0.1 Unitary operator

The unitary operator $\hat{S}$ in (4.13) which is used to transform the Hamiltonian $\hat{H}_2$ in (4.17) gives the corresponding transformed time evolution operator

$$\hat{U}_{\text{trans}}(t) = \exp\left[-i\omega_k t\hat{k}^\dagger\hat{k} + i\frac{g_k^2\omega_m}{\zeta^2}t(\hat{k}^\dagger\hat{k})^2\right]\exp\left[-i\zeta t\hat{a}^\dagger\hat{a} - i\beta t(\hat{a}^\dagger\hat{a})^2\right], \qquad (1)$$

where $\zeta = \omega_m + \beta$. The untransformed operator $\hat{U}(t)$ then becomes

$$\begin{aligned}\hat{U}(t) &= e^{-\hat{S}}\hat{U}_{\text{trans}}(t)e^{\hat{S}} = \exp\left[-i\omega_k t\hat{k}^\dagger\hat{k} + i\frac{g_k^2\omega_m}{\zeta^2}t(\hat{k}^\dagger\hat{k})^2\right] \\ &\quad \exp(-\hat{S})\exp\left[-i\zeta t\hat{a}^\dagger\hat{a} - i\beta t(\hat{a}^\dagger\hat{a})^2\right]\exp(\hat{S}). \qquad (2)\end{aligned}$$

Using the Baker-Campbell-Hausdorff expansion [79] together with making the rotating wave approximation, and also neglecting quadratic and higher order terms in g_c/ζ, (2) simplifies to

$$\begin{aligned}\hat{U}(t) &= \exp\left\{-i[\omega_k t\hat{k}^\dagger\hat{k} - \frac{g_k^2}{\zeta^2}\left[\omega_m t - \sin(\zeta t)](\hat{k}^\dagger\hat{k})^2 + \beta t(\hat{a}^\dagger\hat{a})^2\right]\right\} \\ &\quad \times\exp\left[\frac{g_k}{\zeta}\hat{k}^\dagger\hat{k}(\hat{a}^\dagger - \hat{a}) - \frac{g_k}{\zeta}\hat{k}^\dagger\hat{k}(\hat{a}^\dagger e^{-i\zeta t} - \hat{a}e^{i\zeta t})\right]\exp\left[-i\zeta t\hat{a}^\dagger\hat{a}\right]. \qquad (3)\end{aligned}$$