable projective generators in the category of left A-acts.

The left A-act M is *decomposable* if it is possible to find two nonempty subacts M_1, M_2 of M such that M is left A-isomorphic to the coproduct which is the disjoint union here of M_1 and M_2, otherwise M is *indecomposable*. We think of the indecomposable acts as being the 'atoms' from which other acts can be constructed using coproducts to glue them together. This is made precise in Proposition 2.2.19 below.

Lemma 2.2.17 *Cyclic A-acts are indecomposable.*

Proof Let $P = Ax$ be a cyclic left A-act and the same will be for the cyclic right A-act. Assume that P_1 and P_2 are two subacts of P such that $P = P_1 \cup P_2$ and $P_1 \cap P_2 = \emptyset$. Then $x \in P_1$ or $x \in P_2$. Without loss of generality suppose that $x \in P_1$. From the definition, P_1 is a left $A-$act and so $Ax \subseteq P_1$, but $P_1 \subseteq P = Ax$. Thus $P_1 = Ax$. This means that $P_2 \subseteq P_1$ which is a contradiction. It follows that P is indecomposable. ∎

Lemma 2.2.18 *Let P be a left $A-$act and let*

$$\{P_i \subseteq P : i \in I\}$$

be a set of subacts of P such that P_i is indecomposable for each i and $\bigcap_{i\in I} P_i \neq \emptyset$. Then $\bigcup_{i\in I} P_i$ is an indecomposable subact of P.

Proof Suppose that

$$\bigcup_{i\in I} P_i = M \cup N$$

where M and N are disjoint subacts. Since $\bigcap_{i\in I} P_i \neq \emptyset$, there is $x \in \bigcap_{i\in I} P_i$. Thus $x \in P_i$ for all $i \in I$. The element x must belong to either M or N. Without loss of generality suppose that $x \in M$. Then $x \in P_i \cap M$ for all i. Since $\bigcup_{i\in I} P_i = M \cup N$, we have that $P_i \subseteq M \cup N$ for all i. Therefore

$$P_i = P_i \cap (M \cup N) = (P_i \cap M) \cup (P_i \cap N)$$

for all i. But P_i is indecomposable and $x \in P_i \cap M$. Therefore $P_i \cap N = \emptyset$ for each $i \in I$. Hence $\bigcup_{i\in I} P_i \cap N = \emptyset$. It follows that $N = \emptyset$ and so we have proved that $\bigcup_{i\in I} P_i$ is indecomposable. ∎

Proposition 2.2.19 *Every left A-act P can be uniquely written as a coproduct of indecomposable subacts of P.*

Proof Let $x \in P$. Define U_x to be the set of all indecomposable subacts of P containing x. The set U_x is non-empty because $x \in Ax \subseteq P$, and Ax is indecomposable by Lemma 2.2.17. All the subacts in U_x contain x. It follows that $\bigcap U_x \neq \emptyset$. By Lemma 2.2.18, $S_x = \bigcup U_x$ is an indecomposable subact of P containing x. Let $x, y \in P$. Suppose that $S_x \cap S_y \neq \emptyset$ and let $z \in S_x \cap S_y$. We shall prove that $S_x = S_z$ and, by symmetry, it follows that $S_y = S_z$, and so $S_x = S_y$. Now S_x is an indecomposable subact of P containing z and so $S_x \in U_z$. Hence $S_x \subseteq S_z$. Now $z \in S_x$ and $S_x = \bigcup U_x$. Thus there is $U \in U_x$ such that $z \in U$. Now U is an indecomposable subact of P containing z and so $U \in U_z$ but also U contains x . It follows that $\bigcup U_z$ is an indecomposable subact of P containing x. Thus $\bigcup U_z \in U_x$ giving $S_z \subseteq S_x$. Hence $S_x = S_z$, as claimed. We have proved that $S_x = S_z = S_y$. If we choose $P' \subseteq P$ such that for each distinct $x, y \in P'$ we have that $S_x \cap S_y = \emptyset$, then $P = \amalg_{x \in P'} S_x$. ■

2.2.6 Projectives

The notion of a projective object plays a key role in the theory of Morita equivalence. At the end of this section, we shall have described all projectives in the category of left A-acts in an explicit way. We begin with a description of the free left A-acts.

Proposition 2.2.20

1. *Let X be a set and let A be a monoid. Put $F(X) = A \times X$ and define a left action by $a \cdot (b, x) = (ab, x)$ for each $a \in A$ and $(b, x) \in A \times X$. Then $F(X)$ is a free left A-act with basis $\{1\} \times X$.*

2. *Let $F(X)$ be a free left A-act with basis $\{1\} \times X$. If φ is any map from $\{1\} \times X$ to a left A-act M, then there exists a unique left A-homomorphism $\psi\colon F(X) \to M$ such that $\psi|_X = \varphi$.*

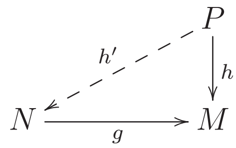
Proof (1) It is easy to check that $F(X)$ is a left A-act. Each $(a, x) \in F(X)$ can be written $(a, x) = a(1, x)$. Put $X' = \{1\} \times X$. Then $F(X) = AX'$ and each element of $F(X)$ can be written uniquely as a product of an element of

A and an element of X'. It follows that $F(X)$ is a free left A-act with basis X'.

(2) Define $\psi\colon F(X) \to M$ by $\psi((a,x)) = a\varphi(x)$. The result now follows. ■

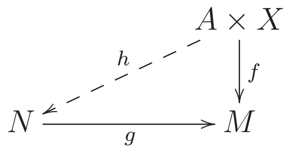
We usually think of $F(X)$ as having basis X by identifying X and X'.

An object P in a category C is said to be *projective* if for each arrow $h\colon P \to M$ and every epimorphism $g : N \to M$ there is an arrow $h'\colon P \to N$ making the following diagram commute



Lemma 2.2.21 *Every free left A-act in the category A-**Act** is projective.*

Proof Let $F(X)$ be a free left A-act. Let f be a left A-morphism from $F(X)$ to M. Let $g\colon N \to M$ be an epimorphism. We know that the epimorphisms are surjective in this category. Thus for each $(1,x) \in X'$, we may choose $n_x \in N$ such that $g(n_x) = f(1,x)$. Define $h(1,x) = n_x$. Then by Proposition 2.2.20, we may extend h to a left A-homomorphism from $F(X)$ to N. By construction $g(h(1,x)) = f(1,x)$, and so, again by Proposition 2.2.20, we have that $gh = f$.



Thus $F(X)$ is projective. ■

We say that a category C has *enough projectives* if and only if for each object M in C there is a projective object P in C and an epimorphism $f\colon P \to M$.

Lemma 2.2.22 *The category A-**Act** has enough projectives.*

Proof Let M be a left A-act. Let X be any set such that $M = AX$. Then $A \times X$ is the free left A-act generated by X, and so from Lemma 2.2.21 $A \times X$ is projective. Define $f\colon A \times X \to M$ by $f(a,x) = ax$. This is a surjective left A-homomorphism. It follows that the category has enough projectives. ■

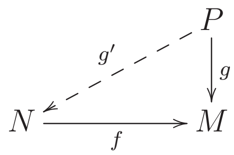
Proposition 2.2.23 *Let P be a left A-act. Then the following conditions are equivalent:*

1. *P is projective.*

2. *The functor $\hom_A(P, -)$: A-**Act** $\to$ **Set** preserves epimorphisms.*

3. *Every epimorphism $M \to P$ has a right inverse.*

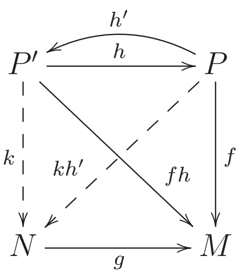
Proof (1) $\Rightarrow$ (2). Let P be projective and let $f\colon N \to M$ be an epimorphism. By definition $\hom(P, f)\colon \hom(P, N) \to \hom(P, M)$ is given by $g \mapsto fg$, left multiplication by f. Let $g \in \hom(P, M)$. Since P is projective there is an arrow $g'\colon P \to N$ making the following diagram commute



But $g' \in \hom(P, N)$ and satisfies $g = fg'$. It follows that $\hom(P, f)$ is surjective and so an epimorphism.

(2) $\Rightarrow$ (3). Let $f\colon M \to P$ be an epimorphism. By (2), the function $\hom(P, f) : \hom(P, M) \to \hom(P, P)$ is an epimorphism. Thus given $1_P \in \hom(P, P)$, there is an arrow $f'\colon P \to M$ such that $ff' = 1_P$. It follows that f has a right inverse.

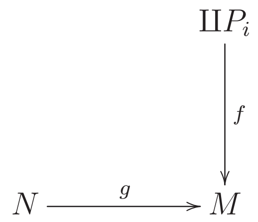
(3) $\Rightarrow$ (1). Let P be such that (3) holds. We prove that P is projective. Let $f\colon P \to M$ be a a left A-homomorphism and let $g\colon N \to M$ be an epimorphism. In Lemma 2.2.22, we proved that this category has enough projectives and so there is a projective P' and an epimorphism $h\colon P' \to P$. But from (3), h has a right inverse $h'\colon P \to P'$ so that $hh' = 1_P$. We have that $fh\colon P' \to M$ and we have an epimorphism $g\colon N \to M$. But P' is projective, and so there is a left A-homomorphism $k\colon P' \to N$ such that $gk = fh$.



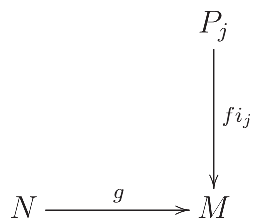
Observe that $kh'\colon P \to N$ and that $g(kh') = fhh' = f$. Therefore P is projective. ■

Proposition 2.2.24 *Let P_i be a set of left A-acts where $i \in I$. Then the coproduct $\amalg_{i\in I} P_i$ is projective if and only if P_i is projective for each $i \in I$.*

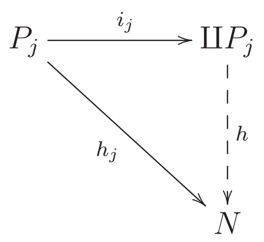
Proof Suppose that all the P_i are projective. Consider the following diagram



where g is an epimorphism. From the definition of the coproduct, there are left A-homomorphisms $i_j\colon P_j \to \amalg P_i$. Thus for each j, we have the following diagram



But each P_j is projective and so there is an A-homomorphism $h_j\colon P_j \to N$ making the diagram above commute. Thus $gh_j = fi_j$ for each j. We now use the definition of the coproduct. Since we have A-homomorphisms $h_j\colon P_j \to N$ there is a unique$A$-homomorphism $h\colon \amalg P_i \to N$ making the following diagram commute



Thus $hi_j = h_j$ for all j. But $ghi_j = fi_j$ for each j. We deduce that $gh = f$, again from the definition of coproducts.

To prove the converse, suppose that $\amalg P_i$ is projective. We shall prove that P_j is projective for each j. In what follows I denotes the one element set on which A acts by fixing the unique element. Choose j and suppose we have the following diagram

$$\begin{array}{ccc} & & P_j \\ & & \big\downarrow{\scriptstyle f} \\ N & \xrightarrow{\;g\;} & M \end{array}$$

where g is an A-epimorphism. Let $i : I \to I$ be the identity A-homomorphism. Let $k : \amalg_{i \neq j} P_i \to I$ be the unique A-homomorphism mapping all elements to the only element of I. From the A-homomorphisms $g : N \to M$ and $i : I \to I$ we have an A-homomorphism $g \amalg i : N \amalg I \to M \amalg I$. From the A-homomorphisms $f : P_j \to M$ and $k : \amalg_{i \neq j} P_i \to I$ there is an A-homomorphism $f \amalg k : P_j \amalg (\amalg_{i \neq j} P_i) \to M \amalg I$ Thus we have a diagram

$$\begin{array}{ccc} & & P_j \amalg (\amalg_{i \neq j} P_i) \\ & \swarrow{\scriptstyle \overline{q}} & \big\downarrow{\scriptstyle f \amalg k} \\ N \amalg I & \xrightarrow[g \amalg i]{} & M \amalg I \end{array}$$

and an A-homomorphism $\overline{q}$ making the above diagram commute because $P_j \amalg (\amalg_{i \neq j} P_i) = \amalg P_i$ is projective. We now define $q : P_j \to N$ such that $q \amalg k = \overline{q}$, this q exists because if we look to the last diagram which is commute and take an element of $P_j \amalg (\amalg_{i \neq j} P_i)$ then this element will be either from $P_j \times \{j\}$ or from the disjoint union of P_i where $i \neq j$. If the element is in $P_j \times \{j\}$ then the composition will give another commutative daigram amonge P_j and N and M. Thus such q exists. Then

$$(g \amalg i)(q \amalg k) = f \amalg k.$$

We have the following

$$P_j \amalg (\amalg_{i \neq j} P_i) \xrightarrow{q \amalg k} N \amalg I \xrightarrow{g \amalg i} M \amalg I$$

$$P_j \amalg (\amalg_{i \neq j} P_i) \xrightarrow{f \amalg k} M \amalg I$$

and so from the property of the coproduct of arrows we have

$$(f \amalg k) i_j = i_M f$$

$$(g \amalg i) i_N = i_M g$$

$$(q \amalg k) i_j = i_N q$$

It follows that

$$(g \amalg i)(q \amalg k) i_j = (f \amalg k) i_j$$

and so

$$(g \amalg i) i_N q = i_M f$$

it follows that $i_M g q = i_M f$. Thus for each $p \in P_j$ we have $gq(p) \in M$ and so

$$gq(p) = i_M(gq(p)) = i_M gq(p) = i_M f(p) = i_M(f(p)) = f(p).$$

We have proved that $gq = f$ and so P_j is projective for each j. ■

Proposition 2.2.25 *A projective left A-act P is indecomposable if and only if P is cyclic.*

Proof If P is cyclic then it is indecomposable by Lemma 2.2.17. Suppose, therefore, that P is indecomposable projective. Let

$$\hom(A, P) = \{f_i : i \in I\}.$$

Define

$$f : \amalg_{i \in I}(A \times \{i\}) \to P$$

by $f(a, i) = f_i(a)$ for each $i \in I$ and $a \in A$. Then f is a left A-homomorphism. We prove that f is an epimorphism. Let $x \in P$. Define $h : A \to P$ by $h(a) = ax$ for each $a \in A$. Then h is a left A-homomorphism and so $h = f_i$ for some $i \in I$. Hence

$$f(1, i) = f_i(1) = h(1) = 1x = x$$

and so f is an epimorphism. Since P is projective and f is an epimorphism, there is by Proposition 2.2.23 a left A-homomorphism

$$g : P \to \amalg_{i \in I}(A \times \{i\})$$

such that $fg = 1_P$. Observe that g is injective. Thus P is isomorphic to a subact of $\amalg_{i \in I}(A \times \{i\})$. This will be of the form $\amalg(B_i \times \{i\})$ where B_i is a subact of A, possibly empty. However if there is more than one $B_i \neq \emptyset$ then P would be isomorphic to a coproduct of more than two acts, but P is indecomposable. Thus the image of P under g will be contained within one of the $A \times \{i\}$. This means that $g(P) \subseteq A \times \{i\}$ for some $i \in I$. Therefore

$$P = f(g(P)) \subseteq f(A \times \{i\}) \subseteq P.$$

Thus, $P = f(A \times \{i\})$. But $A \times \{i\} = A(1, i)$ and so $A \times \{i\}$ is cyclic. But the image of a cyclic A-act under an A-homomorphism is cyclic. It follows that P is cyclic. ■

Theorem 2.2.26 *The left A-act P is indecomposable projective if and only if P is isomorphic to a left A-act of the form Ae where e is an idempotent in A.*

Proof We show first that left A-acts of the form Ae where e is an idempotent are indecomposable and projective. They are clearly indecomposable by Lemma 2.2.17. To show that they are projective we shall use Proposition 2.2.23. Let $f \colon M \to Ae$ be an epimorphism. Thus it is a surjective left A-homomorphism. It follows that there is $m \in M$ such that $f(m) = e$. Define $g \colon Ae \to M$ by $g(ae) = aem$ for each $ae \in Ae$. Then g is a left A-homomorphism. We calculate fg on Ae

$$(fg)(ae) = f(g(ae)) = f(aem) = aef(m) = aee = ae^2 = ae.$$

It follows that $fg = 1_{Ae}$ and so P is projective.

To prove the converse, let P be an indecomposable projective. Then by Proposition 2.2.25, we have that P is cyclic. Therefore there is $z \in P$ such that $P = Az$. Define $f \colon A \to Az$ by $f(a) = az$. Then f is a surjective left A-homorphism. Thus by Proposition 2.2.23, there is a left A-homomorphism $g \colon Az \to A$ such that $fg = 1_{Az}$. Let $g(z) = a$. Then

$$z = 1_{Az}(z) = (fg)(z) = f(g(z)) = f(a) = az.$$

Therefore $az = z$. We have that

$$a^2 = aa = ag(z) = g(az) = g(z) = a.$$

It follows that a is an idempotent. We show that g is a bijection. Suppose that $bz, cz \in Az$ are such that $g(bz) = g(cz)$. Then $f(g(bz)) = f(g(cz))$ and so $(fg)(bz) = (fg)(cz)$ giving $bz = cz$. Also $g(Az) = Aa$. Hence $g\colon Az \to Aa$ is a left A-isomorphism, as required. ■

We can deduce the following theorem from Proposition 2.2.19 and Proposition 2.2.24 and Theorem 2.2.26.

Theorem 2.2.27 *In A-**Act**, P is projective if and only if P is isomorphic to $\coprod_{i \in I} Ae_i$ where $e_i^2 = e_i \in A, i \in I$.*

2.2.7 Generators

A left A-act G in the category A-**Act** is a *generator* if for each pair of left A-acts X and Y and for each pair of left A-homomorphisms $f, g\colon X \to Y$ such that $f \neq g$ then there exists a left A-homomorphism $h\colon G \to X$ such that $fh \neq gh$.

Lemma 2.2.28 *In the category A-**Act**, the left A-act A is a generator.*

Proof Let X and Y be two left A-acts and let $f, g\colon X \to Y$ be two left A-homomorphisms such that $f \neq g$. It follows that there is $x \in X$ such that $f(x) \neq g(x)$. Define $h\colon A \to X$ by $h(a) = ax$ for each $a \in A$. This is a left A-homomorphism. Now observe that

$$(fh)(1) = f(h(1)) = f(1x) = 1f(x) = f(x)$$

and

$$(gh)(1) = g(h(1)) = g(1x) = g(x).$$

But $f(x) \neq g(x)$ and so $(fh)(1) \neq (gh)(1)$. It follows that $fh \neq gh$ and so A is a generator. ■

Lemma 2.2.29 *Let G be a generator in A-**Act**. For each left A-act M, put $\hom(G, M) = \{f_i\colon i \in I\}$. Then there is an epimorphism $f\colon \amalg_{i \in I} G \times \{i\} \to M$.*

Proof The first step of the proof, which was omitted by Knauer [19] and corrected in Lemma II.3.12 of [16], is to show that $\hom(G, M)$ is non-empty. But this follows from the fact that the two injections from M into $M \coprod M$ are distinct and so can be separated by the generator G.

Define f by $f(a, i) = f_i(a)$ for each $(a, i) \in G \times \{i\}$. This is clearly a left A-homomorphism. We prove now that f is an epimorphism. Let $h_1, h_2 \colon M \to N$ be two left A-homomorphisms such that $h_1 f = h_2 f$. Suppose that $h_1 \neq h_2$. Then since G is a generator there is $f_i \colon G \to M$ such that $h_1 f_i \neq h_2 f_i$. In particular, there is an element $a \in G$ such that $h_1 f_i(a) \neq h_2 f_i(a)$. Thus $h_1 f(a, i) \neq h_2 f(a, i)$ for some $(a, i) \in G \times \{i\}$. It follows that $h_1 f \neq h_2 f$, which is a contradiction. We have proved that $h_1 = h_2$ and so f is an epimorphism. ∎

Proposition 2.2.30 *In the category A-**Act**, the following conditions are equivalent:*

1. *G is a generator.*

2. *The functor $\hom_A(G, -)\colon A$-**Act** $\to$ **Set** is faithful.*

3. *There exists an epimorphism $f \colon G \to A$.*

Proof (1) ⇒ (2). Suppose that $f, g \colon X \to Y$ such that $f \neq g$. Then there is $h \colon G \to X$ such that $fh \neq gh$. Thus $\hom(G, f)(h) \neq \hom(G, g)(h)$ and so $\hom(G, f) \neq \hom(G, g)$. It follows that $\hom(G, -)$ is faithful.

(2) ⇒ (1). Let X and Y be two left A-acts and let $f, g \colon X \to Y$ be two left A-homomorphisms such that $f \neq g$. Then $\hom(G, f) \neq \hom(G, g)$ and so there is an arrow $h \colon G \to X \in \hom(G, X)$ such that $fh \neq gh$. It follows that G is a generator.

(3) ⇒ (1). Suppose that there exists an epimorphism $f \colon G \to A$. Let X and Y be two left A-acts, and $g_1, g_2 \colon X \to Y$ two left A-homomorphisms such that $g_1 \neq g_2$. Since A is a generator, there is a left A-homomorphism $h \colon A \to X$ such that $g_1 h \neq g_2 h$. Now consider the map $hf \colon G \to X$. If $g_1 hf = g_2 hf$ then $g_1 h = g_2 h$ since f is an epimorphism and so right cancellable. It follows that $g_1 hf \neq g_2 hf$. Hence G is a generator.

(1) ⇒ (3). Let G be a generator. Then by Lemma 2.2.29, there exists an epimorphism $g \colon \coprod_{i \in I} G \times \{i\} \to A$. But from Theorem 2.2.26, A is indecomposable projective. Therefore from Proposition 2.2.23, there is a left

A-homomorphism $h: A \to \coprod_{i \in I} G \times \{i\}$ such that $gh = 1_A$. Since A is indecomposable, there is $i \in I$ such that $h(A) \subseteq G \times \{i\}$. Define $f: G \times \{i\} \to A$ by $f(x,i) = g(x,i)$. Then $fh: A \to A$ and for each $a \in A$ we have $h(a) \in G \times \{i\}$. Therefore

$$(fh)(a) = f(h(a)) = g(h(a)) = (gh)(a) = a$$

and so $fh = 1_A$. It follows that f is an epimorphism. ∎

Proposition 2.2.31 *Let G be a left A-act generated by $Z = \{z_i : i \in I\}$. Then G is a generator if and only if for each $i \in I$, there is an $a'_i \in A$ such that*

1. *if $b, c \in A$ and $bz_i = cz_j$ then $ba'_i = ca'_j$.*
2. *for some $j \in I$, there is $a'' \in A$ such that $a''a'_j = 1$.*

Proof Let G be a generator. Then by Proposition 2.2.30 there is an epimorphism $f: G \to A$. Define $f(z_i) = a'_i$.

(1) holds. Suppose that $bz_i = cz_j$. Then applying f we get that $ba'_i = ca'_j$.

(2) holds. There is some $a \in G$ such that $f(a) = 1$. By assumption $a = a''z_j$. Thus $1 = a''a'_j$.

To prove the converse, we assume that (1) and (2) hold and prove that G is a generator. To do this we shall construct an epimorphism $f: G \to A$ and the result will follow by Proposition 2.2.30. Define $f: G \to A$ by $f(az_i) = aa'_i$. We need to show that this function is well-defined. Suppose that $az_i = bz_j$. Then by (1), we have that $aa'_i = ba'_j$. Thus f is well-defined and it is easy to check that f is a left A-homomorphism. It remains to prove that f is an epimorphism. Let $a \in A$. Then $a = a1$. By (2), there is $j \in I$ and $b \in A$ such that $ba'_j = 1$. Observe that $f(bz_j) = ba'_j = 1$. Thus $f(abz_j) = a1 = 1$, and so f is surjective. ∎

Corollary 2.2.32 *Consider the left A-act Ae, where e is an idempotent. Then the following are equivalent:*

1. *Ae is a generator.*
2. *There exist $a, b \in A$ such that $eb = b$ and $ab = 1$.*
3. *$A = AeA$.*

Proof (1)⇒(2). Suppose that Ae is a generator. By Proposition 2.2.31(1), there exists $b \in A$ such that whenever $a_1 e = a_2 e$ we have that $a_1 b = a_2 b$. But $1e = ee$ and so $1b = eb$ giving $eb = b$. By (2), there exists $a \in A$ such that $ab = 1$.

(2)⇒(1). Conversely, suppose that there exist $a, b \in A$ such that $eb = b$ and $ab = 1$. Then $a_1 e = a_2 e$ implies that $a_1 eb = a_2 eb$ giving $a_1 b = a_2 b$. Thus Proposition 2.2.31(1) holds. Condition (2) is immediate.

(2)⇒(3). Suppose that $eb = b$ and $ab = 1$. Let $x \in A$ be arbitrary. Then $x = x1 = xab = (xa)eb \in AeA$. Thus $A = AeA$.

(3)⇒(2). Conversely, suppose that $A = AeA$. Then $1 = xey$. Put $a = x$ and $ey = b$. Then $eb = b$ and $ab = 1$. ■

Theorem 2.2.33 *The left A-act P is an indecomposable projective generator if and only if there exists an idempotent e in A such that $A = AeA$ and P is isomorphic as a left A-act to Ae.*

Proof Let e be an idempotent such that $A = AeA$. We prove that Ae is an indecomposable projective generator. By Theorem 2.2.26, Ae is an indecomposable projective left A-act, and it is a generator by Corollary 2.2.32.

To prove the converse, let P be an indecomposable projective generator. Then by Theorem 2.2.26, P is isomorphic to a left A-act of the form Ae where e is an idempotent. By Corollary 2.2.32, we have that $A = AeA$. ■

2.2.8 Morita equivalence of monoids

The goal of this section is to obtain a purely algebraic characterization of when two monoids are Morita equivalent.

Lemma 2.2.34 *Let $F: A$-**Act** $\rightarrow B$-**Act** be a functor. Then $F(A)$ is a (B, A)-biact.*

Proof Clearly $F(A)$ is a left B-act. We shall show that it is also a right A-act. Let $a \in A$. Define $\rho_a: A \rightarrow A$ by $(x)\rho_a = xa$. Clearly, this is a left A-homomorphism. It follows that $F(\rho_a): F(A) \rightarrow F(A)$ is a left B-homomorphism. If $v \in F(A)$ define $v \cdot a = (v)F(\rho_a)$. If $a = 1$ then ρ_1 is the identity on A and so $F(\rho_1)$ is the identity on $F(A)$. Thus $v \cdot 1 = v$. Let $a, a' \in A$. Then $\rho_{aa'} = \rho_a \rho_{a'}$. Thus $F(\rho_{aa'}) = F(\rho_a)F(\rho_{a'})$. It follows that

$$v \cdot aa' = (v)F(\rho_{aa'}) = (v)F(\rho_a)F(\rho_{a'}) = (v \cdot a) \cdot a'.$$

We have therefore shown that $F(A)$ is a right A-act. It remains to show that it is a (B, A)-biact. Let $a \in A$, $b \in B$ and $v \in F(A)$. Then

$$(b \cdot v) \cdot a = (b \cdot v)F(\rho_a) = b \cdot ((v)F(\rho_a)) = b \cdot (v \cdot a)$$

using the fact that $F(\rho_a)$ is a left B-homomorphism. ■

The next result is the key since it enables us to replace arbitrary functors satisfying certain conditions by equivalent tensor functors. It is the monoid version of a theorem in module theory known as the Watts-Gabriel theorem. We follow the proof of Proposition V.1.4 of [16].

Theorem 2.2.35 *Let $F: A$-**Act** $\rightarrow B$-**Act** be a functor that preserves colimits. Then F is equivalent to $F(A) \otimes -$.*

Proof By Lemma 2.2.34, the tensor functor $F(A) \otimes -$, of Theorem 2.2.15, is well-defined. By Theorem 2.2.16, it preserves colimits.

We show first that there is a natural transformation ϕ from $F(A) \otimes -$ to F. Let X be a left A-act. We shall define a function

$$\phi_X: F(A) \otimes X \rightarrow F(X).$$

Let $v \in F(A)$ and $x \in X$. Define $\rho_x: A \rightarrow X$ by $(a)\rho_x = ax$. This is a left A-homomorphism. It follows that $F(\rho_x): F(A) \rightarrow F(X)$ is a left B-homomorphism. We define a map $(v, x) \mapsto (v)F(\rho_x)$. We show that (va, x) and (v, ax) have the same image. By definition from Lemma 2.2.34, $v \cdot a = (v)F(\rho_a)$. Thus (va, x) maps to $((v)F(\rho_a))F(\rho_x)$. But $\rho_a \rho_x = \rho_{ax}$. Thus $((v)F(\rho_a))F(\rho_x) = (v)F(\rho_{ax})$ which is what (v, ax) is mapped to. We have defined a balanced map and so there is a well-defined function

$$\phi_X(v \otimes x) = (v)F(\rho_x)$$

which is a left B-homomorphism because $F(\rho_X)$ is a left B homomorphism. It is routine to check that ϕ is a natural transformation.

So far, we have not used any of the other properties of either F or of the category A-**act**. We use those to show that ϕ is actually a natural isomorphism. Let X be an arbitrary left A-act. By Lemma 2.2.28, the left A-act A is a generator of the category A-**act**. By Lemma 2.2.29, there is an epimorphism

$$e\colon \coprod_I A \to X$$

for some non-empty indexing set I. By Lemma 1.9.2, every epimorphism in the category A-**act** is a coequalizer. There is therefore a left A-act Y and left A-homomorphisms f and g such that the following is a coequalizer

$$Y \underset{g}{\overset{f}{\rightrightarrows}} \coprod_I A \xrightarrow{e} X$$

By Lemma 2.2.29, there is an epimorphism

$$e'\colon \coprod_J A \to Y$$

for some non-empty indexing set J. By Lemma 1.7.8, the following diagram is a coequalizer

$$\coprod_J A \underset{ge'}{\overset{fe'}{\rightrightarrows}} \coprod_I A \xrightarrow{e} X$$

Put $G = F(A) \otimes -$. Both F and G preserve colimits and so both

$$\coprod_J F(A) \underset{F(ge')}{\overset{F(fe')}{\rightrightarrows}} \coprod_I F(A) \xrightarrow{F(e)} F(X)$$

and

$$\coprod_J G(A) \underset{G(ge')}{\overset{G(fe')}{\rightrightarrows}} \coprod_I G(A) \xrightarrow{G(e)} G(X)$$

are coequalizers. By Proposition 2.2.13, the component $\phi_A\colon F(A) \otimes A \to F(A)$ is an isomorphism. Thus $\phi_A\colon G(A) \to F(A)$ is an isomorphism. Using the fact that ϕ is a natural transformation, we therefore have the following diagram

$$\begin{array}{ccccc}
\coprod_J G(A) & \overset{G(fe')}{\underset{G(ge')}{\rightrightarrows}} & \coprod_I G(A) & \xrightarrow{G(e)} & G(X) \\
\downarrow{\scriptstyle \coprod_J \phi_A} & & \downarrow{\scriptstyle \coprod_I \phi_A} & & \downarrow{\scriptstyle \phi_X} \\
\coprod_J F(A) & \overset{F(fe')}{\underset{F(ge')}{\rightrightarrows}} & \coprod_I F(A) & \xrightarrow{F(e)} & F(X)
\end{array}$$

which satisfies the conditions of Lemma 1.7.9. Thus ϕ_X is an isomorphism and so ϕ is a natural isomorphism, as required. ■

We may now prove the main theorem.

Theorem 2.2.36 (Banaschewski-Knauer) *The monoids A and B are Morita equivalent if and only if there is an idempotent $e \in A$ such that $A = AeA$ and B is isomorphic to eAe.*

Proof Suppose first that A and B are Morita equivalent. Then by definition there are functors $S\colon A$-**Act** $\to B$-**Act** and $T\colon B$-**Act** $\to A$-**Act** which form part of an equivalence of categories.

Equivalences preserve colimits by Lemma 1.8.1 and so by Theorem 2.2.35, T is equivalent to the functor $T(B)\otimes -$. Put $P = T(B)$. By Theorem 2.2.16, $P\otimes -$ is a left adjoint to $\hom(P, -)$, so $\hom(P, -)$ is a right adjoint to $P\otimes -$. But S is equivalent to T and T is equivalent to $P \otimes -$, thus S is equivalent to $P \otimes -$ which means that S is a right adjoint to $P \otimes -$. By Lemma 1.8.2, S is equivalent to $\hom(P, -)$.

Thus the monoid $S(P)$ is isomorphic to the monoid $\hom(P, P)$. Because S and T form part of an equivalence of categories, B is isomorphic to $S(T(B)) = S(P)$. It follows that B is isomorphic to $\hom(P, P)$. Since B is an indecomposable projective generator and $P = T(B)$ and T is a weak equivalence, it follows that P is an indecomposable projective generator. Thus by Theorem 2.2.33, there is an idempotent $e \in A$ such that $A = AeA$ and P is isomorphic to Ae. Hence B is isomorphic to $\hom(Ae, Ae)$. By Lemma 2.2.2, $\hom(Ae, Ae)$ is isomorphic to eAe. Thus B is isomorphic to eAe where e is an idempotent such that $A = AeA$.

To prove the converse, let A and B be monoids such that B is isomorphic to eAe where e is an idempotent in A such that $A = AeA$. To prove that A and B are Morita equivalent it is enough to prove that A and eAe are

Morita equivalent since isomorphic monoids are Morita equivalent. Define the function

$$F\colon A\textbf{-Act} \to eAe\textbf{-Act}$$

as follows. For each left A-act X it is clear that eX is a left eAe-act. Define $F(X) = eX$. Let $f\colon X \to Y$ be a left A-homomorphism. Then $f\mid_{eX}\colon eX \to eY$ is a well-defined left eAe-homomorphism. Define $F(f) = f\mid_{eX}$.

We show first that F is faithful. Let $f, g\colon X \to Y$ be left A-homomorphisms such that $f\mid_{eX} = g\mid_{eX}$. By assumption, we may write $1 = aeb$ for some $a, b \in A$. Let $x \in X$. Then $f(x) = f(1x) = f(aebx) = af(ebx)$. But $f(ebx) = g(ebx)$ by assumption. It now follows that $f(x) = g(x)$ and so $f = g$, as required.

Next, we show that F is full. Let $\bar{f}\colon eX \to eY$ be a left eAe-homomorphism. Let $1 = aeb$. Define $f\colon X \to Y$ by $f(x) = a\bar{f}(ebx)$. Let $c \in A$. Then by definition $cf(x) = ca\bar{f}(ebx)$ and $f(cx) = a\bar{f}(ebcx)$. Now

$$a\bar{f}(ebcx) = a\bar{f}(ebcaebx) = aebcae\bar{f}(ebx) = cae\bar{f}(ebx) = ca\bar{f}(ebx),$$

and so f is a left A-homomorphism. Finally, if $ex = x$ then

$$f(x) = a\bar{f}(ebx) = a\bar{f}(ebex) = aebe\bar{f}(x) = \bar{f}(x)$$

and so $f\mid_{eX} = (\bar{f})$, as required.

It remains to show that it is essentially surjective. Let X be a left eAe-act. Form the tensor product $Y = Ae \otimes X$. Then Y is a left A-act. But $F(Y) = eAe \otimes X$ which is isomorphic to X by Proposition 2.2.13. It follows that F is essentially surjective. ∎

The above result looks non-symmetric but the following result shows that it is symmetric as it must be if it is a characterization of Morita equivalence.

Proposition 2.2.37 *Let A and B be monoids. Suppose that there is an idempotent $e \in A$ such that $A = AeA$ and B is isomorphic to eAe. Then there is an idempotent $f' \in B$ such that $B = Bf'B$ and A is isomorphic to $f'Bf'$.*

Proof We may write $1 = aeb$. Put $x = ae$ and $y = eb$. Then $1 = xy$. Put $f = yx$. Then f is an idempotent. Observe that $xyx = x$ and $yxy = y$. It follows that $1_A \mathcal{D} f$. Also $f \leq e$. It follows that $f \in E(eAe)$. It is easy to check that $eAe = eAefeAe$ and that $f(eAe)f = fAf$. Now eAe is isomorphic to B. Under this isomorphism f is mapped to f'. Thus $B = Bf'B$. But $feAef = fAf$. Let $h: fAf \to A$ defined by $h(faf) = h(yxayx) = xyaxy = a$. Then we can show that h is an isomorphism. Thus $feAef$ is isomorphic to A. It follows that $f'Bf'$ is isomorphic to A. ■

We want Morita equivalence of monoids to be weaker than isomorphism. The following example shows that it is.

Example 2.2.38 Let M be the closed interval $[0,1]$ of real numbers. Let ι_x be the identity map of $[0,x]$ onto itself for each $x \in [0,1]$. Let β be defined on M by $\beta(y) = \frac{1}{2}y$ for each $y \in M$. This is a partial bijection and so we may define $\alpha = \beta^{-1}$ which means that the domain of α is $[0, \frac{1}{2}]$. Now let A be the inverse monoid of partial bijections generated by the maps α and β and ι_x for all $x \in [3/4, 1]$. The map ι_1 is the identity of A and $\alpha\beta = \iota_1$. Now $\iota_{\frac{3}{4}}$ is an idempotent and satisfies $\iota_{\frac{3}{4}}\beta = \beta$. Thus by Theorem 2.2.36, the monoid A is Morita equivalent to the monoid $B = \iota_{3/4}A\iota_{3/4}$.

We show that these two inverse monoids are not isomorphic by showing that their semilattices of idempotents are not isomorphic. Every idempotent in A is of the form ι_x where $[0,x]$ is the domain of some $f \in A$. Therefore $x \leq \frac{1}{2}$ or $\frac{3}{4} \leq x \leq 1$. And so the idempotents in A which are different from the identity form a chain which has no upper bound. But the idempotents in B are $\iota_{\frac{3}{4}}$ and ι_x where $x \leq \frac{1}{2}$. Thus the idempotents different from the identity have the upper bound $\iota_{\frac{1}{2}}$.

Chapter 3

Morita equivalence of semigroups

In the previous chapter, we described the Morita theory of monoids. The goal of this chapter is to describe the Morita theory of semigroups with local units. In Section 3.1, we describe the known results on the Morita theory of semigroups with local units. In particular, we describe three different algebraic characterizations of Morita equivalence: two use categories and the third uses enlargements. This theory was initiated by Talwar [40, 41, 42] and substantially developed by Lawson [25]. The remaining sections contain results which are mainly new. Our main goal is to obtain characterizations of Morita equivalence that use Rees matrix semigroups. The motivation for doing this comes from Theorem 2.1.1 where Morita equivalent unital rings are characterized using matrices. In Section 3.2, we describe how the classical Rees theorem can be generalized to certain kinds of semigroups Morita equivalent to monoids. In Section 3.3, we give a new proof of a result by Laan and Márki [20] characterizing Morita equivalence in terms of Rees matrix covers. In Section 3.4, our main goal is to describe the Morita theory of inverse semigroups using Rees matrix semigroups.

3.1 Background results

In this section, we shall outline the Morita theory of semigroups with local units described in Lawson [25].

3.1.1 Motivation

We begin by showing that the obvious way of defining the Morita equivalence of semigroups does not work. It is based on [5].

If the semigroup S acts on the left on the set X we say that X is a *left S-act*. Left S-homomorphisms will be written with their arguments on the left. Thus if $f\colon M \to N$ is a left S-homomorphism, its value at m is denoted by $(m)f$. We denote by $S-\mathbf{Act}$ the category of left S-acts and left S-homomorphisms. We can adjoin an identity to S to get a monoid S^1. If $S \times X \to X$ is a left action of the semigroup S then we can extend it to a left action $S^1 \times X \to X$ of the monoid S^1.

Lemma 3.1.1 *The category S-**Act** is isomorphic to the category S^1-**Act**.*

Proof Define

$$F\colon S\text{-}\mathbf{Act} \to S^1\text{-}\mathbf{Act}$$

by $F(M) = M$ and $F(f) = f$ We can do this because each left S-act M can be extended to a left S^1-act M such that $1 \cdot x = x$ for each $x \in M$, and if $f : M \to N$ is a left S-homomorphism then it can be extended to a left S^1-homomorphism. Define

$$G\colon S^1\text{-}\mathbf{Act} \to S\text{-}\mathbf{Act}$$

by restriction. Then clearly F and G define an isomorphism of categories. ■

Lemma 3.1.2 *Let S and T be semigroups. If the category S^1-**Act** is isomorphic to the category T^1-**Act** then S and T are isomorphic.*

Proof By the definition, the monoids S^1 and T^1 are Morita equivalent. Hence by Theorem 2.2.36, there is an idempotent $e \in S^1$ such that T^1 is isomorphic to eS^1e and there are elements $a, b \in S^1$ such that $ae = a, eb = b, ab = 1$. But $ab = 1$ only if $a = b = 1$ because the identity is adjoined. Thus $e = 1$ and so T^1 is isomorphic to S^1. Thus S is isomorphic to T. ■

The above lemma implies that the obvious definition of Morita equivalence of semigroups leads to an isomorphism between the semigroups. This is not what we want from Morita equivalence. Therefore a different approach has to be taken to defining the Morita equivalence of semigroups.

3.1.2 Definition

The last section showed that we cannot define the Morita equivalence of semigroups in the obvious way. The correct way was found by Talwar [40, 41, 42] and we explain his approach including some simplifications due to Lawson [25]. Talwar was motivated by some generalizations of the classical Morita theory to classes of non-unital rings described in [1, 2].

A semigroup S has *local units* if for each $s \in S$ there exist idempotents e and f such that $es = s = sf$. Both monoids and regular semigroups are semigroups with local units.

Let S be a semigroup with local units. A left S-act X is said to be *left unitary* if and only if $SX = X$. If S has local units and X is a unitary left S-act, then it is easy to check that for each $x \in X$ there exists an idempotent $e \in S$ such that $ex = x$. The unitary left S-acts with the S-homomorphisms between them form a full subcategory of $S - \mathbf{Act}$, which is denoted by $S - \mathbf{UAct}$. If M and N are left S-acts then $\hom_S(M, N)$ denotes the set of all left S-homomorphisms from M to N. If M is an S-biact then $\hom_S(M, N)$ becomes a *left* S-act when we define $s \cdot f$ by $(m)(s \cdot f) = (ms)f$. In particular, $\hom_S(S, M)$ is a left S-act.

We can extend the results on tensors described in Chapter 2 to semigroups with local units. Let X be a left S-act. Form the tensor product $S \otimes X$. The action induces a map $\mu_X \colon S \otimes X \to X$ given by $\mu_X(s \otimes x) = sx$. This map is surjective if and only if X is left unitary. If it is also injective then we say that X is *closed*. The full subcategory of $S - \mathbf{Act}$ consisting of all the closed left acts is denoted by $S - \mathbf{FAct}$. Define right S-acts dually, and define (S, T)-biacts in the usual way. A biact is *unitary* if it is left and right unitary. A biact is *closed* if it is closed as a left and as a right act.

Let S and T be two semigroups with local units. We say that S and T are *Morita equivalent* if the categories $S - \mathbf{FAct}$ and $T - \mathbf{FAct}$ are equivalent. If S and T are both monoids then by Proposition 2.2.13, this definition reduces to the usual definition of Morita equivalence for monoids given in Chapter 2.

Lawson [26] remarks that Talwar takes for granted the result that in the category $S - \mathbf{FAct}$ all epimorphisms are surjections and supplies a proof.

To state Talwar's main theorem, we need the following definition.

A 6-tuple $(S, T, P, Q, \langle -, - \rangle, [-, -])$, where S and T are semigroups, is said to be a *Morita context* if the following conditions are satisfied:

(M1) P is an (S, T)-biact, and Q is a (T, S)-biact.

(M2) $\langle -,-\rangle\colon P\otimes Q\to S$ is an (S,S)-homomorphism and $[-,-]\colon Q\otimes P\to T$ is a (T,T)-homomorphism.

(M3) For all $p,p'\in P, q,q'\in Q$ the following two conditions are satisfied:

(i) $\langle p,q\rangle p' = p[q,p']$.

(ii) $q\langle p,q'\rangle = [q,p]q'$.

We say that a Morita context $(S,T,P,Q,\langle -,-\rangle,[-,-])$ is *unitary* if and only if S and T are semigroups with local units, P and Q are closed as left acts, and the biacts P and Q are unitary.

Theorem 3.1.3 *Let S and T be semigroups with local units. Then S and T are Morita equivalent if and only if there is a unitary Morita context $(S,T,P,Q,\langle -,-\rangle,[-,-])$ with surjective mappings.*

Talwar was able to prove a number of results about the Morita equivalence of semigroups with local units, but he does not provide any workable algebraic criteria. How this can be done is described in the next section.

3.1.3 Algebraic characterizations

In this section, we describe three different algebraic characterizations of Morita equivalence together with some important consequences. All proofs can be found in [26].

The first characterization uses categories. Let S be a semigroup. Define

$$C(S) = \{(e,s,f)\in E(S)\times S\times E(S)\colon esf = s\}$$

with partial product $(e,s,f)(f,t,i) = (e,st,i)$. This is a category called the *Cauchy completion* of S.

Theorem 3.1.4 (First algebraic characterization) *Let S and T be semigroups with local units. Then S and T are Morita equivalent if and only if the categories $C(S)$ and $C(T)$ are equivalent*

The easy half of the proof of this theorem uses the characetrization of projective indecomposable closed actions which is essentially the same as in the monoid case Theorem 2.2.26.

The second characterization uses semigroups built from categories. Recall that a category is *strongly connected* if for each ordered pair of identities e and f there is an arrow from e to f. We shall build semigroups from strongly connected categories using the following technique. Let C be a strongly connected category. A *consolidation* for C is a function p: $C_o \times C_o \to C$, $p(e,f) = p_{e,f}$, where $p_{e,f}$ is an arrow from f to e and $p_{e,e} = e$. Given a category C equipped with a consolidation p we can define a binary operation $\circ$ on C by $x \circ y = xp_{e,f}y$ where x has domain e and y has codomain f. The structure $(C, \circ)$ will be denoted by C^p.

A semigroup S is *regular* if for each $s \in S$ there exists $t \in S$ such that $s = sts$ and $t = tst$. A category C is said to be *regular* if for each $a \in C$ there exists at least one element $a' \in C$ such that $a = aa'a$ and $a' = a'aa'$.

Lemma 3.1.5 *Let C be a strongly connected category and let p be a consolidation on C. Then C^p is a semigroup with local units. In addition, if C is regular then C^p is regular.*

Proof Let $x \in C$ be an arrow from e to f. Then $x \circ e = xp_{e,e}e = xe = x$. Similarly, $f \circ x = x$. Thus C^p is a semigroup with local units.

Suppose now that C is regular. Given x an arrow from e to f there is an arrow x' from f to e such that $x = xx'x$ and $x' = x'xx'$. But $x \circ x' \circ x = xp_{e,e}x'p_{f,f}x = xx'x = x$. Similarly $x' = x' \circ x \circ x'$. Thus C a regular category implies C^p is a regular semigroup. ∎

Let S and T be semigroups with local units. The following definition is equivalent to the one given by Lawson [25] and was suggested by Lauri Tart to Lawson (private communication). A homomorphism $\theta\colon S \to T$ between semigroups with local units is said to be a *local isomorphism* if the following two conditions are satisfied:

(LI1) $\theta \mid eSf\colon eSf \to \theta(e)T\theta(f)$ is an isomorphism for all $e, f \in E(S)$.

(LI2) For each $i \in E(T)$ there exists $e \in E(S)$ such that $i\mathcal{D}\theta(e)$.

Lemma 3.1.6 *If $\theta\colon S \to T$ is a surjective local isomorphism then S and T are Morita equivalent.*

Proof Define $\Theta\colon C(S) \to C(T)$ by $\Theta(e,s,f) = (\theta(e), \theta(s), \theta(f))$. Then this defines an equivalence of categories and so by the first algebraic characterization above the semigroups S and T are Morita equivalent. ∎

Theorem 3.1.7 (Second algebraic characterization) *Let S and T be semigroups with local units. Then S and T are Morita equivalent if and only if there is a consolidation q on $C(S)$ and a local isomorphism*

$$\psi\colon C(S)^q \to T.$$

The third characterization uses enlargements. Let S be a subsemigroup of a semigroup T. We say that T is an *enlargement* of S if

$$S = STS \text{ and } T = TST.$$

Enlargements were introduced in [22] and their theory developed in [26]. Let S, T and R be semigroups with local units. We shall say that R is a *joint enlargement* of S and T if it is an enlargement of subsemigroups S' and T' which are isomorphic to S and T respectively.

Theorem 3.1.8 (Third algebraic characterization) *Let S and T be semigroups with local units. Then S and T are Morita equivalent if and only if S and T have a joint enlargement which can be chosen to be regular if S and T are both regular.*

There are two further results which are very useful. The first was proved by Talwar and is a half-way house between the Morita theory of monoids and the Morita theory of semigroups with local units.

Proposition 3.1.9 *Let S be a monoid and T a semigroup with local units. Then S and T are Morita equivalent if and only if there is an idempotent e in T such that $T = TeT$ and eTe is isomorphic to S. Thus, in particular, T is an enlargement of S.*

The second result is useful in showing that two semigroups are not Morita equivalent and in guessing when they are. Recall that if S is a semigroup and e is an idempotent then eSe is called a *local submonoid*. A semigroup is said to have a property *locally* if each local submonoid has that property.

Proposition 3.1.10 *Let S and T be semigroups with local units which are Morita equivalent.*

1. *Each local submonoid of S is isomorphic to a local submonoid of T, and vice-versa.*

2. *S is regular if and only if T is regular.*

3. *The cardinalities of the sets of regular $\mathcal{D}$-classes in S and T are the same.*

4. *The posets of two-sided ideals in S and T are order-isomorphic.*

5. *The posets of principal two-sided ideals in S and T are order-isomorphic.*

3.1.4 Semigroup background

In this section, we shall gather together the results from semigroup theory that we shall need in this chapter.

The set of regular elements of a semigroup S is denoted by $\text{Reg}(S)$ and the set of idempotents is denoted by $E(S)$. The usual order on the idempotents is defined by $e \leq f$ if and only if $e = ef = fe$.

A semigroup S is a *band* if all its elements are idempotents. A *normal band* S is a band satisfying $xyzx = xzyx$ for each $x, y, z \in S$.

Proposition 3.1.11 (Exercises 1.4, Question 12 of [12]) *Let S be a semigroup. Then the set of regular elements forms a regular subsemigroup if and only if the product of any two idempotents is regular.*

The element t in a regular semigroup S is called an *inverse* of s if $tst = t$ and $sts = s$. The set of inverses of s is denoted by $V(s)$. If each element has a unique inverse then the semigroup is said to be *inverse*. The standard reference on regular semigroup theory is [15]. The following are basic results from [15] that will be useful.

Lemma 3.1.12

1. *Let $s\mathcal{L}e$ where e is an idempotent. Then there is $s' \in V(s)$ such that $s's = e$, and dually for the $\mathcal{R}$ relation.*

2. *Let $e\mathcal{R}s\mathcal{L}f$ where e and f are idempotents. Then there is a unique inverse s' of s such that $s's = f$ and $ss' = e$.*

3. *If a is a regular element in eSf where e and f are idempotents then $V(a) \cap fSe \neq \emptyset$.*

A regular semigroup is said to be *orthodox* if its idempotents form a subsemigroup. Inverse semigroups are orthodox. An orthodox locally inverse semigroup is called a *generalized inverse semigroup*. They are the orthodox semigroups whose idempotents form a normal band which was first defined by M. Yamada in [43].

Let S be a regular semigroup. Then the intersection of all congruences ρ on S such that S/ρ is inverse is a congruence denoted by γ; it is called the *minimum inverse congruence*.

Lemma 3.1.13 (Theorems 6.2.4 and 6.2.5 of [15]) *Let S be an orthodox semigroup. Then the following are equivalent:*

1. $s \, \gamma \, t$.
2. $V(s) \cap V(t) \neq \emptyset$.
3. $V(s) = V(t)$.

Lemma 3.1.14 (Lemma 1.3 [29]) *Let $\theta \colon S \to T$ be a surjective homomorphism between regular semigroups. Then it is a local isomorphism if and only if $\theta \mid eSe \colon eSe \to \theta(e)T\theta(e)$ is an isomorphism for each idempotent $e \in S$.*

Lemma 3.1.15 (Proposition 1.4 [29]) *Let S be a regular semigroup. Then the natural homomorphism to S/γ is a local isomorphism if and only if S is a generalized inverse semigroup.*

The following are Propositions 1 and 2 of [26].

Lemma 3.1.16

1. *Let T be an enlargement of S where $S^2 = S$. Then every idempotent in T is $\mathcal{D}$-related to an idempotent in S.*
2. *Let T be an enlargement of S and let $\theta \colon T \to W$ be a surjective homomorphism. Then W is an enlargement of $\theta(S)$.*

The following is well-known and is included for the sake of compeleteness.

Lemma 3.1.17 *Let e be an idempotent in a semigroup S. Then S is an enlargement of eSe if and only if $S = SeS$*

Proof Let S be an enlargement of eSe. Then $S = S(eSe)S = Se(SeS) \subseteq SeS$. Thus $S = SeS$, as required. Conversely, suppose that $S = SeS$. Then $S(eSe)S = (SeS)eS = SeS = S$ and $eSe(S)eSe = e(SeS)eSe = e(SeS)e = eSe$. Thus S is an enlargement of eSe. ■

Definitions from semigroups can sometimes be extended to categories. An element a' is called an *inverse* of a in a ctegory C if $a = aa'a$ and $a' = a'aa'$. A category is *inverse* if it is regular and each element has a unique inverse. Results about inverse semigroups can be extended to inverse categories. For example, a category is inverse if and only if it is regular and the idempotents in each local monoid commute. In addition, a partial order can be defined on an inverse category for pairs of arrows belonging to the same hom-set in the same way as in the inverse semigroup case. A reference for inverse categories is [11].

In a category C, we define $a\mathcal{L}b$ iff $Ca = Cb$. We define $\mathcal{R}$ dually. As in the semigroup case, the relations $\mathcal{L}$ and $\mathcal{R}$ commute and so we may define the relation $\mathcal{D}$ to be their product.

Lemma 3.1.18 *In a category C the idempotent e splits if and only if it is $\mathcal{D}$-related to an identity.*

Proof Suppose that the idempotent e splits. Then there are arrows x and y such that $e = xy$ and $yx = i$, an identity. Observe that $xyx = x$. Then

$$eC = xyC \subseteq xC = xyxC \subseteq xyC = eC.$$

Thus $e\mathcal{R}x$. Also

$$Cx = Cxyx \subseteq Cyx = Ci = Cyx \subseteq Cx.$$

Thus $x\mathcal{L}i$. It follows that $e\mathcal{D}i$.

Conversely, suppose that $e\mathcal{D}i$ where e is an idempotent and i is an identity. Then $e\mathcal{R}x\mathcal{L}i$ for some arrow x. Let $e = xy$ and $x = eb$ Let $i = cx$ and let $x = di$. But i is an identity and so $x = d$. Observe that $ex = x$. Also $ce = cxy = iy = y$. Thus $yx = cex = cx = i$. Thus the idempotent e splits. ■

The following is well-known and we prove it for the sake of completeness.

Lemma 3.1.19 *Let S be a semigroup with local units. Then $C(S)$ is strongly connected and every idempotent splits.*

Proof The identities in $C(S)$ are the elements of the form (e,e,e) where e is an idempotent in S. Thus given two identities (e,e,e) and (f,f,f) the arrow (f,fe,e) goes from the first to the second. Thus $C(S)$ is strongly connected.

The idempotents in $C(S)$ are the elements of the form (e,f,e) where $f \leq e$. Put $x = (e,f,f)$ and $y = (f,f,e)$. Then $xy = (e,f,e)$ and $yx = (f,f,f)$. Thus every idempotent splits. ■

The following is also well-known and included for the sake of completeness.

Lemma 3.1.20 *Let C and D be equivalent categories.*

1. *C is strongly connected if and only D is strongly connected.*
2. *Every idempotent in C splits if and only if every idempotent in D splits.*

Proof

(1) Let $F: C \to D$ be a weak equivalence. Suppose that D is strongly connected. Let e and f be any two identities in C. Then by assumption there is an arrow in D from $F(e)$ to $F(f)$. Thus there is an arrow in C from e to f using the fact that F is full.

(2) Let e be an idempotent in C based at the identity i. Then $F(e)$ is an idempotent in D. By assumption there is an identity j in D and arrows x and y such that $F(e) = xy$ and $yx = j$. Since F is a weak equivalence there exists an isomorphism z in D and an identity j' in C such that z is an arrow from j to $F(j')$. It follows that zy is an arrow from $F(i)$ to $F(j')$ and that xz^{-1} is an arrow from $F(j')$ to $F(i)$. Also $(xz^{-1})(zy) = F(e)$ and $(zy)(xz^{-1}) = F(j')$. It follows from the fact that F is full and faithful that there are arrows u and v in C such that $e = uv$ and $vu = j'$. Thus every arrow in C splits. ■

Let $\theta: A \to B$ be a homomorphism. We say that regular elements lift along θ if for each regular element $b \in B$ there is a regular element $a \in A$ such that $\theta(a) = b$.

Lemma 3.1.21

1. *Regular elements lift along surjective local isomorphisms.*
2. *The composition of local isomorphisms is a local isomorphism.*

Proof (1) Let $\alpha: S \to T$ be a surjective local isomorphism. Let $t \in T$ be regular. Then there is $t' \in T$ such that $tt't = t$ and $t'tt' = t'$. But α is surjective and so there is $s, s' \in S$ such that $t = \alpha(s)$ and $t' = \alpha(s')$. Thus

$$\alpha(ss's) = \alpha(s)\alpha(s')\alpha(s) = tt't = t = \alpha(s)$$

and

$$\alpha(s'ss') = \alpha(s')\alpha(s)\alpha(s') = t'tt' = t' = \alpha(s')$$

Since S has local units then $s \in eSf$ and $s' \in e'Sf'$ for some idempotents e, f, e', f' in S. But α is a local isomorphism, and so $ss's = s$ and $s'ss' = s'$. It follows that s is regular in S and so regular elements lift along θ.

(2) Let $\alpha: S \to T$ and $\beta: T \to U$ be local isomorphisms. We prove that $\beta\alpha: S \to U$ is a local isomorphism. This is straightforward. ∎

3.2 Generalizations of the Rees theorem

In this section, we shall look at the simplest case of Morita equivalence where we are interested in when a semigroup with local units is Morita equivalent to a monoid. This was discussed in the paper by Lawson and Márki [26]. The results of this section can be seen as trying to generalize the classical Rees-Suschkewitsch theorem. Therefore we begin by explaining how this theorem is a part of Morita theory.

3.2.1 The Rees-Suschkewitsch theorem

In this section, we shall explain one of the first theorems in semigroup theory. We refer to [15] for mathematical details and [13, 14] for historical background. We do not give any proofs here because, in the next section, we shall generalize the results of this section and give all proofs in the more general case. The main definition we need is that of a Rees matrix semigroup.

Let S be a semigroup, I and Λ non-empty sets, and P the $\Lambda \times I$ matrix with entries $p_{\lambda i}$ from S. Put $M(S; I, \Lambda; P)$ equal to the set $I \times S \times \Lambda$ with the multiplication

$$(i, s, \lambda)(j, t, \mu) = (i, sp_{\lambda j}t, \mu).$$

Then $M(S; I, \Lambda; P)$ is a semigroup called a *Rees matrix semigroup*.

A semigroup is said to be *simple* if it has no non-trivial ideals. An idempotent e is said to be *primitive* if $f \leq e$ implies that $f = e$. A semigroup S is said to be *completely simple* if it is simple and has a primitive idempotent. The following is the first substantial theorem of semigroup theory [37]. It is an analogue of the Artin-Wedderburn theorem in ring theory.

Theorem 3.2.1 (Rees-Suschkewitsch) *A semigroup is completely simple if and only if it is isomorphic to a Rees matrix semigroup over a group.*

The Rees theorem, as the above result is often called, has motivated a lot of semigroup theory. It can best be understood in terms of Morita theory. The following can be deduced from [40] and [25].

Theorem 3.2.2 *Let S be a semigroup with local units. Then the following are equivalent.*

1. *S is completely simple.*
2. *S is regular and locally a group.*
3. *There is an idempotent e such that $S = SeS$ and eSe is a group.*
4. *S is Morita equivalent to a group.*

3.2.2 Locally unipotent monoids

The goal of this section is to state and prove a direct generalization of the Rees theorem explained in the previous section. We have seen that a semigroup with local units is Morita equivalent to a group if and only if it is isomorphic to a Rees matrix semigroup over a group. Our goal in this section is to see to what extent this result can be generalized when the group is replaced by a *unipotent monoid*: that is, a monoid with exactly one idempotent. The following result is an immediate consequence of Proposition 3.1.9, Lemma 3.1.16, and Proposition 2.3.2 of [15].

Lemma 3.2.3 *Let S be a semigroup with local units. It is Morita equivalent to a unipotent monoid if and only if there is an idempotent $e \in S$ such that $S = SeS$ and eSe is a unipotent monoid. In which case, we have the following.*

1. *All the idempotents in S are $\mathcal{D}$-related.*

2. *The regular elements of S form a single $\mathcal{D}$-class,*

Observe that it does not follow by the above result that the regular elements form a subsemigroup.

To explore this class of semigroups in more depth, we need to start with some examples which are not necessarily completely simple. We construct a wide-ranging class of such examples next.

Let T be a monoid, let I, Λ be nonempty sets, and let P be a $\Lambda \times I$ matrix over T with the following properties:

(C1) For each $i \in I$ there is $\lambda \in \Lambda$ such that $p_{\lambda i}$ is invertible in T.

(C2) For each $\mu \in \Lambda$ there is $j \in I$ such that $p_{\mu j}$ is invertible in T.

Then $M = M(T; I, \Lambda; P)$ is called a *classical Rees matrix semigroup over the monoid T*.

Proposition 3.2.4 *Let $M = M(T; I, \Lambda; P)$ be a classical Rees matrix semigroup over the unipotent monoid T. Then M has local units and is an enlargement of a unipotent submonoid T' that is isomorphic to T which implies that M is Morita equivalent to the unipotent monoid T.*

Proof We begin by locating the idempotents of M. Suppose $(i, t, \lambda)^2 = (i, t, \lambda)$. Then $tp_{\lambda i}t = t$. But $tp_{\lambda i}$ and $p_{\lambda i}t$ are idempotents and so, since T is unipotent, must be equal to the identity of T. It follows that t is an invertible element in T with inverse $p_{\lambda i}$ and so $t = p_{\lambda i}^{-1}$. Conversely, let $p_{\lambda i}$ be invertible. Then it is easy to check that $(i, p_{\lambda i}^{-1}, \lambda)$ is an idempotent. Thus

$$E(M) = \{(i, p_{\lambda i}^{-1}, \lambda) : i \in I, \lambda \in \Lambda, p_{\lambda i} \text{ is an invertible element in T}\}.$$

Next we show that M has local units. Let (i, t, λ) be any element in M. Since M is classical there is for each $i \in I$ a $\mu \in \Lambda$ such that $p_{\mu i}$ is invertible. Thus $(i, p_{\mu i}^{-1}, \mu)$ is an idempotent in M. But

$$(i, p_{\mu i}^{-1}, \mu)(i, t, \lambda) = (i, p_{\mu i}^{-1} p_{\mu i} t, \lambda) = (i, t, \lambda).$$

Thus each element in M has a left idempotent identity. A similar argument shows that each element in M has a right idempotent identity. We have therefore proved that M has local units.

Let $(i, p_{\lambda i}^{-1}, \lambda)$ be an arbitrary idempotent and let (j, t, μ) be arbitrary. Observe that

$$(j, t, \mu) = (j, t, \lambda)(i, p_{\lambda i}^{-1}, \lambda)(i, p_{\lambda i}^{-1}, \mu).$$

It follows that $M = M(i, p_{\lambda i}^{-1}, \lambda)M$. We prove that the local submonoid $(i, p_{\lambda i}^{-1}, \lambda)M(i, p_{\lambda i}^{-1}, \lambda)$ is isomorphic to the unipotent monoid T which will prove the claim by Proposition 3.1.9. Define

$$\theta\colon (i, p_{\lambda i}^{-1}, \lambda)M(i, p_{\lambda i}^{-1}, \lambda) \to T$$

by $\theta(i, t, \lambda) = p_{\lambda i}t$. Observe that

$$\theta((i, t, \lambda)(i, s, \lambda)) = \theta(i, tp_{\lambda i}s, \lambda) = p_{\lambda i}tp_{\lambda i}s = \theta(i, t, \lambda)\theta(i, s, \lambda)$$

and $\theta(i, p_{\lambda i}^{-1}, \lambda) = p_{\lambda i}p_{\lambda i}^{-1} = 1$ and so θ is a monoid homomorphism.

Let $t \in T$. Put $t' = p_{\lambda i}^{-1}t$ and observe that $t = p_{\lambda i}t'$. It follows readily that $(i, t', \lambda) \in (i, p_{\lambda i}^{-1}, \lambda)M(i, p_{\lambda i}^{-1}, \lambda)$ and that $\theta(i, t', \lambda) = p_{\lambda i}t' = t$. Thus θ is surjective. Finally, suppose that $(i, t, \lambda), (i, s, \lambda) \in (i, p_{\lambda i}^{-1}, \lambda)M(i, p_{\lambda i}^{-1}, \lambda)$ are such that $\theta(i, t, \lambda) = \theta(i, s, \lambda)$ then $p_{\lambda i}t = p_{\lambda i}s$. Then $p_{\lambda i}^{-1}p_{\lambda i}t = p_{\lambda i}^{-1}p_{\lambda i}s$ which means $t = s$. ■

The next lemma establishes an additional property of the classical Rees matrix semigroup over a unipotent monoid.

Lemma 3.2.5 *Let $M = M(T; I, \Lambda; P)$ be a classical Rees matrix semigroup over the unipotent monoid T. Then any two idempotent left identities (resp. right identities) of an element of M are $\mathcal{R}$-related (resp. $\mathcal{L}$-related).*

Proof Let $(i, t, \lambda) \in M$ and let $(j, p_{\mu j}^{-1}, \mu)$ and $(k, p_{\nu k}^{-1}, \nu)$ be two idempotent left identities of (i, t, λ). Then from $(j, p_{\mu j}^{-1}, \mu)(i, t, \lambda) = (i, t, \lambda)$ we get that $(j, p_{\mu j}^{-1}p_{\mu i}t, \lambda) = (i, t, \lambda)$ and so $j = i$. Similarly $k = i$. Thus the two idempotent left identities of (i, t, λ) are $(i, p_{\mu i}^{-1}, \mu)$ and $(i, p_{\nu i}^{-1}, \nu)$. These are $\mathcal{R}$-related because

$$(i, p_{\mu i}^{-1}, \mu)(i, p_{\nu i}^{-1}, \nu) = (i, p_{\nu i}^{-1}, \nu)$$

and

$$(i, p_{\nu i}^{-1}, \nu)(i, p_{\mu i}^{-1}, \mu) = (i, p_{\mu i}^{-1}, \mu).$$

■

The lemma above motivates the following definition. Let S be a semigroup with local units. Then S is a semigroup with *strong local units* if

(SLU1) Whenever e and f are idempotents such that $ex = x$ and $fx = x$ then $e\mathcal{R}f$.

(SLU2) Whenever e and f are idempotents such that $xe = x$ and $xf = x$ then $e\mathcal{L}f$.

From the above definition and lemmas we deduce the following result.

Corollary 3.2.6 *Classical Rees matrix semigroups over unipotent monoids have strong local units and are Morita equivalent to unipotent monoids.*

Our goal is to determine to what extent classical Rees matrix semigroups over unipotent monoids can be used to describe all semigroups with local units which are Morita equivalent to unipotent monoids.

Theorem 3.2.7 *Let S be a semigroup with local units. Then S is Morita equivalent to a unipotent monoid T if and only if there is a classical Rees matrix semigroup $M = M(T; I, \Lambda; P)$ over T and a surjective local isomorphism $\theta \colon M \to S$ which induces a bijection between* $\mathrm{Reg}(M)$ *and* $\mathrm{Reg}(S)$.

Proof We prove the easy direction first. Suppose that there is a classical Rees matrix semigroup $M = M(T; I, \Lambda; P)$ over a unipotent monoid T and a surjective local isomorphism $\theta \colon M \to S$ which induces a bijection between $\mathrm{Reg}(M)$ and $\mathrm{Reg}(S)$. Then by the proof of Proposition 3.2.4, the semigroup M is an enlargement of a local submonoid T' isomorphic to T. Thus by Lemma 3.1.16, we have that S is an enlargement of $\theta(T')$. But θ is a local isomorphism and so S is an enlargement of a monoid isomorphic to T. Thus from Proposition 3.1.9 S is Morita equivalent to T.

We now prove the converse. By Proposition 3.1.9, there is an idempotent $e \in S$ such that $S = SeS$ and eSe is a unipotent monoid. By Lemma 3.2.3, the set $\mathrm{Reg}(S)$ forms a single $\mathcal{D}$-class. Put $I = \mathrm{Reg}(S)/\mathcal{R}$ and $\Lambda = \mathrm{Reg}(S)/\mathcal{L}$. For each $\mathcal{H}$-class in L_e choose an element r_i, and for each $\mathcal{H}$-class in R_e choose an element q_λ. In the case of $\mathcal{H}_e$ we may choose e to be both r_0 and λ_0. The data we have defined forms the basis of a co-ordinatization of the *regular elements* of S and is illustrated below.

e		q_λ	
r_i			

From the fact that $r_i \mathcal{L} e$, we have that $r_i e = r_i$. Similarly $eq_\lambda = q_\lambda$. The proof procedes by a series of steps.

(1) $S = \bigcup_{i\in I, \lambda\in\Lambda} r_i S q_\lambda$.
Let $s \in S$. Since S has local units there exist $e_s, f_s \in E(S)$ such that $s = e_s s f_s$. But e_s and f_s are regular elements in S and so there is $i \in I$ such that $e_s \mathcal{R} r_i$ and there is $\lambda \in \Lambda$ such that $f_s \mathcal{L} q_\lambda$. By Lemma 3.1.12, there is $q'_\lambda \in V(q_\lambda)$ such that $f_s = q'_\lambda q_\lambda$ and $r'_i \in V(r_i)$ such that $e_s = r_i r'_i$. It follows that

$$s = r_i r'_i s q'_\lambda q_\lambda = r_i (r'_i s q'_\lambda) q_\lambda$$

as required.

(2) $S = \bigcup_{i\in I, \lambda\in\Lambda} r_i eSe q_\lambda$.
Let $s \in S$. From (1) we have $s = r_i t q_\lambda$. But $r_i = r_i e$ and $q_\lambda = eq_\lambda$. Thus $s = r_i(ete)q_\lambda$ and $ete \in eSe$, as required.

(3) Let q, r be two regular elements. If there is an idempotent f such that $qf = q$ and $fr = r$ then we will show that qr is regular.
Since q, r are regular, there are inverses $q_1 \in V(q)$ and $r_1 \in V(r)$. Put $q' = fq_1$ and $r' = r_1 f$. Then it is easy to check that $q' \in V(q)$ and $r' \in V(r)$. Now $q'q = fq'qf$ is an idempotent in fSf. But fSf is unipotent since all local submonoids of S are unipotent by Proposition 3.1.10 and so f is the only idempotent it contains. Thus $q'q = f$. By similar reasoning $rr' = f$. It is routine now to check that $r'q' \in V(qr)$. Thus qr is regular, as required.

(4) Define $p_{\lambda i} = q_\lambda r_i$. Then $M(eSe; I, \Lambda; P)$ is a classical Rees matrix semigroup over a unipotent monoid.
We have that $ep_{\lambda i}e = p_{\lambda i}$ and so $p_{\lambda i} \in eSe$. Let $\lambda \in \Lambda$. Then $\mathcal{L}_{q_\lambda}$ contains an idempotent f. But $f \mathcal{R} r_i$ for some $i \in I$ by our co-ordinatization. Thus $q_\lambda f = q_\lambda$ and $fr_i = r_i$. Therefore by (3), $q_\lambda r_i$ is a regular element in eSe. But eSe is a unipotent monoid and so $q_\lambda r_i$ is invertible. Similarly, for each $j \in I$ there is $\mu \in \Lambda$ such that $p_{\mu j}$ is invertible.

(5) Define $\theta \colon M(eSe; I, \Lambda; P) \to S$ by $\theta(i, t, \lambda) = r_i t q_\lambda$. Then θ is surjective local isomorphism.
The map is surjective by (2). To show that it is a homomorphism we calcu-

late. Let $(i,t,\lambda),(j,s,\mu)\in M$. Then

$$\theta((i,t,\lambda)(j,s,\mu))=\theta(i,tp_{\lambda j}s,\mu)$$

$$=r_itq_\lambda r_jsq_\mu=\theta(i,t,\lambda)\theta(j,s,\mu).$$

It remains to show that θ is local isomorphism. Let $(i,p_{\mu i}^{-1},\mu)$ and $(j,p_{\lambda j}^{-1},\lambda)$ be idempotents and let (i,s,λ) and (i,t,λ) belong to $(i,p_{\mu i}^{-1},\mu)M(j,p_{\lambda j}^{-1},\lambda)$. Suppose that $\theta(i,s,\lambda)=\theta(i,t,\lambda)$. Then $r_isq_\lambda=r_itq_\lambda$. It follows that $q_\mu r_isq_\lambda r_j=q_\mu r_itq_\lambda r_j$; that is $p_{\mu i}sp_{\lambda j}=p_{\mu i}tp_{\lambda j}$. But $p_{\mu i}$ and $p_{\lambda j}$ are both invertible and so $s=t$, as required.

(6) Let $s=r_itq_\lambda$ where $t\in eSe$. Then s is regular if and only if t is invertible. Suppose t is invertible. From (3) since $r_ie=r_i$ and $et=t$ then r_it is regular Similarly $s=r_itq_\lambda$ is regular.

Suppose that s is regular. By Lemma 3.1.12, we can find $r_i'\in V(r_i)$ such that $r_i'r_i=e$ and $q_\lambda'\in V(q_\lambda)$ such that $q_\lambda q_\lambda'=e$. Then $t=r_i'sq_\lambda'$.

Put $r_ir_i'=f$, an idempotent. Then $r_i'f=r_i'r_ir_i'=r_i'$ and $fs=r_ir_i'(r_itq_\lambda)=r_itq_\lambda=s$ and so from (3) $r_i's$ is regular. Similarly we can prove that $r_i'sq_\lambda'$ is regular. Thus t is regular in S. Suppose t' is an inverse of t in S. Then $et'e$ will be an inverse of t in eSe. Thus t is regular in eSe. But eSe is a unipotent monoid and so t is invertible in eSe.

(7) Let $r\mathcal{L}e\mathcal{R}q$. If $s\in \mathrm{Reg}(rSq)$ then $r\mathcal{R}s\mathcal{L}q$.
Let s be a regular element in rSq. Then $s=rgq$ where g is an invertible element in eSe. It is clear that $s\in rS$. Let $r'\in V(r)$ such that $r'r=e$, and let $q'\in V(q)$ such that $qq'=e$. Then $sq'=rgqq'=rg$ and so $r=sq'g^{-1}$. It follows that $r\in sS$. Hence $s\mathcal{R}r$. Similarly we can find that $s\mathcal{L}q$. It follows that $r\mathcal{R}s\mathcal{L}q$.

(8) The restriction of θ to the regular elements of M is a bijection from $\mathrm{Reg}(M)$ to $\mathrm{Reg}(S)$.
Homomorphisms always map regular elements to regular elements. Let s be a regular element in S. By (2), we have that $s=r_itq_\lambda$ for some $i\in I$ and $\lambda\in\Lambda$ and by (6), t is invertible in eSe. Since $\lambda\in\Lambda$ and $i\in I$, then there is $j\in I$ and $\mu\in\Lambda$ such that $p_{\lambda j}$ and $p_{\mu i}$ are invertible in eSe. Thus $(j,p_{\lambda j}^{-1}t^{-1}p_{\mu i}^{-1},\mu)$ is an element in M satisfying

$$(i,t,\lambda)(j,p_{\lambda j}^{-1}t^{-1}p_{\mu i}^{-1},\mu)(i,t,\lambda)=(i,t,\lambda).$$

It follows that (i, t, λ) is regular element in M. Thus θ restricted to the regular elements is surjective. To show that this restriction is injective, let $(i, g_1, \lambda), (j, g_2, \mu) \in \text{Reg}(M)$ such that $\theta(i, g_1, \lambda) = \theta(j, g_2, \mu)$. By (6), g_1 and g_2 are invertible elements in eSe. Then $r_i g_1 q_\lambda = r_j g_2 q_\mu = x$, say. Since x is regular, we may apply (7) to deduce that $r_i \mathcal{R} x \mathcal{L} q_\lambda$ and $r_j \mathcal{R} x \mathcal{L} q_\mu$. Thus by (7), we deduce that $i = j$ and $\lambda = \mu$. It now follows easily that $g_1 = g_2$. ■

A special case of the above theorem is worth singling out.

Corollary 3.2.8 *A semigroup with local units is isomorphic to a classical Rees matrix semigroup over a unipotent monoid if and only if it is Morita equivalent to a unipotent monoid and has strong local units.*

Proof By Corollary 3.2.6, classical Rees matrix semigroups over unipotent monoids are Morita equivalent to unipotent monoids and have strong local units.

Now let S be a semigroup with strong local units which is Morita equivalent to a unipotent monoid. We shall use the proof of Theorem 3.2.7. It remains to prove that θ is an isomorphism. Suppose that $r_i S q_\lambda \cap r_j S q_\mu \neq \emptyset$. Let $s \in r_i S q_\lambda \cap r_j S q_\mu$. Then $s = r_i t_1 q_\lambda = r_j t_2 q_\mu$ where $t_1, t_2 \in eSe$. Let $r_i' \in V(r_i)$. Then $r_i r_i' r_i = r_i$ and so $r_i r_i' s = s$. Let $r_j' \in V(r_j)$. Then we have also $r_j r_j' s = s$. By our assumption of strong local units, it follows that $r_i r_i' \mathcal{R} r_j r_j'$. Thus $r_i \mathcal{R} r_j$ which means $i = j$. By a similar argument, we may show that $\lambda = \mu$. We have proved that for each element $s \in S$ there are unique $i \in I$, $\lambda \in \Lambda$ and $t \in eSe$ such that $x = r_i t q_\lambda$. Hence θ is injective.■

3.2.3 The unambiguous case

The following definition is due to Birget [6]. A semigroup S has an *unambiguous $\mathcal{L}$-order* if $S^1 a \subseteq S^1 b, S^1 c$ implies that $S^1 b \subseteq S^1 c$ or $S^1 c \subseteq S^1 b$. *Unambiguous $\mathcal{R}$-order* is defined dually. A semigroup is said to be *unambiguous* if it has both an unambiguous $\mathcal{L}$-order and an unambiguous $\mathcal{R}$-order.

Lemma 3.2.9 *Let $M = M(S; I, \Lambda; P)$ be a classical Rees matrix semigroup over a monoid S. Then M is unambiguous if and only if S is unambiguous.*

Proof Let S be unambiguous. We prove that M is unambiguous. Suppose that

$$M(i, a, \lambda) \subseteq M(j, b, \lambda), M(k, c, \lambda).$$

Then

$$(i, a, \lambda) = (i, x, \mu)(j, b, \lambda) = (i, y, \nu)(k, c, \lambda).$$

Hence

$$a = (xp_{\mu j})b = (yp_{\nu k})c.$$

But S is unambiguous and so $b = uc$ or $c = ub$ for some $u \in S$. Without loss of generality we assume the former. The Rees matrix semigroup is classical and so given $k \in I$ there exists $\xi \in \Lambda$ such that $p_{\xi k}$ is invertible. Observe that

$$(j, b, \lambda) = (j, uc, \lambda) = (j, up_{\xi k}^{-1}, \xi)(k, c, \lambda).$$

It follows that $M(j, b, \lambda) \subseteq M(k, c, \lambda)$.

Let M be unambiguous. We prove that S is unambiguous. Suppose that $Sa \subseteq Sb, Sc$. Then $a = xb = yc$. Let $i \in I$ be any element which we now fix. Then since M is classical there exists $\mu \in \Lambda$ such that $p_{\mu i}$ is invertible. Observe that

$$(i, a, \mu) = (i, xp_{\mu i}^{-1}, \mu)(i, b, \mu) = (i, yp_{\mu i}^{-1}, \mu)(i, c, \mu).$$

But M is unambiguous and so, without loss of generality, we may assume that $(i, b, \mu) = (i, u, \xi)(i, c, \mu)$ for some $\xi \in \Lambda$ and some $u \in S$. Thus $b = (up_{\xi i})c$. ■

Corollary 3.2.10 *Completely simple semigroups are unambiguous.*

Proof If G is a group and $g \in G$ is any element then $Gg = G$ and $gG = G$. It follows that groups are unambiguous and so by Lemma 3.2.9 and Theorem 3.2.1, we have that completely simple semigroups are unambiguous. ■

Lemma 3.2.11 *Let S be a semigroup with local units. Suppose that S is locally unipotent. If S is unambiguous, then S has strong local units.*

Proof Suppose that $es = s = fs$ where e and f idempotents. Then $s \in eS, fS$ and so $sS \subseteq eS, fS$. By unambiguity, we have that $eS \subseteq fS$ or $fS \subseteq eS$. Suppose, without loss of generality, the former. Then $fe = e$. Observe that ef is an idempotent and that $fef = ef$. Thus $ef \in fSf$. By assumption, fSf is a unipotent monoid with unique idempotent f. It follows that $ef = f$. We have therefore proved that $eS = fS$ and so $e\mathcal{R}f$. Thus (SLU1) holds. By symmetry (SLU2) holds and we have shown that S has strong local units. ■

The following is a direct generalization of the classical Rees theorem and follows from Corollary 3.2.9, and Lemmas 3.2.11 and Corollary 3.2.8.

Theorem 3.2.12 *Let S be an unambiguous semigroup with local units. Then S is Morita equivalent to an unambiguous unipotent monoid if and only if S is isomorphic to a classical Rees matrix semigroup over an unambiguous unipotent semigroup.*

3.2.4 Unambiguous locally inverse semigroups

This section was motivated by Theorem 2.8 of [28] which in turn relies on the Local Structure Theorem Theorem 2.4 of [28]. It states that if T is a regular semigroup such that $T = TeT$ for some idempotent where eTe is inverse then T is actually isomorphic to some regular Rees matrix semigroup $RM(eTe; I, \Lambda; P)$, which will be defined and explained in details it Theorem 3.2.15, where the sets $I, \Lambda \subseteq E(T)$ may be chosen such that uT is a maximal principal right ideal of T for each $u \in I$ and dually, and distinct maximal principal left (respectively, right) ideals are disjoint.

We shall need a definition from [6, 7]. A semigroup S is said to have the *Dedekind height $\mathcal{L}$-property* if each element $a \in S$ is contained in only finitely many distinct principal left ideals. The *Dedekind height $\mathcal{R}$-property* is defined dually. A semigroup is said to be *Dedekind* if it satisfies both properties. We shall be interested in locally inverse semigroups that are unambiguous and Dedekind.

Let T be a regular semigroup that is unambiguous and Dedekind. Let Ta be an arbitrary principal left ideal. The partially ordered set of all principal left ideals of T that contain Ta is finite since T is Dedekind and so contains maximal elements. Therefore each principal left ideal of T is contained in a maximal principal left ideal. Let Ta, Tb be two maximal principal left ideals. Suppose that $Ta \cap Tb \neq \emptyset$. Let $c \in Ta \cap Tb$. Then $Tc \subseteq Ta, Tb$. Thus Ta and Tb are comparable. But both are maximal and so $Ta = Tb$. It follows that such semigroups in the case where T is Morita equivalent to an inverse monoid can be described by means of Theorem 2.8 [28].

To further motivate this class we shall use the following lemma.

Lemma 3.2.13 *Let $M = M(S; I, \Lambda; P)$ be a classical Rees matrix semigroup* *over the semigroup with local units S. Then M is Dedekind if and only if S is Dedekind.*

Proof Observe that $M(i,s,\lambda) \subseteq M(j,t,\lambda)$ implies that $s = (xp_{\mu j})t$ for some $x \in S$ and some $\mu \in \Lambda$. Thus $Ss \subseteq St$. Suppose that $Ss = St$. Then $t = ys$ for some $y \in S$. Choose $\nu \in \Lambda$ such that $p_{\nu i}$ is invertible. Then $(j,t,\lambda) = (j, yp_{\nu i}^{-1}, \nu)(i,s,\lambda)$. Thus $M(i,s,\lambda) = M(j,t,\lambda)$. It follows that the set of principal left ideals of M containing $M(i,s,\lambda)$ is mapped injectively to the set of principal left ideals of S containing s. It follows that if S is Dedekind then so too is M.

Let $Ss \subseteq St$ in S. Then $s = xt$ for some $x \in S$. Let $\lambda \in \Lambda$ be chosen and then fixed. Since M is classical there exists $i \in I$ such that $p_{\lambda i}$ is invertible. Observe that $(i,s,\lambda) = (i, xp_{\lambda i}^{-1}, \lambda)(i,t,\lambda)$. Thus $M(i,s,\lambda) \subseteq M(i,t,\lambda)$. Suppose that $M(i,s,\lambda) = M(i,t,\lambda)$. Then it is immediate that $Ss = St$. It follows that the set of principal left ideals of S containing Ss is mapped injectively into the set of principal left ideals of M containing $M(i,s,\lambda)$ where both i and λ are fixed. It follows that if M is Dedekind so too is S. ■

Examples 3.2.14

1. Groups are clearly both unambiguous and Dedekind. It follows from the Rees theorem that completely simple semigroups are unambiguous, Dedekind and locally inverse.

2. The fundamental four-spiral semigroup Sp_4 [8] also belongs to this class. An inverse monoid S is called an *inverse ω-monoid* if its semilattice of idempotents is isomorphic to the set $\mathcal{N}$ with the dual ordering. This means that $E(S) = \{e_n : n \in \mathcal{N}\}$ where $e_m \leq e_n$ iff $m \geq n$. The bicyclic monoid B is the fundamental bisimple inverse ω-monoid; see Section 5.4 of [23]. It is given by the monoid presentation $B = \langle p, q : pq = 1 \rangle$. Then
$$\mathrm{Sp}_4 = M(B; \{1,2\}, \{1,2\}; \begin{pmatrix} 1 & q \\ 1 & 1 \end{pmatrix}).$$

These two examples suggest studying the class of unambiguous, Dedekind locally inverse semigroups in more detail. Our goal is to construct all such semigroups which are Morita equivalent to monoids.

We begin with an important way of constructing new regular semigroups from old.

Theorem 3.2.15 (Lemma 2.1 of [28] and Lemma 2.6 of [31]) *Let $M(S; I, \Lambda; P)$ be a Rees matrix semigroup over the regular semigroup S. Then its set $R = RM(S; I, \Lambda; P)$ of regular elements forms a regular subsemigroup. Each local submonoid of R is isomorphic to a local submonoid of S.*

The semigroup $R = RM(S; I, \Lambda; P)$ is called a *regular Rees matrix semigroup.*

Proposition 3.2.16 *Let $RM(S; I, \Lambda; P)$ be a regular Rees matrix semigroup over the regular monoid S. If there exists $(\lambda, i) \in \Lambda \times I$ such that $p_{\lambda i} \mathcal{J} 1$, where 1 is the identity of S, then $R = RM(S; I, \Lambda; P)$ is Morita equivalent to the monoid S.*

Proof Let $1 = xp_{\lambda i}y$. Consider the triple (i, yx, λ). It is easy to check that it is an idempotent. We prove next that

$$R = R(i, yx, \lambda)R.$$

Let $(j, a, \mu) \in R$. Then

$$(j, a, \mu) = (j, ax, \lambda)(i, yx, \lambda)(i, y, \mu)$$

and so $R = R(i, yx, \lambda)R$.

We need to show now that $(i, yx, \lambda)R(i, yx, \lambda)$ is isomorphic to S. Define $\theta \colon S \to M(S; I, \Lambda; P)$ by $\theta(s) = (i, ysx, \lambda)$. It is easy to check that θ is an injective homomorphism. Since s is regular $\theta(s)$ is regular and so it is in fact a mapping into R. Observe that $\theta(s) \in (i, yx, \lambda)RM(i, yx, \lambda)$. It remains only to show that θ is surjective onto this local submonoid. But this is straightforward. The result now follows from Proposition 3.1.9. ■

Proposition 3.2.17 *Let S be regular. Put $R = RM(S; I, \Lambda; P)$.*

1. *If S is unambiguous then R is unambiguous.*

2. *If S is Dedekind then R is Dedekind.*

Proof (1) Assume that S is unambiguous. Suppose that

$$R(i, a, \lambda) \subseteq R(j, b, \lambda), R(k, c, \lambda)$$

where we may assume that (i, a, λ), (j, b, λ) and (k, c, λ) are all idempotents since R is regular. It follows that

$$(i, a, \lambda) = (i, a, \lambda)(j, b, \lambda) = (i, a, \lambda)(k, c, \lambda).$$

Thus

$$a = (ap_{\lambda j})b = (ap_{\lambda k})c.$$

By assumption, and without loss of generality, we may assume that $b = dc$. But then

$$(j, b, \lambda)(k, c, \lambda) = (j, bp_{\lambda k}c, \lambda) = (j, dcp_{\lambda k}c, \lambda) = (j, dc, \lambda) = (j, b, \lambda),$$

as required. The result now follows by symmetry.

(2) Assume that S is Dedekind. Let (j, a, λ) be any idempotent. Consider the set

$$\{R(i, x, \lambda) \colon R(j, a, \lambda) \subseteq R(i, x, \lambda)\}.$$

We prove that it is finite. We may assume that (i, x, λ) is also an idempotent. Observe that $R(j, a, \lambda) \subseteq R(i, x, \lambda)$ iff $(j, a, \lambda) = (j, a, \lambda)(i, x, \lambda)$ iff $a = ap_{\lambda i}x$ iff $p_{\lambda j}a = p_{\lambda j}ap_{\lambda i}x$ iff $Sp_{\lambda j}a \subseteq Sp_{\lambda i}x$. It now follows from the fact that S is Dedekind that R is Dedekind. ■

It is proved in Proposition 5.2.13 of [23] that an inverse semigroup whose idempotents form a finite chain is in fact a finite chain of groups.

Proposition 3.2.18 *Let S be an inverse monoid.*

1. *S is unambiguous if and only if the idempotents of S form a chain.*

2. *Assume that S is unambiguous. Then S is Dedekind if and only if the semilattice of S is either a finite chain, in which case S is a chain of groups, or S is an inverse ω-monoid.*

Proof Observe that in an inverse semigroup $Se \subseteq Sf$ iff $e \leq f$.

(1) Suppose that S is unambiguous. Let $e, f \in E(S)$. Then $Sef = Sfe \subseteq Se, Sf$. Thus without loss of generality we may assume that $Se \subseteq Sf$. But then $e \leq f$. It follows that the semilattice of idempotents is a linearly ordered set. Conversely, suppose that $E(S)$ is a linearly ordered set. Suppose that $Sa \subseteq Sb, Sc$. Then $Sa = Sa^{-1}a$, $Sb = Sb^{-1}b$ and $Sc = Sc^{-1}c$. Suppose that $b^{-1}b \leq c^{-1}c$. Then $Sb^{-1}b \subseteq Sc^{-1}c$.

(2) Suppose that S is unambiguous and Dedekind and that the semilattice of idempotents is infinite. Then above each idempotent can only be a finite number of idempotents. It follows that the idempotents must form an ω-chain. The converse is clear. ■

We have the following immediate corollary of the above results.

Proposition 3.2.19 *Let S be an inverse monoid and let $RM(S; I, \Lambda; P)$ be a regular Rees matrix semigroup over S. Suppose that there exists $(\lambda, i) \in \Lambda \times I$ such that $p_{\lambda i} \mathcal{J} 1$. If S is either a chain of groups or an inverse ω-monoid then $RM(S; I, \Lambda; P)$ is a locally inverse semigroup which is unambiguous, Dedekind and Morita equivalent to S.*

The above proposition tells us how to construct a family of unambiguous, Dedekind locally inverse semigroups which are Morita equivalent to inverse monoids. We now prove that all such semigroups arise in this way.

Proposition 3.2.20 *Let T be an unambiguous, Dedekind, regular semigroup for which there is an idempotent e such that $T = TeT$ and eTe is inverse. Then T is isomorphic to a regular Rees matrix semigroup $RM(eTe; I, \Lambda; P)$ where there exists $(\lambda, i) \in \Lambda \times I$ such that $p_{\lambda i} \mathcal{J} e$. The inverse monoid eTe is both unambiguous and Dedekind.*

Proof We shall use Theorem 2.8 of [28]. Because T is Dedekind, each principal left ideal is contained in a maximal principal left ideal, and dually. Since T is regular, each principal left ideal has an idempotent generator, and dually. Let $\{e_i : i \in I\}$ be a set of representatives of the idempotent generators of the maximal principal right ideals. Similarly, let $\{f_\lambda : i \in \Lambda\}$ be a set of representatives of the idempotent generators of the maximal principal left ideals. By unambiguity, distinct maximal principal left ideals are disjoint, and dually. Each idempotent of T is $\mathcal{D}$-related to an idempotent in eTe by Lemma 3.1.16. Thus for each $u \in E(T)$ we may find r_u and $r'_u \in V(r_u)$ such that $u = r_u r'_u$ and $r'_u r_u \leq e$. If $u = e_i$ we write $r_i = r_{e_i}$ and if $u = f_\lambda$ we write $r_\lambda = r_{f_\lambda}$. Put $p_{\lambda i} = r'_\lambda r_i \in eTe$. By Theorem 2.8 [28], the semigroup T is isomorphic to the regular Rees matrix semigroup $RM(eTe; I, \Lambda; P)$. There exists $i \in I$ and $\lambda \in \Lambda$ such that $e = e_i e f_\lambda$. But

$$e = e^2 = (e_i e f_\lambda)(e_i e f_\lambda) = (e e_i e r_\lambda r'_\lambda)(r_i r'_i e f_\lambda e)$$

$$= (ee_ier_\lambda)p_{\lambda i}(r_i'ef_\lambda e)$$

but $r_\lambda = r_\lambda(r_\lambda' r_\lambda)e = r_\lambda e$ and $r_i' = e(r_i' r_i)r_i' = er_i'$ and so for some $t, s \in T$ we have

$$e = (ete)p_{\lambda i}(ese).$$

It follows that $e\mathcal{J}p_{\lambda i}$ in eTe. It remains to prove the last two assertions. We prove first that eTe is umambiguous. Let $i, j \in E(eTe)$. Then $ij = ji \leq i, j$. It follows that $Tij = Tji \subseteq Ti, Tj$. Without loss of generality, we may assume by unambiguity that $Ti \subseteq Tj$. Thus $i = ij$. But then $i \leq j$ in the inverse semigroup eTe. Finally, we prove that eTe is Dedekind. Let $i \leq i_1 \leq i_2 \leq \ldots$. Then $Ti \subseteq Ti_1 \subseteq \ldots$. But T is Dedekind so only finitely many of these principal left ideals can be distinct. Thus there are only finitely many distinct idempotents above i. ■

We may combine the above two propositions in the following theorem which can be viewed as a generalization of the classical Rees theorem.

Theorem 3.2.21 *Let T be an unambiguous, Dedekind, regular semigroup. Then T is Morita equivalent to an inverse monoid if and only if T is isomorphic to a regular Rees matrix semigroup $RM(S; I, \Lambda; P)$ satisfying the following two conditions:*

1. *There exists $(\lambda, i) \in \Lambda \times I$ such that $p_{\lambda i}\mathcal{J}1$.*
2. *S is an inverse monoid with either a finite chain or ω-chain of idempotents.*

To finish off this section, we shall look at completely simple images of semigoups of the form $RM(S; I, \Lambda; P)$ where S is an inverse monoid. Recall that σ is the minimum group congruence on the inverse semigroup S and it is defined by $s\sigma t$ if and only if there is an element u such that $u \leq s$ and $u \leq t$. On a regular semigroup, the natural partial order is given by $s \leq t$ if and only if $s = et = tf$ for some idempotents e and f [36, 29]. We may deduce from Theorem 4.2 in [36], that the finest primitive congruence β on a regular semigroup without zero which is locally inverse is given by $s\beta t$ if and only if there exists $u \leq s, t$.

Theorem 3.2.22 *Let $R = RM(S; I, \Lambda; P)$ where S is an inverse monoid. Put $G = S/\sigma$. Define $q_{\lambda i} = \sigma(p_{\lambda i})$ and put Q equal to the resulting matrix. Define*

$$\theta\colon RM(S; I, \Lambda; P) \to M(G; I, \Lambda; Q)$$

by $\theta(i, s, \lambda) = (i, \sigma(s), \lambda)$. *Then* θ *is a surjective homomorphism to a completely simple semigroup and the kernel of this map is the minimum completely simple congruence on* $RM(S; I, \Lambda; P)$.

Proof The fact that θ is a surjective homomorphism is easy to prove. Let $(i, s, \lambda), (i, t, \lambda) \in R$. Then

$$s = sp_{\lambda k}xp_{\mu i}s \text{ and } t = tp_{\lambda l}yp_{\nu i}t$$

for some $x, y \in S$ and some (k, μ) and (l, ν). Suppose that $\theta(i, s, \lambda) = \theta(i, t, \lambda)$. Then in the inverse semigroup S there is an element $u \leq s, t$. We prove that $(i, u, \lambda) \leq (i, s, \lambda), (i, t, \lambda)$. By symmetry, it is enough to prove that $(i, u, \lambda) \leq (i, s, \lambda)$. Consider the element $(i, up_{\lambda k}x, \mu)$. First, we show that it is an idempotent. Since $s = sp_{\lambda k}xp_{\mu i}s$ and $u \leq s$ which means $u = es = sf = esf$ for some idempotents $e, f \in S$. Thus $u = esf = esp_{\lambda k}xp_{\mu i}sf = up_{\lambda k}xp_{\mu i}u$. It follows that $(i, up_{\lambda k}x, \mu)$ is an idempotent. Next observe that $(i, u, \lambda) = (i, up_{\lambda k}x, \mu)(i, s, \lambda)$ because $u = up_{\lambda k}xp_{\mu i}s$. It is immediate that (i, u, λ) is regular. By symmetry we deduce that $(i, u, \lambda) \leq (i, s, \lambda)$. ■

3.3 Morita equivalence via Rees matrix semigroups

Laan and Márki [20] found the following Rees matrix characterization of when two semigroups S and T with local units are Morita equivalent: there is a Rees matrix semigroup $M(T; I, \Lambda; P)$ over T such that $T = T\mathrm{im}(P)T$, where $\mathrm{im}(P)$ is the set of elements that occur in P, and there is a local isomorphism from M to S. We shall give a different proof of this theorem that generalizes the co-ordinatization approach used in the previous section.

Let $M = M(T; I, \Lambda; P)$ be a Rees matrix semigroup. Put

$$\overline{M} = E(M)ME(M),$$

a semigroup with local units. We say that $\overline{M}$ is a *Rees matrix semigroup with local units*. We want to use such Rees matrix semigroups to construct semigroups Morita equivalent to T. This will require us to impose conditions on P. We shall say that $M = M(T; I, \Lambda; P)$ is *proper* if the following condition holds: for each $e \in E(T)$ there exists $(\lambda, i) \in \Lambda \times I$ such that $TeT \subseteq Tp_{\lambda i}T$. The Rees matrix semigroups that we used in the previous section were proper in this sense. We first show the connection between this definition and the work of Laan and Márki.

Lemma 3.3.1 *Let T be a semigroup with local units and let $M = M(T; I, \Lambda; P)$ be a Rees matrix semigroup over T. Then M is proper if and only if $T = T\mathrm{im}(P)T$.*

Proof Suppose that M is proper. Let $t \in T$ be arbitrary. Then since T has local units there is an idempotent e such that $t = et$. By assumption, given e there exists $(\lambda, i) \in \Lambda \times I$ such that $TeT \subseteq Tp_{\lambda i}T$. It follows that $e = xp_{\lambda i}y$ for some $x, y \in T$. Thus $t = et = xp_{\lambda i}yt$ and so $t \in T\mathrm{im}(P)T$. Conversely, suppose that $T = T\mathrm{im}(P)T$. Let $e \in E(T)$. Then $e = xp_{\lambda i}y$ for some $(\lambda, i) \in \Lambda \times I$ and elements $x, y \in T$. Thus $TeT \subseteq Tp_{\lambda i}T$. ■

Our first goal is to obtain an alternative characterization of when a Rees matrix semigroup is proper.

Lemma 3.3.2 *Let T be a semigroup with local units. Let e be an idempotent such that $TeT \subseteq TpT$ and let $e = xpy$ where $x \in eT$ and $y \in Te$. Then $(yx)p(yx) = yx$.*

Proof We have that $(yx)p(yx) = y(xpy)x = yex = yx$. ■

In any semigroup T, define

$$R(b) = \{a \in T : a = aba\}.$$

If $R(b)$ is non-empty it consists of regular elements. If $a \in R(b)$ then both ab and ba are idempotents. Let $M = M(T; I, \Lambda; P)$ be a Rees matrix semigroup. Define

$$R(P) = \{a \in T : a = ap_{\lambda i}a \text{ for some } p_{\lambda i}\}.$$

Lemma 3.3.3 *$M = M(T; I, \Lambda; P)$ is proper if and only if each idempotent of T is $\mathcal{D}$-related to an element of $R(P)$.*

Proof Suppose that M is proper. Let $e \in E(T)$. By assumption, there exists $(\lambda, i) \in \Lambda \times I$ such that $TeT \subseteq Tp_{\lambda i}T$. Let $e = xp_{\lambda i}y$ where $x \in eT$

and $y \in Te$. Then by Lemma 3.3.2, we have that $yx \in R(p_{\lambda i})$. It follows that $p_{\lambda i}yxp_{\lambda i} \in V(yx)$. Thus $p_{\lambda i}yx \,\mathcal{D}\, yxp_{\lambda i} \,\mathcal{D}\, yx$. Now $e = xp_{\lambda i}y = x(p_{\lambda i}y)$. Observe that

$$x(p_{\lambda i}y)x = (xp_{\lambda i}y)x = ex = x \text{ and } (p_{\lambda i}y)x(p_{\lambda i}y) = p_{\lambda i}y(xp_{\lambda i}y) = p_{\lambda i}ye = p_{\lambda i}y.$$

Thus $p_{\lambda i}y \in V(x)$. It follows that $e\mathcal{D}p_{\lambda i}yx$. Hence

$$e \,\mathcal{D}\, yx \in R(p_{\lambda i}).$$

To prove the converse, suppose that each idempotent of T is $\mathcal{D}$-related to an element of $R(P)$. Let $e \in E(T)$. Then by assumption there exists $z \in R(p_{\lambda i})$ for some $(\lambda, i) \in \Lambda \times I$ such that $e\mathcal{D}z$. Now $e\mathcal{D}z$ implies $e\mathcal{J}z$ and so $TeT = TzT$. But $z = zp_{\lambda i}z$. Thus $TzT = Tzp_{\lambda i}zT \subseteq Tp_{\lambda i}T$. Thus $TeT \subseteq Tp_{\lambda i}T$, and so M is proper. ■

The above results now enable us to prove the following. It was originally proved using different methods by Laan and Márki.

Proposition 3.3.4 *Let T be a semigroup with local units. Let $M = M(T; I, \Lambda; P)$ be a proper Rees matrix semigroup. Then $\overline{M}$ is Morita equivalent to T.*

Proof Observe that $C(M) = C(\overline{M})$. To avoid confusion we shall denote the elements of $C(M)$ by square brackets. A typical element of $C(M)$ has the form

$$\mathbf{s} = [(i, a, \lambda), (i, s, \mu), (j, b, \mu)]$$

where $a = ap_{\lambda i}a$ and $b = bp_{\mu j}b$ and $ap_{\lambda i}s = s$ and $sp_{\mu j}b = s$. We shall prove that $C(M)$ and $C(T)$ are equivalent categories which proves that $\overline{M}$ and T are Morita equivalent.

If $[(i, a, \lambda), (i, s, \mu), (j, b, \mu)] \in C(M)$ then $ap_{\lambda i}$ and $bp_{\mu j}$ are idempotents and $(ap_{\lambda i})sp_{\mu j}(bp_{\mu j}) = sp_{\mu j}$. Thus $(ap_{\lambda i}, sp_{\mu j}, bp_{\mu j})$ is an element of $C(T)$. We may therefore define $\Psi\colon C(M) \to C(T)$ by

$$\Psi[(i, a, \lambda), (i, s, \mu), (j, b, \mu)] = (ap_{\lambda i}, sp_{\mu j}, bp_{\mu j}).$$

We shall prove that Ψ is an equivalence of categories.

(1) Ψ is a functor.
The identities in $C(M)$ are the elements of the form

$$[(i, a, \lambda), (i, a, \lambda), (i, a, \lambda)]$$

where $a = ap_{\lambda i}a$. By definition

$$\Psi[(i,a,\lambda),(i,a,\lambda),(i,a,\lambda)] = (ap_{\lambda i}, ap_{\lambda i}, ap_{\lambda i})$$

which is an identity in $C(T)$. Let

$$\mathbf{s} = [(i,a,\lambda),(i,s,\mu),(j,b,\mu)]$$

and

$$\mathbf{t} = [(j,b,\mu),(j,t,\xi),(k,c,\xi)]$$

be a pair of composable elements in $C(M)$. Then

$$\mathbf{st} = [(i,a,\lambda),(i,sp_{\mu j}t,\xi),(k,c,\xi)]$$

and so

$$\Psi(\mathbf{st}) = (ap_{\lambda i}, sp_{\mu j}tp_{\xi k}, cp_{\xi k}).$$

On the other hand

$$\Psi(\mathbf{s})\Psi(\mathbf{t}) = (ap_{\lambda i}, sp_{\mu j}, bp_{\mu j})(bp_{\mu j}, tp_{\xi k}, cp_{\xi k})$$

which is just

$$(ap_{\lambda i}, sp_{\mu j}tp_{\xi k}, cp_{\xi k}).$$

(2) Ψ is faithful.
Suppose that

$$\Psi[(i,a,\lambda),(i,s,\mu),(j,b,\mu)] = \Psi[(i,a,\lambda),(i,t,\mu),(j,b,\mu)].$$

Then $sp_{\mu j} = tp_{\mu j}$. But $sp_{\mu j}b = tp_{\mu j}b$ and so $s = t$, as required.

(3) Ψ is full.
Let $(f,t,e) \in C(T)$ where

$$\Psi[(i,a,\lambda),(i,a,\lambda),(i,a,\lambda)] = (e,e,e)$$

and

$$\Psi[(j,b,\mu),(j,b,\mu),(j,b,\mu)] = (f,f,f).$$

Thus $ap_{\lambda i} = e$ and $bp_{\mu j} = f$. But

$$[(j,b,\mu),(j,ta,\lambda),(i,a,\lambda)] \in C(M).$$

and

$$\Psi[(j,b,\mu),(j,ta,\lambda),(i,a,\lambda)] = (f,t,e),$$

as required.

(4) Ψ is essentially surjective.
Let (e,e,e) and (f,f,f) be identities in $C(T)$. We prove first that there is an isomorphism in $C(T)$ from (f,f,f) to (e,e,e) if and only if $e\mathcal{D}f$. Suppose that $e\mathcal{D}f$. Then there exists x and $y \in V(x)$ such that $xy = e$ and $yx = f$. Observe that (e,x,f) and (f,y,e) are well-defined elements of $C(T)$ and that they are mutually invertible. Conversely, (e,x,f) is an isomorphism in $C(T)$ if and only if there is an element (f,y,e) such that $xy = e$ and $yx = f$. Observe that $x \in eTf$ and $y \in fTe$. Thus $y \in V(x)$ and $xy = e$ and $yx = f$. It follows that $e\mathcal{D}f$.

Let (e,e,e) be an arbitrary identity in $C(T)$. Then e is an idempotent in T. From Lemma 3.3.3 there exists (λ,i) and $a \in R(p_{\lambda i})$ such that $e\mathcal{D}a$. Thus $e\mathcal{D}ap_{\lambda i}$. It follows that there is an isomorphism between (e,e,e) and $\Psi[(i,a,\lambda),(i,a,\lambda),(i,a,\lambda)]$ and so Ψ is essentially surjective. ■

Proposition 3.3.5 *Let S and T be semigroups with local units. If S is Morita equivalent to T then S is a locally isomorphic image of a proper Rees matrix semigroup over T.*

Proof Let S and T be Morita equivalent. We shall construct a co-ordinatization of S which we shall use to construct a proper Rees matrix semigroup over T.

By Theorem 3.1.8, the third algebraic characterization, there is a semigroup with local units U such that U is an enlargement of both S and T. By Lemma 3.1.16(1), each regular $\mathcal{D}$-class of U contains a regular $\mathcal{D}$-class of S and a regular $\mathcal{D}$-class of T.

Let $\mathcal{E}$ be an idempotent transversal of the regular $\mathcal{D}$-classes of T. Let $D(U)_e$ be the $\mathcal{D}$-class in U containing e. This will necessarily be a regular $\mathcal{D}$-class. Let I^e index the $\mathcal{R}$-classes in $D(U)_e$ that contain elements from S. Let Λ^e index the $\mathcal{L}$-classes in $D(U)_e$ that contain elements from S. For each $i \in I^e$ choose an element r_i such that $r_i\mathcal{L}e$ and r_i is $\mathcal{R}$-related to some element in S. Observe that r_i is regular and $r_ie = r_i$. Dually, for each $\lambda \in \Lambda^e$ choose an element q_λ such that $q_\lambda\mathcal{R}e$ and q_λ is $\mathcal{L}$-related to some element of S. Observe that q_λ is regular and that $eq_\lambda = q_\lambda$. Put $I = \bigcup_{e\in\mathcal{E}} I^e$ and $\Lambda = \bigcup_{e\in\mathcal{E}} \Lambda^e$. Define P by $p_{\lambda i} = q_\lambda r_i$. Observe that $p_{\lambda i} \in T$ because if

$r_i e = r_i$ and $eq_\lambda = q_\lambda$ where $e \in \mathcal{E}$ then $ep_{\lambda i}e = p_{\lambda i} \in eUe \subseteq T$ since U is an enlargement of T. We have therefore defined a Rees matrix semigroup $M = M(T; I, \Lambda; P)$.

Let $e \in \mathcal{E}$. Then $e\mathcal{D}f$ where $f \in E(S)$ since each idempotent in T is $\mathcal{D}$-related to an idempotent in S. By assumption, there exists r_i where $i \in I^e$ and q_λ where $\lambda \in \Lambda^e$ such that $r_i \mathcal{R} f \mathcal{L} q_\lambda$. It follows that $q_\lambda r_i \mathcal{H} e$. Thus $p_{\lambda i}$ is a regular element $\mathcal{D}$-related to e and so M is a proper Rees matrix semigroup.

(1) $S = \bigcup_{i \in I, \lambda \in \Lambda} r_i T q_\lambda$. Let $s \in S$.
Since S has local units there exist $e_s, f_s \in E(S)$ such that $s = e_s s f_s$. But e_s and f_s are regular elements in S and so there is $i \in I$ such that $e_s \mathcal{R} r_i$ and there is $\lambda \in \Lambda$ such that $f_s \mathcal{L} q_\lambda$. By Lemma 3.1.12, there is $q'_\lambda \in V(q_\lambda)$ such that $f_s = q'_\lambda q_\lambda$ and $r'_i \in V(r_i)$ such that $e_s = r_i r'_i$. It follows that

$$s = r_i r'_i s q'_\lambda q_\lambda = r_i (r'_i s q'_\lambda) q_\lambda.$$

Thus $s = r_i t q_\lambda$. Suppose that $i \in I^e$ and $\lambda \in \Lambda^f$. Then $r_i e = r_i$ and $f q_\lambda = q_\lambda$ where $e, f \in \mathcal{E}$. Thus $s = r_i(etf)q_\lambda$. But $etf \in TUT \subseteq T$ since U is an enlargement of T.

It follows from the above calculation that the function

$$\theta \colon M(T; I, \Lambda; P) \to S$$

defined by $\theta(i, t, \lambda) = r_i t q_\lambda$ is surjective.

(2) θ is a homomorphism.
Let $(i, t, \lambda), (j, s, \mu) \in M$. Then

$$\theta((i, t, \lambda)(j, s, \mu)) = \theta(i, t p_{\lambda j} s, \mu)$$

$$= r_i t q_\lambda r_j s q_\mu = \theta(i, t, \lambda)\theta(j, s, \mu).$$

(3) Every idempotent in S is the image of an idempotent from M.
Let $f \in E(S)$. Then $f\mathcal{D}e$ for some $e \in \mathcal{E}$. We have $f = ff$ and so there is $i \in I$ such that $f \mathcal{R} r_i$ and there is $\lambda \in \Lambda$ such that $f \mathcal{L} q_\lambda$. By Lemma 3.1.12, there is $q'_\lambda \in V(q_\lambda)$ such that $f = q'_\lambda q_\lambda$ and $r'_i \in V(r_i)$ such that $f = r_i r'_i$. It follows that

$$f = r_i r'_i q'_\lambda q_\lambda = r_i (e r'_i q'_\lambda e) q_\lambda.$$

Put $t = er'_i q'_\lambda e$. Thus $\theta(i,t,\lambda) = f$. It is easy to check that (i,t,λ) is an idempotent.

(4) θ is a surjective local isomorphism.
Let (i,a,λ) and (j,b,μ) be idempotents. Then a typical element of $(i,a,\lambda)M(j,b,\mu)$ is of the form $(i, ap_{\lambda k}cp_{\xi j}b, \mu)$ where $k \in I$ and $\xi \in \Lambda$ are arbitrary. The image of such an element under θ is $x = r_i ap_{\lambda k}cp_{\xi j}bq_\mu$. Observe that $ap_{\lambda k}cp_{\xi j}b = (aq_\lambda)x(r_j b)$. Thus θ restricted to $(i,a,\lambda)M(j,b,\mu)$ is injective. Now let $r_i aq_\lambda sr_j bq_\mu \in \theta(i,a,\lambda)S\theta(j,b,\mu)$. Then $(i, aq_\lambda sr_j b, \mu) \in (i,a,\lambda)M(j,b,\mu)$ and $\theta(i, aq_\lambda sr_j b, \mu) = r_i aq_\lambda sr_j bq_\mu$. Thus

$$\theta\colon (i,a,\lambda)M(j,b,\mu) \to \theta(i,a,\lambda)S\theta(j,b,\mu)$$

is a bijection. It is evident from what we have proved that the restriction $\theta\colon \overline{M} \to S$ is a surjective local isomorphism. ■

The proof of the following is now obtained from Proposition 3.3.5 and Lemma 3.1.6 and Proposition 3.3.4. The version of the theorem that applies to regular semigroups follows from Lemma 3.1.21.

Theorem 3.3.6 *Let S and T be semigroups with local units. Then S and T are Morita equivalent if and only if S is a locally isomorphic image of a proper Rees matrix semigroup with local units over T. If, in addition, both S and T are regular, then the Rees matrix semigroup can be chosen to be a regular Rees matrix semigroup.*

3.4 Inverse semigroups

The main goal of this section is to describe all the inverse semigroups Morita equivalent to a given inverse semigroup S. We do this in Section 3.4.4 in terms of a special class of Rees matrix semigroups. In Section 3.4.1, we give a new proof of Theorem 3.1.7, the second algebraic characterization of Morita equivalence, and in Section 3.4.2 we specialize this result to the case of inverse semigroups. In Section 3.4.3, we apply this special case to characterizing those semigroups Morita equivalent to semigroups with commuting idempotents. This can be seen to complete work begun by Khan and Lawson [17, 18].

3.4.1 Consolidations: the general case

The goal of this section is to give another proof of the second algebraic characterization of Morita equivalence in terms of consolidations. Our proofs only use category theory and not enlargements.

Recall that a category C is *strongly connected* if for each pair of identities e and f the set $eCf \neq \emptyset$. A *consolidation* q in a strongly connected category C is a function $q: C_o \times C_o \to C$ such that $q_{e,f} \in eCf$ and $q_{e,e} = e$. A consolidation q is said to be *regular* if $q_{e,f}$ is regular for all identities e and f.

Categories of the form $C(S)$ are strongly connected by Lemma 3.1.19. It is convenient to identify the identities of $C(S)$ with the idempotents of S. Consolidations on such categories can be constructed very simply as the following lemma shows.

Lemma 3.4.1 *Let S be a semigroup with local units. Let $q: E(S) \times E(S) \to S$ be a function such that $q(e,e) = e$ and $q(e,f) \in eSf$. Given such a function, define $q_{e,f} = (e, q(e,f), f)$. Then q is a consolidation on $C(S)$ and every consolidation on $C(S)$ arises in this way.*

If q is a consolidation in C then C becomes a semigroup with local units when we define $x \circ y = xq_{e,f}y$ whenever $x \in Ce$ and $y \in fC$ where e and f are identities. We denote C with respect to this semigroup operation by C^q. Observe that if xy is defined in the category C then $x \circ y = xy$ since $q_{e,e} = e$ for all identities e. If e is an identity in C then e is an idempotent in C^q. However, in general, C^q will have other idempotents: $x \in eCf$ is an idempotent if and only if $x = xq_{f,e}x$. Idempotents in C^q of the form e where e is an identity in the category C will also be called *identities*. Observe that arrows $\mathcal{D}$-related in the category C continue to be $\mathcal{D}$-related in the semigroup C^q.

Lemma 3.4.2 *Let C be a strongly connected category with consolidation q. If every idempotent in C splits then every idempotent in C^q is $\mathcal{D}$-related to an identity.*

Proof Let a be an idempotent in C^q where $a \in eCf$. Then $a = aq_{e,f}a$. Put $b = q_{e,f}aq_{e,f}$. Then $aba = aq_{e,f}aq_{e,f}a = aq_{e,f}a = a$ and similarly $b = bab$. Put $i = ba$. Then i is an idempotent in fCf and $i\mathcal{D}a$. By Lemma 3.1.18 there is an identity j such that $j\mathcal{D}i$. It follows that $a\mathcal{D}j$ in the semigroup C^q.■

Lemma 3.4.3 *Let $\Theta: C \to D$ be a weak equivalence functor between two strongly connected categories. Let q be a consolidation on D.*

1. *There is a consolidation p on C, called the* pullback consolidation, *such that $\Theta: C^p \to D^q$ is a homomorphism satisfying (LI1).*

2. *If q is regular then p is regular.*

3. *If all idempotents in D split then Θ is a local isomorphism.*

Proof (1). Let e and f be arbitrary identities in C. Then $\Theta(e)$ and $\Theta(f)$ are identities in D. There is therefore an element $q_{\Theta(f),\Theta(e)}$. Since the functor Θ is full and faithful there is a unique element $p_{f,e}$ such that $\Theta(p_{f,e}) = q_{\Theta(f),\Theta(e)}$. We now calculate $\Theta(x \circ y)$ where $x \in Ce$ and $y \in fC$ where e and f are identities. We have that

$$\Theta(x \circ y) = \Theta(xp_{e,f}y) = \Theta(x)\Theta(p_{e,f})\Theta(y) = \Theta(x) \circ \Theta(y).$$

Thus $\Theta: C^p \to D^q$ is a homomorphism.

Let a and b be idempotents in C^p where $a \in eCf$ and $b \in iCj$. Then $a = ap_{f,e}a$ and $b = bp_{j,i}b$. The elements in $a \circ C^p \circ b$ are of the form $ap_{f,g}xp_{h,i}b$ where $x \in gCh$. The elements in $\Theta(a) \circ D^q \circ \Theta(b)$ are of the form $\Theta(a)q_{\Theta(f),l}yq_{m,\Theta(i)}\Theta(b)$ where $y \in lDm$. The fact that Θ is full and faithful now implies that (LI1) holds.

(2). Let e and f be identities in C and suppose that $\Theta(e) \overset{x}{\leftarrow} \Theta(f)$ is regular with an inverse y. Suppose that $e \overset{x'}{\leftarrow} f$ and $f \overset{y'}{\leftarrow} e$ are such that $\Theta(x') = x$ and $\Theta(y') = y$. Then $\Theta(x'y'x') = xyx = x$. By uniqueness, $x'y'x' = x'$ and similarly $y'x'y' = y'$. In particular, x' is regular. The result now follows.

(3). It remains to show that (LI2) holds which is where we use the fact that idempotents in D split. By Lemma 3.4.2, every idempotent in D^q is $\mathcal{D}$-related to an identity. But Θ is essentially surjective and so each identity in D is isomorphic to the image under θ of an identity of C. But isomorphic identities in D are $\mathcal{D}$-related in D^q. It follows that (LI2) holds. ∎

The *standard consolidation* s on $C(S)$ is defined by $s_{e,f} = (e, ef, e)$.

Lemma 3.4.4 *With the above definitions, the map $\nu: C(S)^s \to S$, given by $(e, s, f) \mapsto s$, is a surjective local isomorphism.*

Proof Clearly, ν is a surjective homomorphism. It remains to show that ν satisfies (LI1). Let (e, a, f) and (i, b, j) be idempotents in $C(S)^s$. Then both a and b are idempotents in S. The elements of $(e, a, f) \circ C(S)^s \circ (i, b, j)$ therefore have the form (e, acb, j) where c is an arbitrary element of S. It follows that ν induces a bijection from $(e, a, f) \circ C(S)^s \circ (i, b, j)$ to aSb. ■

Recall that a *weak equivalence* is a functor that is full, faithful and essentially surjective.

Proposition 3.4.5 *There is a weak equivalence $\Theta: C(S) \to C(T)$ if and only if there is a local isomorphism $\theta: C(S)^p \to T$ for some consolidation p defined on $C(S)$.*

Proof Let $\Theta: C(S) \to C(T)$ be a weak equivalence. We endow $C(T)$ with the standard consolidation s. Let p be the pullback consolidation defined on $C(S)$ by Lemma 3.1.19 and 0Lemma 3.4.3. Then $\Theta: C(S)^p \to C(T)^s$ defines a local isomorphism by Lemm 3.1.19 and Lemma 3.4.3. But $\nu: C(T)^s \to T$ is a local isomorphism by Lemma 3.4.4. Thus by Lemma 3.1.21, there is a local isomorphism from $C(S)^p$ to T.

Now let $\theta: C(S)^p \to T$ be a local isomorphism for some consolidation p defined on $C(S)$. The identity (e, e, e) is an idempotent in $C(S)^p$ and so $\theta(e, e, e)$ is an idempotent in T. Clearly $(e, e, e)(e, s, f)(f, f, f) = (e, s, f)$ in $C(S)$ and so in $C(S)^p$ and so $\theta(e, e, e)\theta(e, s, f)\theta(f, f, f) = \theta(e, s, f)$. It follows that $(\theta(e, e, e), \theta(e, s, f), \theta(f, f, f))$ is a well-defined element in $C(T)$. We may therefore define $\Theta: C(S) \to C(T)$ by $\Theta(e, s, f) = (\theta(e, e, e), \theta(e, s, f), \theta(f, f, f))$. It is routine to check that Θ is a functor, and it is full and faithful because (LI1) holds.

We now show that Θ is essentially surjective. Let (e, e, e) be an identity in $C(T)$. Then e is an idempotent in T. By (LI2), there is an idempotent (i, a, j) in $C(S)^p$ such that $\theta(i, a, j)\mathcal{D}e$. However, by Lemma 3.4.2 every idempotent in $C(S)^p$ is $\mathcal{D}$-related to an identity (j, j, j). Thus $e\mathcal{D}\theta(j, j, j)$. Let $t, t' \in T$ such that $e = tt'$ and $t't = \theta(j, j, j)$. Then $(e, t, \theta(j, j, j)) \in C(T)$ is an isomorphism. Hence Θ is essentially surjective. ■

The above proposition was first proved in [25] where enlargements were used. The advantage of this new proof is that it is more direct.

By Theorem 3.1.4, the first algebraic characterization of Morita equivalent semigroups, and Proposition 3.4.5, we have the following, which is the second characterization of Morita equivalent semigroups (Theorem 3.1.7). This was originally proved by Lawson using enlargements.

Theorem 3.4.6 *Let S and T be semigroups with local units. Then S and T are Morita equivalent if and only if there is a local isomorphism $\theta \colon C(S)^p \to T$ for some consolidation p defined on $C(S)$.*

We conclude this section with some results connected to regularity.

Lemma 3.4.7 *The standard consolidation on the semigroup S is regular if and only if the regular elements of S form a subsemigroup.*

Proof By definition, $s_{e,f} = (e, ef, f)$. By Proposition 3.1.11, the regular elements of S form a subsemigroup if and only if ef is regular for all idempotents e and f. Suppose that ef is regular. Thus by Lemma 3.1.12, there exists $a \in V(ef) \cap fSe$. Thus (f, a, e) is an inverse of (e, ef, f) in the category $C(S)$. Thus the standard consolidation s is regular.

Conversely, suppose that the standard consolidation is regular. Then the arrows (e, ef, f) have inverses in $C(S)$ which implies that ef is regular. It follows again by Proposition 3.1.12, that the regular elements of S form a regular subsemigroup. ■

Proposition 3.4.8 *Let S and T be semigroups with local units. If S is Morita equivalent to T whose regular elements form a subsemigroup then for all $e, f \in E(S)$ the set $eSf \cap \text{Reg}(S)$ is non-empty.*

Proof Let $\Theta \colon C(S) \to C(T)$ be a weak equivalence and let s be the standard consolidation on $C(T)$ and let q be the pullback consolidation on $C(S)$. By Lemma 3.4.7, the standard consolidation is regular, and by Lemma 3.4.3, the pullback consolidation is also regular. By Lemma 3.4.1, this implies that $eSf \cap \text{Reg}(S)$ is non-empty for all idempotents e and f. ■

The condition in the proposition above first arose in [23].

3.4.2 Consolidations: the inverse case

We shall now specialize the results of the previous section to inverse semigroups. Let $\Theta\colon C \to D$ be a full and faithful functor between two inverse categories. Let $fCe \neq \emptyset$ where e and f are identities. Then Θ induces an order isomorphism between the partially ordered sets fCe and $\Theta(f)D\Theta(e)$.

A consolidation q on a strongly connected inverse category C is said to be *McAlister* if it satisfies the following two additional conditions:

(MC1) $q_{e,f}^{-1} = q_{f,e}$.

(MC2) $q_{e,f}q_{f,g} \leq q_{e,g}$.

Lemma 3.4.9 *Let S be an inverse semigroup. Then $C(S)$ is a strongly connected inverse category in which all idempotents split and whose standard consolidation is McAlister.*

Proof By Lemma 3.1.20, the category $C(S)$ is strongly connected and all idempotents split. It is a regular category in which the local monoids are inverse. Thus it is an inverse category. The standard consolidation on $C(S)$ is easily seen to be McAlister. ■

The proof of the following is straightforward because the definition of a McAlister consolidation is purely algebraic.

Lemma 3.4.10 *Let $\Theta\colon C \to D$ be a weak equivalence functor between two strongly connected inverse categories. Let q be a McAlister consolidation on D. Then its pullback consolidation p on C is also McAlister.*

The next result tells us of the pivotal role played by McAlister consolidations in the inverse case. The proof follows immediately by Lemma 3.4.10 and 3.4.3.

Theorem 3.4.11 *Let S and T be inverse semigroups. Then S and T are Morita equivalent if and only if there is a local isomorphism $\theta\colon C(S)^p \to T$ for some McAlister consolidation p defined on $C(S)$.*

Our goal now is to characterize abstractly the categories $C(S)$ where S is inverse.

Theorem 3.4.12 *An inverse category I is equivalent to a category of the form $C(S)$ where S is an inverse semigroup if and only if the following three conditions hold:*

1. *I is strongly connected.*
2. *Every idempotent in I splits.*
3. *I is equipped with a McAlister consolidation.*

Proof We prove the easy direction first. Suppose that I is equivalent to a category of the form $C(S)$ where S is an inverse semigroup. Thus there is a weak equivalence from I to $C(S)$. By Lemmas 3.1.19 and 3.1.20, I is strongly connected and every idempotent splits. By Lemma 3.4.9, the standard consolidation on $C(S)$ is McAlister. Thus by Lemma 3.4.10, the pullback consolidation on I is McAlister.

We now prove the hard direction. Let I be a strongly connected inverse category in which all idempotents split equipped with a McAlister consolidation q. Define $x \circ y = xq_{e,f}y$ if $\text{dom}(x) = e$ and $\text{cod}(y) = f$. We denote the set I equipped with the binary operation $\circ$ by I°. Recall that a regular semigroup is a generalized inverse semigroup if it is orthodox and locally inverse. It follows by Theorem 6.7 from [22] that I° is a generalized inverse semigroup. Let γ be the minimum inverse congruence on I°. This is described by Lemma 3.1.13. Put $S = I^\circ/\gamma$, an inverse semigroup. We prove that I is equivalent to $C(S)$.

Let $e \xleftarrow{a} f$ in the category I. Then e and f being identities in I yield idempotents $\gamma(e)$ and $\gamma(f)$. The category product in I is extended by the new binary operation $\circ$ and so $eaf = a$ implies that $e \circ a \circ f = a$. It follows that $(\gamma(e), \gamma(a), \gamma(f)) \in C(S)$. We may therefore define $\Theta : I \to C(S)$ by $\Theta(a) = (\gamma(e), \gamma(a), \gamma(f))$. This is clearly a functor and it is full and faithful because γ is a local isomorphism by Lemma 3.1.15.

It remains to show that F is essentially surjective. Let (f, f, f) be an identity in $C(S)$. Then f is an idempotent in S. It follows that there is an idempotent a in I° such that $\gamma(a) = f$. But I is an inverse category and so $a^{-1}a$ is an idempotent in I and clearly $\gamma(a^{-1}a) = f$. We may therefore assume that a is an idempotent in the inverse category I. Idempotents split in I and so there is an identity i and elements x and y such that $xy = a$ and $yx = i$. Consider the triples $(f, \gamma(x), \gamma(i))$ and $(\gamma(i), \gamma(y), f)$ which are well-defined elements of $C(S)$. Observe that $(f, \gamma(x), \gamma(i))(\gamma(i), \gamma(y), f) = (f, f, f)$ and

$(\gamma(i), \gamma(y), f)(f, \gamma(x), \gamma(i)) = (\gamma(i), \gamma(i), \gamma(i))$. It follows that (f, f, f) is isomorphic to $(\gamma(i), \gamma(i), \gamma(i))$, and so F is a essentially surjective. ■

3.4.3 Semigroups whose idempotents locally commute

In this section, we shall revisit the papers by Khan and Lawson [17, 18]. Our goal is to characterize those semigroups which are Morita equivalent to semigroups whose idempotents commute. Inverse semigroups play an important role. Our first result provides a first necessary condition. It is immediate by Proposition 3.1.10(1).

Lemma 3.4.13 *Let S be a semigroup with local units Morita equivalent to a semigroup T with local units and commuting idempotents. Then S has locally commuting idempotents.*

The above result is important because it singles out the class of semigroups with locally commuting idempotents. We shall now describe some of their properties. The following result is well-known but we include it for the sake of completeness.

Lemma 3.4.14 *Let S be a semigroup in which the set of idempotents forms a commutative subsemigroup. Then the set of regular elements forms an inverse semigroup.*

Proof Let x and y be regular elements in S. Let $x' \in V(x)$ and $y' \in V(y)$. Then $xy(y'x')xy = x(yy')(x'x)y = x(x'x)(yy')y = xy$ since idempotents commute. Thus the set of regular elements forms a regular subsemigroup in which the idempotents commute and so it forms an inverse semigroup. ■

When we pass from semigroups in which the idempotents commute to those in which they only locally commute then we cannot prove the analogue of the above result because the set of regular elements of such a semigroup need not form a subsemigroup. However it is possible to prove something about the multiplicative properties of regular elements. The following was proved as Proposition 2.3(iii) of [18], but because of its importance we state and prove it again here.

Lemma 3.4.15 *Let S be a semigroup with locally commuting idempotents. If x and y are regular elements such that $xe = x$ and $ey = y$ then xy is regular.*

Proof Let $x' \in V(x)$. Then from $xe = x$ we get that $x'xe = x'x$. By standard semigroup theory or direct verification, we have that $ex'x$ is an idempotent and $x'x\mathcal{L}ex'x \leq e$. Since $x\mathcal{L}ex'x$ there exists by Lemma 3.1.12 an $x'' \in V(x)$ such that $x''x = ex'x$. We have therefore proved that if $xe = x$ then there exists $x' \in V(x)$ such that $x'x \leq e$. By a dual argument, from $ey = y$ there exists $y' \in V(y)$ such that $yy' \leq e$. With these choices of inverses we have that $xy(y'x')xy = x(yy')(x'x)y$. But $x'x, yy' \leq e$ and so, by assumption, these idempotents commute. Thus $xy(y'x')xy = x(yy')(x'x)y = x(x'x)(yy')y = xy$. Thus xy is a regular element, as claimed. ■

Lemma 3.4.16 *Let S be a semigroup with locally commuting idempotents. Then the regular elements of $C(S)$ form an inverse category.*

Proof By Lemma 3.4.15, it is easy to check that the regular elements of $C(S)$ form a subcategory which contains all the object of this category. A regular category is inverse if and only if the local monoids are inverse which is the case here. ■

If S is a semigroup with locally commuting idempotents we denote by $I(S)$ its associated inverse category.

Lemma 3.4.17 *Let S be a semigroup in which the idempotents commute. Then $I(S) = C(\mathrm{Reg}(S))$.*

Proof We proved in Lemma 3.4.14 that $\mathrm{Reg}(S)$ was an inverse semigroup. Let $(e, a, f) \in I(S)$. Then for some b we have that $(e, a, f) = (e, a, f)(f, b, e)(e, a, f)$ and so $a = aba$. It follows that a is regular and so $(e, a, f) \in C(\mathrm{Reg}(S))$. The reverse inclusion is immediate. ■

Lemma 3.4.18 *Let S and T be semigroups with locally commuting idempotents. Then an equivalence between $C(S)$ and $C(T)$ leads to an equivalence between $I(S)$ and $I(T)$.*

Proof We prove a slightly more general result. Let $F\colon C \to D$ be a full, faithful and essentially surjective functor between two categories whose regular elements form wide inverse subcategories $I(C)$ and $I(D)$, respectively. The image of a regular element of C under F is a regular element of D. Thus F restricts to a functor from $I(C)$ to $I(D)$ which is clearly faithful, and dense because identities are regular. It remains to check that it is full. Let $b \in I(D)$ be such that $F(f) \xleftarrow{b} F(e)$. Then there is a unique element $f \xleftarrow{a} e$ such that $F(a) = b$. Let b' be the inverse of b. Then $F(e) \xleftarrow{b'} F(f)$. Thus there is a unique element $e \xleftarrow{a'} f$ such that $F(a') = b'$. Observe that $f \xleftarrow{aa'a} e$ and $F(aa'a) = bb'b = b$. Thus from the fact that F is faithful we have that $aa'a = a$. Similarly $a'aa' = a'$. It follows that a is regular and so F restricted to $I(C)$ is also full. ■

The key result which we deduce from the above is the following.

Proposition 3.4.19 *Let S be a semigroup with locally commuting idempotents. If S is Morita equivalent to a semigroup with commuting idempotents then there is an inverse semigroup T' such that $I(S)$ is equivalent to $C(T')$.*

Proof Let S be Morita equivalent to T a semigroup with commuting idempotents. By Lemma 3.4.18, we have that $I(S)$ is equivalent to $I(T)$. But by Lemma 3.4.17, $I(T) = C(T')$ where $T' = \text{Reg}(S)$ is an inverse semigroup.■

The above proposition will become one-half of our final characterization of semigroups Morita equivalent to semigroups with commuting idempotents.

Let S be a semigroup. Define the relation $\leq$ on the set of regular elements of S as follows. Let $s, t \in \text{Reg}(S)$. Then $s \leq t$ iff $R_s \leq R_t$ and $s = ft$ for some idempotent $f \in E(R_s)$. It is proved in Proposition 2.1 of [18] that this is a partial order.

Proposition 3.4.20 (Proposition 2.2 of [18]) *Let S be a semigroup and $s, t \in \text{Reg}(S)$. Then the following are equivalent.*

1. *$s \leq t$.*
2. *For each $f \in E(R_t)$ there exists $e \in E(R_s)$ such that $e \leq f$ and $s = et$.*
3. *For each $f' \in E(L_t)$ there exists $e' \in E(L_s)$ such that $e' \leq f'$ and $s = te'$.*

4. $s = et = tf$ for some idempotents $e, f \in S$.

Proposition 3.4.21 (Proposition 2.3 of [18]) *Let S be a semigroup with locally commuting idempotents. If $a, b, c, d \in \operatorname{Reg}(S)$ are such that ac and bd are regular and where $a \leq b$ and $c \leq d$ then $ac \leq bd$.*

Let S be a semigroup having locally commuting idempotents. Let E be the set of idempotents in S. A function $p : E \times E \to S$ where $p(u, v) = p_{u,v}$ for each $u, v \in E$ is called a *McAlister sandwich function* if and only if it satisfies the following three conditions:

(MS1) $p_{u,v} \in uSv$ and $p_{u,u} = u$.

(MS2) $p_{u,v} \in V(p_{v,u})$.

(MS3) $p_{u,v}p_{v,f} \leq p_{u,f}$.

Condition (MS3) is well-defined because all the $p_{u,v}$ are regular, and the product of $p_{u,v}$ and $p_{v,f}$ is regular from Lemma 3.4.15.

The link between McAlister sandwich functions and McAlister consolidations on inverse categories is provided by the following.

Proposition 3.4.22 *Let S be a semigroup with locally commuting idempotents. Then S has a McAlister sandwich function if and only if the inverse category $I(S)$ is equipped with a McAlister consolidation.*

Proof Suppose that S is equipped with a McAlister sandwich function p. Define a consolidation q on $I(S)$ to be $q_{e,f} = (e, p_{e,f}, f)$. This lives in $I(S)$ by (MS2) and is a consolidation by (MS1). It remains to show that the second condition of McAlister consolidation holds. We have that

$$q_{e,f}q_{f,g} = (e, p_{e,f}p_{f,g}, g).$$

By (MS3), we have that $p_{e,f}p_{f,g} \leq p_{e,g}$. Put $p_{g,e}p_{e,g} = g' \leq g$ and $p_{e,g}p_{g,e} = e' \leq e$. Then by Proposition 3.4.20, there exist $i \leq g'$ and $j \leq e'$ such that $p_{e,f}p_{f,g} = p_{e,g}i$ and $p_{e,f}p_{f,g} = jp_{e,g}$ and $i\mathcal{L}p_{e,f}p_{f,g}\mathcal{R}j$. Let $z \in V(p_{e,f}p_{f,g})$ be such that $zp_{e,f}p_{f,g} = i$ and $p_{e,f}p_{f,g}z = j$. It follows that in the inverse category $I(S)$ we have $(e, p_{e,f}p_{f,g}, g)^{-1} = (g, z, e)$. If we calculate $(e, p_{e,g}, g)(e, p_{e,f}p_{f,g}, g)^{-1}(e, p_{e,f}p_{f,g}, g)$ we get $(e, p_{e,f}p_{f,g}, g)$. Thus

$$(e, p_{e,f}p_{f,g}, g) \leq (e, p_{e,g}, g),$$

as required. We have therefore shown that there is a McAlister consolidation defined on $I(S)$.

Let q be a McAlister consolidation defined on $I(S)$. Observe that for this to be defined it is necessary that $I(S)$ be strongly connected. Let $q_{e,f} = (e, p_{e,f}, f)$. By the definition of a consolidation we have that $p_{e,e} = e$. By construction we have that $p_{e,f}$ is regular and $p_{f,e} \in V(p_{e,f})$. We have that $(e, p_{e,f}, f)(f, p_{f,g}, g) \leq (e, p_{e,g}, g)$. Thus $p_{e,f}p_{f,g} = ip_{e,g} = p_{e,g}j$ where i and j are idempotents. Thus by Proposition 3.4.20, we deduce that (MS3) holds. Thus we have defined a McAlister sandwich function on S. ∎

The following theorem recasts [17, 18] in terms of Morita theory.

Theorem 3.4.23 *Let S and T be semigroups with local units. Suppose T has locally commuting idempotents. Then S is Morita equivalent to T if and only if S has locally commuting idempotents and is equipped with a McAlister sandwich function.*

Proof Suppose first that S is Morita equivalent to a semigroup with commuting idempotents T. By Lemma 3.4.13, we know that S has locally commuting idempotents. By Proposition 3.4.19 and Theorem 3.4.12, the inverse category $I(S)$ is equipped with a McAlister consolidation. It follows by Proposition 3.4.22 that S is equipped with a McAlister sandwich function.

To prove the converse, let S be a semigroup with locally commuting idempotents and equipped with a McAlister sandwich function. Khan and Lawson proved the following in [18]. First, we can use the McAlister sandwich function to define a consolidation on $I(S)$ by Proposition 3.4.22. The resulting semigroup obtained we denote by $C(S)^\bullet$. This semigroup has a normal band of idempotents by Proposition 3.1 of [18]. We may define a congruence δ on $C(S)^\bullet$ in such a way that $T = C(S)^\bullet/\delta$ is a semigroup with commuting idempotents and $\delta^\natural \colon C(S)^\bullet \to T$ is a local isomorphism by Theorem 4.2 [18]. By Theorem 3.1.7, it follows that S is Morita equivalent to T. ∎

If we combine Theorem 3.4.23, Proposition 3.4.22 and Theorem 3.4.12, we obtain the following.

Theorem 3.4.24 *The semigroup with local units S is Morita equivalent to a semigroup with commuting idempotents if and only if the following two conditions hold:*

1. *S has locally commuting idempotents.*

2. *The inverse category $I(S)$ is equivalent to a category of the form $C(T)$ where T is an inverse semigroup.*

It follows from Proposition 3.4.8, that a necessary condition for a semigroup S with locally commuting idempotents to be Morita equivalent to a semigroup with commuting idempotents is that $eSf \cap \mathrm{Reg}(S) \neq \emptyset$ for all idempotents $e, f \in S$. Not all semigroups with locally commuting idempotents satisfy this condition as is shown in [24].

3.4.4 Morita equivalence of inverse semigroups

The goal of this section is to solve the following problem: given an inverse semigroup S how do we construct all inverse semigroups T that are Morita equivalent to S? We shall show how to do this. This section can be seen as a generalization and completion of some of the results to be found in [22].

This section is based on a different characterization of the Morita equivalence of inverse semigroups. Let S and T be inverse semigroups. An *equivalence biset from S to T* consists of an (S,T)-biset X equipped with surjective functions

$$\langle -,- \rangle \colon X \times X \to S\ , \text{ and } [-,-] \colon X \times X \to T$$

such that the following axioms hold, where $x, y, z \in X$, $s \in S$, and $t \in T$:

(E1) $\langle sx, y \rangle = s\langle x, y \rangle$

(E2) $\langle y, x \rangle = \langle x, y \rangle^{-1}$

(E3) $\langle x, x \rangle x = x$

(E4) $[x, yt] = [x, y]t$

(E5) $[x, y] = [y, x]^{-1}$

(E6) $x[x, x] = x$

(E7) $\langle x, y \rangle z = x[y, z]$.

Observe that by (E6) and (E7), we have that $\langle x, x\rangle x = x[x, x] = x$.

It is not hard to see, Theorem 5.1 of [39], that if there is an equivalence biset from S to T then there is a weak equivalence from $C(S)$ to $C(T)$ and so by Theorem 1.1, the inverse semigroups S and T are Morita equivalent. In fact, the converse is true by Theorem 2.14 of [9].

Theorem 3.4.25 *Let S and T be inverse semigroups. Then S and T are Morita equivalent if and only if there is an equivalence biset from S to T.*

Our main tool will be regular Rees matrix semigroups described in Section 3.1. The first problem is that regular Rees matrix semigroups over inverse semigroups are locally inverse but not inverse. To get closer to being an inverse semigroup we need to impose more conditions on the Rees matrix semigroup. First, we shall restrict our attention to *square* Rees matrix semigroups: those semigroups where $I = \Lambda$. In this case, we shall denote our Rees matrix semigroup by $M(S, I, p)$ where $p: I \times I \to S$ is the function giving the entries of the sandwich matrix P. Next, we shall place some conditions on the sandwich matrix P:

(MF1) $p_{i,i}$ is an idempotent for all $i \in I$.

(MF2) $p_{i,i}p_{i,j}p_{j,j} = p_{i,j}$.

(MF3) $p_{i,j} = p_{j,i}^{-1}$.

(MF4) $p_{i,j}p_{j,k} \leq p_{i,k}$.

(MF5) For each $e \in E(S)$ there exists $i \in I$ such that $e \leq p_{i,i}$.

We shall call functions satisfying all these conditions *McAlister functions*. Our choice of name reflects the fact that McAlister was the first to study functions of this kind in [29].

The following is essentially Theorem 6.7 of [22] but we include a full proof for the sake of completeness.

Lemma 3.4.26 *Let $M = M(S, I, p)$ where p satisfies (E1)–(E4).*

1. *(i, s, j) is regular if and only if $s^{-1}s \leq p_{j,j}$ and $ss^{-1} \leq p_{i,i}$.*

2. *If (i, s, j) is regular then one of its inverses is (j, s^{-1}, i).*

3. (i,s,j) *is an idempotent if and only if* $s \leq p_{i,j}$.

4. *The idempotents form a subsemigroup.*

Proof (1). Suppose that (i,s,j) is a regular element. Then there is an element (k,t,l) such that $(i,s,j) = (i,s,j)(k,t,l)(i,s,j)$ and $(k,t,l) = (k,t,l)(i,s,j)(k,t,l)$. Thus, in particular, $s = sp_{j,k}tp_{l,i}s$. Now

$$p_{j,j}s^{-1}s = p_{j,j}s^{-1}sp_{j,k}tp_{l,i}s = s^{-1}sp_{j,j}p_{j,k}tp_{l,i}s$$

using the fact that $p_{j,j}$ is an idempotent. But $p_{j,j}p_{j,k} = p_{j,k}$ and so

$$p_{j,j}s^{-1}s = s^{-1}sp_{j,k}tp_{l,i}s = s^{-1}s.$$

Thus $s^{-1}s \leq p_{j,j}$. By symmetry, $ss^{-1} \leq p_{i,i}$.

(2) This is a straightforward verification.

(3). Suppose that (i,s,j) is an idempotent. Then $s = sp_{j,i}s$. It follows that $s^{-1} = s^{-1}sp_{j,i}ss^{-1} \leq p_{j,i}$ and so $s \leq p_{i,j}$. Conversely, suppose that $s \leq p_{i,j}$. Then $s^{-1} \leq p_{j,i}$ and so $s^{-1} = s^{-1}sp_{j,i}ss^{-1}$ which gives $s = sp_{j,i}s$. This implies that (i,s,j) is an idempotent.

(4). Let (i,s,j) and (k,t,l) be idempotents. Then by (2) above we have that $s \leq p_{i,j}$ and $t \leq p_{k,l}$. Now $(i,s,j)(k,t,l) = (i, sp_{j,k}t, l)$. But $sp_{j,k}t \leq p_{i,j}p_{j,k}p_{k,l} \leq p_{i,l}$. It follows that $(i,s,j)(k,t,l)$ is an idempotent. ■

We may immediately deduce the following from the above lemma.

Proposition 3.4.27 *Let S be an inverse semigroup. If $M = M(S,I,p)$ where p satisfies (E1)–(E4) then $RM(S,I,p)$ is a generalized inverse semigroup.*

Let S be a regular semigroup. Recall that the intersection of all congruences ρ on S such that S/ρ is inverse is a congruence denoted by γ; it is called the minimum inverse congruence. Let S be an orthodox semigroup. Then $s \,\gamma\, t$ if and only if $V(s) = V(t)$.

Lemma 3.4.28 *Let $RM = RM(S,I,p)$ where p satisfies (E1)–(E4). Then $(i,s,j)\gamma(k,t,l)$ if and only if $s = p_{i,k}tp_{l,j}$ and $t = p_{k,i}sp_{j,l}$.*

Proof We use Lemma 3.1.13. Suppose that $(i,s,j)\gamma(k,t,l)$. Then the two elements have the same sets of inverses. Now (j,s^{-1},i) is an inverse of (i,s,j) and so by assumption it is an inverse of (k,t,l). Thus

$$t = tp_{l,j}s^{-1}p_{i,k}t \text{ and } s^{-1} = s^{-1}p_{i,k}tp_{l,j}s^{-1}.$$

It follows that

$$s \leq p_{i,k}tp_{l,j} \text{ and } t^{-1} \leq p_{l,j}s^{-1}p_{i,k}$$

so that

$$t \leq p_{k,i}sp_{j,l}.$$

Now

$$s \leq p_{i,k}tp_{l,j} \leq p_{i,k}p_{k,i}sp_{j,l}p_{l,j} \leq p_{i,i}sp_{j,j} \leq s.$$

Thus $s = p_{i,k}tp_{l,j}$. Similarly, $t = p_{k,i}sp_{j,l}$.

Conversely, suppose that $s = p_{i,k}tp_{l,j}$ and $t = p_{k,i}sp_{j,l}$. We shall prove that $V(i,s,j) \cap V(k,t,l) \neq \emptyset$. To do this, we shall prove that (j,s^{-1},i) is an inverse of (k,t,l). We calculate

$$tp_{l,j}s^{-1}p_{i,k}t = t(p_{l,j}s^{-1}p_{i,k})t = t(p_{k,i}sp_{j,l})^{-1}t = tt^{-1}t = t.$$

Similarly, $s^{-1} = s^{-1}p_{i,k}tp_{l,j}s^{-1}$. The result now follows from Lemma 3.1.13. ■

With the assumptions of the above lemma, put

$$IM(S,I,p) = RM(S,I,p)/\gamma.$$

We call $IM(S,I,p)$ the *inverse Rees matrix semigroup* over S.

Let S be a regular semigroup. Then the natural homomorphism from S to S/γ is a local isomorphism if and only if S is a generalized inverse semigroup by Lemma 3.1.15.

Lemma 3.4.29 *Let S be semigroup with local units. Let $M = M(S,I,p)$ where p satisfies (MF1)–(MF5). Then S is Morita equivalent to $RM(S,I,p)$.*

Proof We shall construct a weak equivalence from $C(RM(S,I,p))$ to $C(S)$. By Theorem 3.1.4, this implies that S is Morita equivalent to $RM(S,I,p)$. A typical element of $C(RM(S,I,p))$ has the form

$$\mathbf{s} = [(i,a,j),(i,s,k),(l,b,k)]$$

where (i,s,j) is regular and (i,a,j) and (l,b,k) are idempotents and

$$(i,a,j)(i,s,k)(l,b,k) = (i,s,k).$$

Observe that both $ap_{j,i}$ and $bp_{k,l}$ are idempotents and that

$$(ap_{j,i})sp_{k,l}(bp_{k,l}) = sp_{k,l}.$$

It follows that

$$(ap_{j,i}, sp_{k,l}, bp_{k,l})$$

is a well-defined element of $C(S)$. We may therefore define

$$\Psi: C(RM(S,I,p)) \to C(S)$$

by

$$\Psi[(i,a,j),(i,s,k),(l,b,k)] = (ap_{j,i}, sp_{k,l}, bp_{k,l}).$$

It is now easy to check that Ψ is full and faithful. Let (e,e,e) be an arbitrary identity of $C(S)$. Then e is an idempotent in S. By (MF5), there exists $i \in I$ such that $e \leq p_{i,i}$. It follows that (i,e,i) is an idempotent in $RM(S,I,p)$. Thus

$$[(i,e,i),(i,e,i),(i,e,i)]$$

is an identity in $C(RM(S,I,p))$. But

$$\Psi[(i,e,i),(i,e,i),(i,e,i)] = (ep_{i,i}, ep_{i,i}, ep_{i,i}) = (e,e,e).$$

Thus every identity in $C(S)$ is the image under Ψ of an identity in $C(RM(S,I,p))$. In particular, Ψ is essentially surjective. ■

We may summarize what we have found so far in the following result.

Proposition 3.4.30 *Let S be an inverse semigroup and let $p: I \times I \to S$ be a McAlister function. Then S is Morita equivalent to the inverse Rees matrix semigroup $IM(S,I,p)$.*

Our goal now is to prove that all inverse semigroups Morita equivalent to S are isomorphic to inverse Rees matrix semigroups $IM(S,I,p)$. We begin with some results about equivalence bisets all of which are taken from [39].

The following is part of Proposition 2.3 [39].

Lemma 3.4.31 *Let $(S, T, X, \langle -, - \rangle, [-, -])$ be an equivalence biset.*

1. *For each $x \in X$ both $\langle x, x \rangle$ and $[x, x]$ are idempotents.*
2. *$\langle x, y \rangle \langle z, w \rangle = \langle x[y, z], w \rangle$.*
3. *$[x, y][z, w] = [x, \langle y, z \rangle w]$.*
4. *$\langle xt, y \rangle = \langle x, yt^{-1} \rangle$.*
5. *$[sx, y] = [x, s^{-1}y]$.*

Lemma 3.4.32 *Let $(S, T, X, \langle, \rangle, [,])$ be an equivalence biset from S to T.*

1. *For each $x \in X$ there exists a homomorphism $\epsilon_x \colon E(S) \to E(T)$ such that $ex = x\epsilon_x(e)$ for all $e \in E(S)$.*
2. *For each $x \in X$ there exists a homomorphism $\eta_x \colon E(S) \to E(T)$ such that $xf = \eta_x(f)x$ for all $e \in E(S)$.*

Proof We prove (1); the proof of (2) follows by symmetry. Define ϵ_x by $\epsilon_x(e) = [ex, ex]$. By Proposition 2.4 of [39], this is a semigroup homomorphism. Next we use the argument from Proposition 3.6 of [39]. We calculate $x[ex, ex]$ as follows

$$x[ex, ex] = \langle x, ex \rangle ex = \langle x, x \rangle ex = e\langle x, x \rangle x = ex,$$

as required. ■

Lemma 3.4.33 *Let $(S, T, X, \langle, \rangle, [,])$ be an equivalence biset from S to T. Define $p \colon X \times X \to S$ by $p_{x,y} = \langle x, y \rangle$. Then p is a McAlister function.*

Proof (MF1) holds. By Lemma 3.4.31(1), $\langle x, x \rangle$ is an idempotent.

(MF2) holds. By Lemma 3.4.31(2), $\langle x, x \rangle \langle x, y \rangle = \langle x[x, x], y \rangle$. But $x[x, x] = x$ by (E6), and so $\langle x, x \rangle \langle x, y \rangle = \langle x, y \rangle$. The other result holds dually.

(MF3) holds. This follows from (E2).

(MF4) holds. By Lemma 3.4.31(2), we have that $\langle x, y \rangle \langle y, z \rangle = \langle x[y, y], z \rangle$. By Lemma 3.4.32, we have that $x[y, y] = \eta_x([y, y])x = fx$. Thus $\langle x[y, y], z \rangle = \langle fx, x \rangle = f\langle x, z \rangle \leq \langle x, z \rangle$.

(MF5) holds. Let $e \in E(S)$. Then since $\langle -, - \rangle$ is surjective, there exists $x, y \in X$ such that $e = \langle x, y \rangle$. But then $e = \langle x, y \rangle \langle y, x \rangle \leq \langle x, x \rangle = p_{x,x}$. ■

Lemma 3.4.34 *Let* $(S, T, X, \langle , \rangle, [,])$ *be an equivalence biset from* S *to* T. *Define* $p: X \times X \to S$ *by* $p_{x,y} = \langle x, y\rangle$. *Form the regular Rees matrix semigroup* $R = RM(S, X, p)$. *Define* $\theta: RM(S, X, p) \to T$ *by* $\theta(x, s, y) = [x, sy]$. *Then* θ *is a surjective homomorphism with kernel* γ.

Proof We show first that θ is a homomorphism. By definition

$$(x, s, y)(u, t, v) = (x, s\langle y, u\rangle t, v).$$

Thus

$$\theta((x, s, y)(u, t, v)) = [x, s\langle y, u\rangle tv],$$

whereas

$$\theta(x, s, y)\theta(u, t, v) = [x, sy][u, tv].$$

By Lemma 3.4.31(3), we have that

$$[x, sy][u, tv] = [x, \langle sy, u\rangle tv]$$

but by (E1), $\langle sy, u\rangle = s\langle y, u\rangle$. It follows that θ is a homomorphism.

Next we show that θ is surjective. Let $t \in T$. Then there exists $(x, y) \in X \times X$ such that $[x, y] = t$. Consider the element $(x, \langle x, x\rangle\langle y, y\rangle, y)$ of $M(S, I, p)$. This is in fact an element of $RM(S, X, p)$. The image of this element under θ is

$$[x, \langle x, x\rangle\langle y, y\rangle y] = [x, \langle x, x\rangle y]$$

since $\langle y, y\rangle y = y$. But by Lemma 3.4.31(5), we have that

$$[x, \langle x, x\rangle y] = [\langle x, x\rangle x, y] = [x, y] = t,$$

as required.

It remains to show that the kernel of θ is γ. Let $(x, s, y), (u, t, v) \in RM(S, X, p)$. Suppose first that $\theta(x, s, y) = \theta(u, t, v)$. By definition, $[x, sy] = [u, tv]$. From Lemma 3.4.26 $ss^{-1} \leq p_{s,s}$ which means $ss^{-1} = p_{s,s}ss^{-1}$, and so $s = ss^{-1}s = p_{s,s}s$. By symmetry $sp_{y,y} = s$. Then

$$s = \langle x, x\rangle s\langle y, y\rangle = \langle x, x\rangle\langle sy, y\rangle = \langle x[x, sy], y\rangle$$

by Lemma 3.4.31(2). But $[x, sy] = [u, tv]$. Thus

$$s = \langle x[u, tv], y\rangle = \langle x, u\rangle\langle tv, y\rangle = \langle x, u\rangle t\langle v, y\rangle.$$

By symmetry and Lemma 3.4.28, we deduce that $(x, s, y)\gamma(u, t, v)$.

Suppose now that $(x, s, y)\gamma(u, t, v)$. Then by Lemma 3.4.28

$$s = \langle x, u\rangle t\langle v, y\rangle \text{ and } t = \langle u, x\rangle s\langle y, v\rangle.$$

Now

$$[x, sy] = [x, \langle x, u\rangle t\langle v, y\rangle y] = [x, \langle x, u\rangle tv[y, y]] = [u[x, x], tv[y, y]] = [x, x][u, tv][y, y]$$

using Lemma 3.4.31. This gives $[x, sy] \leq [u, tv]$. A symmetric argument shows that $[u, tv] \leq [x, sy]$. Hence $[x, sy] = [u, tv]$, as required. ■

We may now state our main theorem which follows by Theorem 3.4.25, Proposition 3.4.30 and Lemma 3.4.34.

Theorem 3.4.35 *Let S be an inverse semigroup. For each McAlister function $p: I \times I \to S$ the inverse Rees matrix semigroup $IM(S, I, p)$ is Morita equivalent to S, and every inverse semigroup Morita equivalent to S is isomorphic to one of this form.*

Let S be an inverse monoid and suppose that $p: I \times I \to S$ is a function satisfying (MF1)–(MF5). Condition (MF5) says that For each $e \in E(S)$ there exists $i \in I$ such that $e \leq p_{i,i}$. Thus, in particular, there exists $i_0 \in I$ such that $1 \leq p_{i_0,i_0}$. But p_{i_0,i_0} is an idempotent and so $1 = p_{i_0,i_0}$. Suppose now that $p: I \times I \to S$ is a function satisfying (MF1)–(MF4) and there exists $i_0 \in I$ such that $1 = p_{i_0,i_0}$. Every idempotent $e \in S$ satisfies $e \leq 1$. It follows that (MF5) holds. Thus in the monoid case, the functions $p: I \times I \to S$ satisfying (MF1)–(MF5) are precisely what we called *normalized, pointed sandwich functions* in [22]. Furthermore, the inverse semigroups Morita equivalent to an inverse monoid are precisely the enlargements of that monoid [9, 25]. Thus the theory developed in pages 446–450 of [22] is the monoid case of the theory we have just developed.

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