0.1 Bogoliubov transformation

Equation (3.50) can be diagonalised by the Bogoliubov transformations

$$\begin{bmatrix} \hat{f} \\ \hat{f}^{\dagger} \end{bmatrix} = \begin{bmatrix} \alpha_1 & -\beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix} \begin{bmatrix} \hat{D}_1 \\ \hat{D}_1^{\dagger} \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} \hat{g} \\ \hat{g}^{\dagger} \end{bmatrix} = \begin{bmatrix} \alpha_2 & -\beta_2 \\ -\beta_2 & \alpha_2 \end{bmatrix} \begin{bmatrix} \hat{D}_2 \\ \hat{D}_2^{\dagger} \end{bmatrix}, \tag{2}$$

where $\alpha_i^2 - \beta_i^2 = 1$ and $[\hat{D}_i, \hat{D}_i^{\dagger}] = 1$ for $i = 1, 2$. Under the above transformations, the diagonalised Hamiltonian takes the form

$$\begin{aligned} H &= \begin{bmatrix} \hat{D}_1^{\dagger} & \hat{D}_1 \end{bmatrix} \begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix} \begin{bmatrix} \hat{D}_1 \\ \hat{D}_1^{\dagger} \end{bmatrix} + \begin{bmatrix} \hat{D}_2^{\dagger} & \hat{D}_2 \end{bmatrix} \begin{bmatrix} \widetilde{D}_{11} & 0 \\ 0 & \widetilde{D}_{22} \end{bmatrix} \begin{bmatrix} \hat{D}_2 \\ \hat{D}_2^{\dagger} \end{bmatrix} \\ &\quad + \begin{bmatrix} \hat{s}^{\dagger} & \hat{s} \end{bmatrix} \begin{bmatrix} \omega/2 & 0 \\ 0 & \omega/2 \end{bmatrix} \begin{bmatrix} \hat{s} \\ \hat{s}^{\dagger} \end{bmatrix} \end{aligned} \tag{3}$$

where

$$D_{11} = (\omega + \kappa/\sqrt{2})\alpha_1^2 + (\kappa/\sqrt{2})\beta_1^2 - \sqrt{2}\kappa\alpha_1\beta_1 \tag{4}$$

$$D_{22} = (\omega + \kappa/\sqrt{2})\beta_1^2 + (\kappa/\sqrt{2})\alpha_1^2 - \sqrt{2}\kappa\alpha_1\beta_1 \tag{5}$$

$$\widetilde{D}_{11} = (\omega - \kappa/\sqrt{2})\alpha_2^2 - (\kappa/\sqrt{2})\beta_2^2 + \sqrt{2}\kappa\alpha_2\beta_2 \tag{6}$$

$$\widetilde{D}_{22} = (\omega - \kappa/\sqrt{2})\beta_2^2 - (\kappa/\sqrt{2})\alpha_2^2 + \sqrt{2}\kappa\alpha_2\beta_2 \tag{7}$$

$$\alpha_1^2 = \frac{1}{2} + \frac{1}{2}\frac{(\omega + \sqrt{2}\kappa)}{\sqrt{(\omega + \sqrt{2}\kappa)^2 - 2\kappa^2}} \tag{8}$$

$$\beta_1^2 = -\frac{1}{2} + \frac{1}{2}\frac{(\omega + \sqrt{2}\kappa)}{\sqrt{(\omega + \sqrt{2}\kappa)^2 - 2\kappa^2}} \tag{9}$$

$$\alpha_2^2 = \frac{1}{2} + \frac{1}{2}\frac{(\omega - \sqrt{2}\kappa)}{\sqrt{(\omega - \sqrt{2}\kappa)^2 - 2\kappa^2}} \tag{10}$$

$$\beta_2^2 = -\frac{1}{2} + \frac{1}{2}\frac{(\omega - \sqrt{2}\kappa)}{\sqrt{(\omega - \sqrt{2}\kappa)^2 - 2\kappa^2}}. \tag{11}$$

Using the diagonalised Hamiltonian (3), the Heisenberg equations of motion for the time-evolved operators $\hat{D}_1^\dagger(t)$, $\hat{D}_2^\dagger(t)$ and $\hat{s}^\dagger(t)$ are

$$\frac{d}{dt}\hat{D}_1^\dagger = i[\hat{H}, \hat{D}_1^\dagger], \tag{12}$$

$$\frac{d}{dt}\hat{D}_2^\dagger = i[\hat{H}, \hat{D}_2^\dagger], \tag{13}$$

$$\frac{d}{dt}\hat{s}^\dagger = i[\hat{H}, \hat{s}^\dagger]. \tag{14}$$

Making use of the fact that $[\hat{D}_i, \hat{D}_j^\dagger] = \delta_{ij}$ we get

$$\hat{D}_1^\dagger(t) = e^{iK_1 t}\hat{D}_1^\dagger(0) \tag{15}$$

$$\hat{D}_2^\dagger(t) = e^{iK_2 t}\hat{D}_2^\dagger(0) \tag{16}$$

$$\hat{s}^\dagger(t) = e^{i\omega t}\hat{s}^\dagger(0), \tag{17}$$

where

$$K_1 = (\omega + \kappa/\sqrt{2})(\alpha_1^2 + \beta_1^2) - 2\sqrt{2}\kappa\alpha_1\beta_1 + (\kappa/\sqrt{2})(\alpha_1^2 + \beta_1^2), \tag{18}$$

$$K_2 = (\omega - \kappa/\sqrt{2})(\alpha_2^2 + \beta_2^2) + 2\sqrt{2}\kappa\alpha_2\beta_2 - (\kappa/\sqrt{2})(\alpha_2^2 + \beta_2^2). \tag{19}$$

Now the explicit form of $\mathbf{F}$ can be constructed by expressing $\hat{a}(t)$, $\hat{b}(t)$, $\hat{c}(t)$ in terms of $\hat{f}(t)$, $\hat{g}(t)$, $\hat{s}(t)$. For instance, the time-evolved operator $\hat{a}(t)$ can be reexpressed as

$$\hat{a}(t) = \frac{1}{2}(\hat{f}(t) + \hat{g}(t)) + \frac{\hat{s}(t)}{\sqrt{2}}. \tag{20}$$

Rewriting $\hat{f}(t)$ and $\hat{g}(t)$ in terms of $\hat{D}_1(t)$, $\hat{D}_2(t)$ and their Hermitian conjugates, equation (20) takes the form

$$\begin{aligned}\hat{a}(t) &= [\frac{\alpha_1^2}{2}\hat{f}(0) + \frac{\alpha_1\beta_1}{2}\hat{f}^\dagger(0)]e^{-iK_1t} - [\frac{\beta_1^2}{2}\hat{f}(0) + \frac{\alpha_1\beta_1}{2}\hat{f}^\dagger(0)]e^{iK_1t} \\ &+[\frac{\alpha_2^2}{2}\hat{g}(0) + \frac{\alpha_2\beta_2}{2}\hat{g}^\dagger(0)]e^{-iK_2t} - [\frac{\beta_2^2}{2}\hat{g}(0) + \frac{\alpha_2\beta_2}{2}\hat{g}^\dagger(0)]e^{iK_2t} + \frac{\hat{s}(0)}{\sqrt{2}}e^{-i\omega t}. \end{aligned} \tag{21}$$

In a similar manner the time-evolved expressions for the other mode operators can also be obtained. Finally $\hat{f}(0)$, $\hat{g}(0)$ and $\hat{s}(0)$ can be reexpressed in terms of $\hat{a}(0)$, $\hat{b}(0)$ and $\hat{c}(0)$ allowing us to get closed form analytical solutions for the time evolved Heisenberg operators $\hat{a}(t)$, $\hat{b}(t)$, $\hat{c}(t)$ and their hermitian conjugates.