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# Limit groups and Makanin-Razborov diagrams for hyperbolic groups

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## Abstract

This thesis gives a detailed description of Zlil Sela's construction of Makanin-Razborov diagrams which describe $\mathrm{Hom}(G,\Gamma)$, the set of all homomorphisms from $G$ to $\Gamma$, where $G$ is a finitely generated group and $\Gamma$ is a hyperbolic group. Moreover, while Sela's construction requires $\Gamma$ to be torsion-free, this thesis removes this condition and addresses the case of arbitrary hyperbolic groups.

Sela's shortening argument, which is the main tool in the construction of the Makanin-Razborov diagrams, relies on the Rips machine, a structure theorem for finitely generated groups acting stably on real trees. As homomorphisms from a f.g. group $G$ to a hyperbolic group $\Gamma$ give rise to stable actions of $G$ on real trees, which appear topologically as limits of the $G$-actions on the Cayley graph of $\Gamma$, the Rips machine and the shortening argument allow us to explore the structure of $\mathrm{Hom}(G,\Gamma)$ and construct Makanin-Razborov diagrams which encode all homomorphisms from $G$ to $\Gamma$.

While Sela's version of the Rips machine allows the formulation of the shortening argument only in the case where $\Gamma$ is torsion-free, Guirardel has presented a generalized version of the Rips machine, which we exploit to generalize the shortening argument and the construction of Makanin-Razborov diagrams to the case of arbitrary hyperbolic groups.

**Acknowledgements**

I would like to thank Richard Weidmann most sincerely for his patient and competent supervision of my work as a Ph.D. student. Richard introduced me to numerous aspects of geometric group theory, always showed honest interest in my work, and acknowledged me as a co-researcher in our joint project which is the basis for this thesis.

I would also like to express my sincere thanks to my former second supervisor Jim Howie for agreeing to be my first supervisor in the final months of my Ph.D. due to Richard's absence. I also gratefully acknowledge his help in providing me with the opportunity to come to Heriot-Watt University to pursue a Ph.D.

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# Contents

# List of Figures

# Introduction

The question of whether one can decide if a system of equations (with constants) in a free group has a solution or not was answered affirmatively by Makanin [M1] who described an algorithm that produces such a solution if it exists and says *No* otherwise. In his groundbreaking work he introduced a rewriting process for systems of equations in the free semigroup. This process was later refined by Razborov to give a complete description of the set of solutions for a system of equations in a free group [Ra1, Ra2]. This description is now referred to as Makanin-Razborov diagrams.

Rips recognized that the Makanin process can be adapted to study group actions on real trees, which gave rise to what is now called the Rips machine, a structure theorem for finitely presented groups acting on real trees similar to Bass-Serre theory for groups acting on simplicial trees, see [G2, GLP, BF0]. This has been generalized to finitely generated groups by Sela [Sel1] and further refined by Guirardel [G]. Recently Dahmani and Guirardel [DG] have in turn used the geometric ideas underlying the Rips theory to provide an alternative version of Makanin's algorithm.

Razborov's original description of the solution set has been refined independently by Kharlampovich and Myasnikov [KM1, KM2] and Sela [Sel2]; this description has been an important tool in their solutions to the Tarski problems regarding the elementary theory of free groups. Kharlampovich and Myasnikov modified Razborov's methods to obtain their own version of the rewriting process while Sela used the Rips machine extensively, bypassing much of the combinatorics of the process.

While it seems that the full potential of the ideas underlying the Makanin process has not yet been realized there are a number of generalizations of the above results. Sela [Sel3] has shown the existence of Makanin-Razborov diagrams for torsion-free hyperbolic groups, which was then generalized to torsion-free relatively hyperbolic groups with finitely generated abelian parabolic subgroups by Groves [Gr]. Makanin-Razborov diagrams for free products have been constructed independently by Jaligot and Sela following Sela's geometric approach and by Casals-Ruiz and Kazachkov [CK2] following the combinatorial approach of Kharlampovich and Myasnikov. Casals-Ruiz and Kazachkov also constructed Makanin-Razborov diagrams for graph groups [CK2].

The aim of this thesis is to give a detailed description of Sela's construction of Makanin-Razborov diagrams for hyperbolic groups and remove the torsion-freeness assumption on $\Gamma$. We define $\Gamma$-*limit groups*, where $\Gamma$ is an arbitrary hyperbolic group, in analogy to Bestvina and Feighn's definition of limit groups in the case where $\Gamma$ is free, construct JSJ-decompositions of (one-ended) $\Gamma$-limit groups and develop the shortening argument in the setup of arbitrary hyperbolic groups to finally prove the existence of Makanin-Razborov diagrams.

The thesis is mainly based on a joint work with Dr. Richard Weidmann, see [RW]. While in [RW] the constructed JSJ-decompositions are not unfolded, this thesis independently proves that the JSJ-decompositions can in fact be unfolded (see chapter 5), a result which has been obtained independently by Guirardel and Levitt.

We are mostly interested in equations without constants; we only comment on the case with constants at the end of the thesis; no new ideas are needed to deal with them. Thus we are interested in finding all tuples

$$(x_1, \ldots, x_n) \in \Gamma^n$$

that satisfy equations

$$w_i(x_1, \ldots, x_n) = 1$$

for $i \in I$ where $w_i$ is some word in the $x_j^{\pm 1}$ and $\Gamma$ is a hyperbolic group. It is clear that these solutions are in 1-to-1 correspondence to homomorphisms from

$$G = \langle x_1, \ldots, x_n | w_i(x_1, \ldots, x_n), i \in I \rangle$$

to $\Gamma$. Thus parametrizing the set of solutions to the above system of equations is equivalent to parametrizing $\mathrm{Hom}(G, \Gamma)$. The goal of this thesis is the proof of the following theorem.

**Theorem 0.1.** *Let $\Gamma$ be a hyperbolic group and $G$ be a finitely generated group. Then there exists a finite directed rooted tree $T$ with root $v_0$ satisfying*

1. *The vertex $v_0$ is labeled by $G$,*
2. *Any vertex $v \in VT$, $v \neq v_0$, is labeled by a group $G_v$ that is fully residually $\Gamma$,*
3. *Any edge $e \in ET$ is labeled by an epimorphism $\pi_e : G_{\alpha(e)} \to G_{\omega(e)}$,*

*such that for any homomorphism $\phi : G \to \Gamma$ there exists a directed path $e_1, \ldots, e_k$ from $v_0$ to some vertex $\omega(e_k)$ such that*

$$\phi = \psi \circ \pi_{e_k} \circ \alpha_{k-1} \circ \ldots \circ \alpha_1 \circ \pi_{e_1}$$

*where $\alpha_i \in Aut(G_{\omega(e_i)})$ for $1 \leq i \leq k$ and $\psi : G_{\omega(e_k)} \to \Gamma$ is locally injective.*

Here a homomorphism is called locally injective if it is injective when restricted to 1-ended and finite subgroups. In particular, it is injective on the vertex groups of any Dunwoody/Linnell decompositon.

The proof broadly follows Sela's proof in the torsion-free case but is also partly inspired by Bestvina and Feighn's exposition of Sela's construction for free groups [BF1] and by Groves's adaption of Sela's work to the relatively hyperbolic case. We further rely on Guirardel's version of the Rips machine.

The proof of the existence of the Makanin-Razborov diagrams has two main aspects. On the one hand, we need to show that every f.g. group $G$ which is fully residually

$\Gamma$ admits a $\Gamma$-*factor set*, a finite set $\{q_1, \dots, q_n\}$ of proper quotient maps such that each homomorphism $\varphi : G \to \Gamma$ factors through some $q_i$ up to precomposition with an automorphism of $G$ (cf. Definition 4.1). This will be proven with the help of the shortening argument in chapter 4 and provides the local finiteness of the diagram.

On the other hand, it needs to be verified that the diagram has finite diameter, i.e. does not contain infinite branches. Therefore it needs to be shown that hyperbolic groups are *weakly equationally Noetherian*, which means that every system of equations without constants in $\Gamma$ is equivalent (i.e., has the same solution set) to a finite subsystem.

If $\Gamma$ is weakly equationally Noetherian, it is easy to see that there does not exist an infinite sequence

$$G_1 \to G_2 \to G_3 \to \dots$$

of non-injective epimorphisms such that all groups $G_i$ are residually $\Gamma$ (cf. Corollary 6.2). As a consequence, the Makanin-Razborov diagram does not have infinite diameter as an infinite branch in the diagram would provide such a sequence. Chapter 6 will give a detailed account of these aspects and the construction of the diagrams, and in chapter 7 we finally prove that all hyperbolic groups are weakly equationally Noetherian.

We start by introducing $\Gamma$-limit groups in chapter 1, these are the groups that occur as vertex groups in the Makanin-Razborov diagram. Moreover, we illustrate how $\Gamma$-limit groups admit actions on real trees if $\Gamma$ is a hyperbolic group and study the stability properties of these actions. In chapter 2 and chapter 3 we discuss the Rips machine and the JSJ-decomposition of $\Gamma$-limit groups, which will later be an important tool in the proof that $\Gamma$-limit groups are equationally Noetherian.

After discussing the shortening argument in chapter 4 as outlined above, we will prove in chapter 5 that the JSJ-decomposition constructed in chapter 3 can be chosen unfolded.

In chapter 6 we then describe the Makanin-Razborov diagrams for hyperbolic groups. In chapter 7 we then discuss Sela's shortening quotients and prove that all hyperbolic groups are in fact equationally Noetherian and the construction in chapter 6 indeed applies to all hyperbolic groups. We conclude by discussing equations with constants in chapter 8.

# Chapter 1

# $\Gamma$-limit groups and their actions on real trees

In this chapter we introduce the concept of a $\Gamma$-limit group. $\Gamma$-limit groups will occur as vertex groups in the Makanin-Razborov diagrams. $\Gamma$-limit groups naturally admit limit actions on metric spaces, and if $\Gamma$ is hyperbolic then these metric spaces will be real trees in the situations we are interested in. In section 1.2, we show in detail how these limit actions arise, and in section 1.3 we prove important stability properties of these actions. We will then study the structure of almost abelian subgroups of $\Gamma$-limit groups. Most of the material is standard except that we have to deal with torsion.

## 1.1 $\Gamma$-limit groups

Throughout this section $\Gamma$ is an arbitrary group. A group $G$ is called *fully residually* $\Gamma$ if for any finite set $S \subset G$ there exists a homomorphism $\varphi : G \to \Gamma$ such that $\varphi|_S$ is injective. Note that any subgroup of $\Gamma$ is fully residually $\Gamma$. A related notion is that of a $\Gamma$-limit group. We follow the definition that Bestvina and Feighn [BF1] introduced in the case where $\Gamma$ is free.

Let $G$ be a group and $(\varphi_i)$ a sequence of homomomorhisms from $G$ to $\Gamma$. We say that this sequence is *stable* if for any $g \in G$ either $\varphi_i(g) = 1$ for almost all $i$ or $\varphi_i(g) \neq 1$ for almost all $i$. If $(\varphi_i)$ is stable then the *stable kernel* of the sequence, denoted by $\underrightarrow{\ker}(\varphi_i)$, is defined as

$$\underrightarrow{\ker}(\varphi_i) := \{g \in G \,|\, \varphi_i(g) = 1 \text{ for almost all } i\}.$$

We then call the quotient $G/\underrightarrow{\ker}(\varphi_i)$ the *$\Gamma$-limit group associated to* $(\varphi_i)$ and the projection $\varphi : G \to G/\underrightarrow{\ker}(\varphi_i)$ *the $\Gamma$-limit map associated to* $(\varphi_i)$.

Moreover we call a quotient map $\varphi : G \to G/N$ a *$\Gamma$-limit map* if it is the $\Gamma$-limit map associated to a stable sequence $(\varphi_i) \subset \mathrm{Hom}(G, \Gamma)$. We will denote the $\Gamma$-limit group $G/N$ by $L_\varphi$.

**Lemma 1.1.** *If $G$ is countable and fully residually $\Gamma$, then $G$ is a $\Gamma$-limit group.*

*Proof.* Choose a surjective function $f : \mathbb{N} \to G$. For each $i \in \mathbb{N}$ let $M_i := \{f(j) \,|\, j \leq i\}$ and choose $\varphi_i : G \to \Gamma$ such that $\varphi_i|_{M_i}$ is injective. Clearly the sequence $(\varphi_i)$ is stable and $\underrightarrow{\ker}(\varphi_i) = \{1\}$. Thus $G = G/\underrightarrow{\ker}(\varphi_i)$ is a $\Gamma$-limit group. □

It turns out that in many situations, in particular in those that we are interested in, the converse is true as well, i.e. $\Gamma$-limit groups are fully residually $\Gamma$. See section 6.1 for a proof in the case where $\Gamma$ is equationally Noetherian.

## 1.2 Limit actions

Let $G$ and $\Gamma$ be f.g. groups and $(\varphi_i) \subset \mathrm{Hom}(G, \Gamma)$ be a stable sequence with $\Gamma$-limit map $\varphi$. In this section we illustrate how the associated $\Gamma$-limit group $L_\varphi$ admits actions on metric spaces that arise as limits of metric $G$-spaces. If $\Gamma$ is hyperbolic and the $\varphi_i$ are pairwise distinct, these $L_\varphi$-spaces turn out to be real trees.

An action of a group $G$ on a metric space $X$ is a homomorphism

$$\rho : G \to \mathrm{Isom}(X)$$

from $G$ to the isometry group of $X$. A (based) $G$-space is then a tuple $(X, x_0, \rho)$ of a metric space $X$, a base point $x_0 \in X$ and an action $\rho$ of $G$ on $X$. If $g \in G$ and $x \in X$, it is convenient to denote the element $\rho(g)(x) \in X$ simply by $gx$ if the action $\rho$ is understood. Moreover, when we want to distinguish the action, we use the notation ${}_{\rho}gx := \rho(g)(x)$ to improve readability.

Let $X = (X, x_0, \rho)$ be a based metric $G$-space. Then the action $\rho$ of $G$ on $X$ induces a pseudo-metric

$$d_\rho : G \times G \to \mathbb{R}_{\geq 0}$$

on $G$, given by

$$d_\rho(g, h) = d_X({}_{\rho}gx_0, {}_{\rho}hx_0).$$

Note that this pseudo-metric depends on the basepoint $x_0$ of $X$. However, the notation $d_\rho$ will not cause any ambiguities.

The pseudo-metric $d_\rho$ is clearly $G$-invariant, i.e. $d_\rho(g, h) = d_\rho(kg, kh)$ for all $g, h, k \in G$. Denote by $\mathcal{A}(G)$ the space of all $G$-invariant pseudo-metrics on $G$, with the compact-open topology (with respect to the discrete topology on $G$). Thus a sequence $(d_i)$ of $G$-invariant pseudo-metrics on $G$ converges in $\mathcal{A}(G)$ iff the sequence $(d_i(1, g))$ converges in $\mathbb{R}$ for all $g \in G$.

Now fix a finite generating set $S_G$ of $G$ and equip $G$ with the word metric $d_{S_G}$ relative to $S_G$.

**Lemma 1.2.** *Let $G$ be a group with finite generating set $S_G$ and $(d_i) \subset \mathcal{A}(G)$ be a sequence of $G$-invariant pseudo-metrics. If there exists $\lambda \in \mathbb{R}$ such that for each $i \in \mathbb{N}$ and $s \in S_G$, $d_i(1, s) \leq \lambda$, then $(d_i)$ has a subsequence which converges in $\mathcal{A}(G)$.*

*Proof.* For $k \in \mathbb{N}$, let $B_k := \{g \in G \,|\, d_{S_G}(1, g) \leq k\}$. If $\lambda$ is as above, it follows that for all $k \in \mathbb{N}$ and $g \in B_k$,

$$d_i(1, g) \leq k\lambda.$$

As $B_k$ is finite, the compactness of the cube $[0, k\lambda]^{|B_k|}$ then implies that there is a subsequence $(d_{i_{j,k}})_{j\in\mathbb{N}} \subset (d_i)$ such that for all $g \in B_k$, the sequence

$$\left(d_{i_{j,k}}(1, g)\right)_{j\in\mathbb{N}} \subset \mathbb{R}$$

converges. Moreover, each sequence $(d_{i_{j,k}})_{j\in\mathbb{N}}$ may be chosen as a subsequence of $(d_{i_{j,k-1}})_{j\in\mathbb{N}}$. Then the diagonal sequence $\left(d_{i_{k,k}}\right)_{k\in\mathbb{N}}$ converges in $\mathcal{A}(G)$. □

We now turn to the case where the pseudo-metrics on $G$ are induced by homomorphisms from $G$ to some hyperbolic group $\Gamma$. Thus the Cayley graph of $\Gamma$ with respect to some finite generating set $S_\Gamma$ is hyperbolic, i.e. $\delta$-hyperbolic for some $\delta \geq 0$.

Throughout this thesis we work with the definition of $\delta$-hyperbolicity of a metric (or pseudo-metric) space $(X, d)$ using the Gromov product: we say that $X$ is *$\delta$-hyperbolic* for some $\delta \geq 0$ if for any $x, y, z, t \in X$,

$$(x|y)_t \geq \min\left((x|z)_t, (y|z)_t\right) - \frac{\delta}{3}. \tag{1.1}$$

where the Gromov product of $x$ and $y$ with respect to $t$ is given by

$$(x|y)_t := \frac{1}{2}\left(d(x, t) + d(y, t) - d(x, y)\right).$$

Note that the constant being $\frac{\delta}{3}$ rather than $\delta$ is slightly non-standard. For a geodesic metric space this choice of the constant implies the $\delta$-thinness of geodesic triangles, i.e. that for any geodesic triangle $[x, y] \cup [y, z] \cup [z, x]$ we have

$$[x, y] \subset N_\delta([y, z] \cup [z, x]),$$

see [Aetal]. The definition via the Gromov product has the advantage that it also applies to (pseudo-)metric spaces that are not geodesic spaces such as the pseudo-metrics induced on a discrete group by an action on a based metric space.

For the remainder of this section, let $\Gamma$ be a hyperbolic group. Fix a finite generating set $S_\Gamma$ and let $X$ be the Cayley graph of $\Gamma$ with respect to $S_\Gamma$. Choose $\delta \geq 0$ such that $X$ is $\delta$-hyperbolic. Every homomorphism $\varphi \in \mathrm{Hom}(G, \Gamma)$ naturally induces an isometric $G$-action $\rho_\varphi$ on $X$, given by

$$\rho_\varphi(g)(x) = \varphi(g)x \text{ for all } x \in X \text{ and } g \in G.$$

Denote by $d_\varphi$ the pseudo-metric on $G$ induced by the based $G$-space $(X, 1, \rho_\varphi)$. The $\delta$-hyperbolicity of $X$ clearly implies that the pseudo-metric space $(G, d_\varphi)$ is also $\delta$-hyperbolic. Lemma 1.4 below illustrates how a limit action of the group $G$ arises from a sequence $(\varphi_i) \subset \mathrm{Hom}(G, \Gamma)$.

We need one more definition, here $|g|$ denotes the word length of the element $g \in \Gamma$ with respect to the generating set $S_\Gamma$.

**Definition 1.3.** The length of a homomorphism $\varphi : G \to \Gamma$ (with respect to $S_G$ and $S_\Gamma$), denoted by $|\varphi|$, is defined by

$$|\varphi| := \sum_{s \in S_G} |\varphi(s)|.$$

**Lemma 1.4.** *Let $G$ be a f.g. group, $\Gamma$ a hyperbolic group and $(\varphi_i)$ a sequence of pairwise distinct homomorphisms from $G$ to $\Gamma$.*

*Then there exists a based real $G$-tree $(T, x_0, \rho)$ such that the induced sequence $\left(\frac{1}{|\varphi_i|} d_{\varphi_i}\right)$ of (scaled) pseudo-metrics on $G$ has a subsequence converging in $\mathcal{A}(G)$ to $d_\rho$, and $T$ is spanned by $Gx_0$.*

*Remark* 1.5. If the sequence $(\varphi_i)$ in Lemma 1.4 is stable, then $\underrightarrow{\ker}(\varphi_i)$ clearly acts trivially on $T$. In this case the $G$-action induces an action of the limit group $L = G/\underrightarrow{\ker}(\varphi_i)$ on $T$ in the obvious way.

*Proof of Lemma 1.4.* Note that for every $k \in \mathbb{R}$, there are only finitely many homomorphisms from $G$ to $\Gamma$ of length less than $k$. As the $\varphi_i$ are pairwise distinct, this implies that

$$\lim_{i \to \infty} |\varphi_i| = \infty.$$

By Lemma 1.2, the sequence $\left(\frac{1}{|\varphi_i|}d_{\varphi_i}\right)$ has a subsequence $\left(\frac{1}{|\varphi_{i_j}|}d_{\varphi_{i_j}}\right)$ converging to a pseudo-metric $d_\infty$ on $G$. Recall that $X$ is $\delta$-hyperbolic. Thus for each $i$, the pseudo-metric $\frac{1}{|\varphi_i|}d_{\varphi_i}$ is $\frac{\delta}{|\varphi_i|}$-hyperbolic. As $\lim\limits_{i\to\infty}\frac{\delta}{|\varphi_i|}=0$, this implies that the limiting

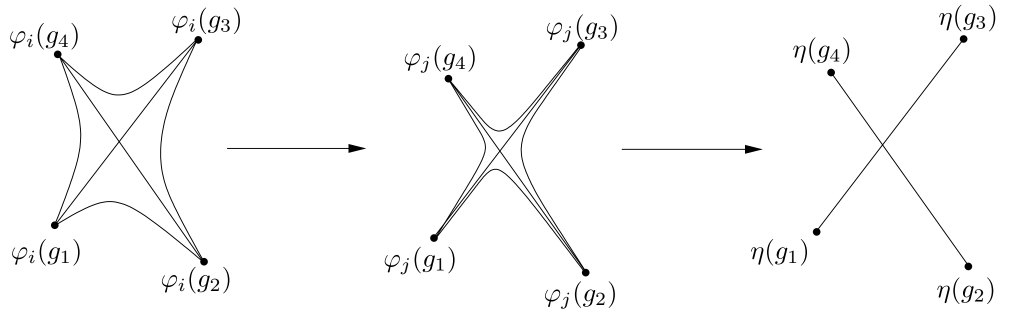



Figure 1.1: A quadrilateral degenerating to a tree

pseudo-metric $d_\infty$ is 0-hyperbolic. In particular $G$ acts on a 0-hyperbolic metric space, namely the space $(\hat{G},\hat{d}_\infty)$ obtained from the pseudo-metric space $(G,d_\infty)$ by metric identification, i.e. by identifying points of distance 0. Now, see Lemma 2.13 of [B], there is a real $G$-tree $T=(T,x_0,\rho)$ satisfying

- $T$ admints a $G$-equivariant isometric embedding $\eta:\hat{G}\to T$, i.e. $\hat{d}_\infty(g,h)=d_T(\eta(g),\eta(h))$ for all $g,h\in\hat{G}$,
- $\eta(\hat{G})$ spans $T$, i.e. no proper subtree of $T$ contains $\eta(\hat{G})$.

The tree $T$ is easily constructed from $(\hat{G},\hat{d}_\infty)$ by first adding segments of length $\hat{d}_\infty(x,y)$ between any two points $x,y\in\hat{G}$ and then identifying the initial segments of $[x,y]$ and $[x,z]$ of length $(y|z)_x$ for any $x,y,z\in\hat{G}$. By construction, the induced pseudo-metric $d_\rho$ of the $G$-action on $T$ with basepoint $\eta(1)$ is precisely $d_\infty$. $\square$

In the following we say that a sequence $(\varphi_i)\subset\mathrm{Hom}(G,\Gamma)$ *converges* (with respect to the generating sets $S_G$ and $S_\Gamma$) if the induced sequence $\left(\frac{1}{|\varphi_i|}d_{\varphi_i}\right)$ of scaled pseudo-metrics converges in $\mathcal{A}(G)$. In the setting of Lemma 1.4 we will refer to the $G$-tree $T$ which induces the limit pseudo-metric as the *limit tree* of the sequence.

Call a group action *trivial* if it has a global fixed point and *non-trivial* otherwise. Note that if a sequence $(\varphi_i)$ converges, the action of $G$ on the limit tree $T$ may be *trivial*. However Theorem 1.9 below shows that the non-triviality of the action can be guaranteed by the right choice of basepoints of the $G$-actions on $X$. Before we show this, we introduce approximating sequences.

**Definition 1.6.** Let $(X_i) = ((X_i, x_i, \rho_i))$ be a sequence of metric $G$-spaces. Assume that the sequence $(d_{\rho_i})$ converges to a pseudo-metric $d_\rho$ induced by the based $G$-space $X = (X, x, \rho)$. For a point $t \in X$, an approximating sequence of $t$ is a sequence $(t_i)$, where $t_i \in X_i$ for each $i$, such that

$$\lim_{i\to\infty} d_{X_i}(t_i, {}_{\rho_i}gx_i) = d_X(t, {}_\rho gx) \tag{1.2}$$

for each $g \in G$.

Note that every point ${}_\rho gx$ in the orbit of the basepoint $x$ is approximated by the sequence $({}_{\rho_i}gx_i)$. In particular, the sequence $(x_i)$ of basepoints approximates $x$. In general a point of a limit space may not have an approximating sequence, however, in the setting of Lemma 1.4 the following lemma implies that any point of the limit tree has an approximating sequence.

**Lemma 1.7.** *Let $(X_i) = (X_i, x_i, \rho_i)$ be a sequence of geodesic $G$-spaces, where each $X_i$ is $\delta_i$-hyperbolic and*

$$\lim_{i\to\infty} \delta_i = 0.$$

*Assume that $(d_{\rho_i})$ converges to $d_\rho$ where $T = (T, x, \rho)$ is a $G$-tree spanned by ${}_\rho Gx$. Then the following hold.*

1. *Every $t \in T$ has an approximating sequence.*

2. *If $(t_i)$ and $(\bar{t}_i)$ are approximating sequences for some $t \in T$, then*

$$\lim_{i\to\infty} d_{X_i}(t_i, \bar{t}_i) = 0.$$

3. *If* $(t_i)$ *is an approximating sequence for* $t$ *then* $({}_{\rho_i}gt_i)$ *is an approximating sequence for* ${}_{\rho}gt$.

4. *If* $(t_i)$ *and* $(y_i)$ *are approximating sequences for* $t$ *and* $y$ *then*

$$\lim_{i\to\infty} d_{X_i}(t_i, y_i) = d_T(t, y).$$

*Proof.* We will not explicitly mention the different actions $\rho_i$ and $\rho$ in the proof. Let $t \in T$. As $T$ is spanned by $Gx$ there exist $g_1, g_2 \in G$ such that $t \in [g_1x, g_2x]$. Fix such $g_1, g_2$. For each $i$ choose $t_i \in [g_1x_i, g_2x_i] \subset X_i$ s.th.

$$\frac{d_{X_i}(t_i, g_1x_i)}{d_{X_i}(g_1x_i, g_2x_i)} = \frac{d_T(t, g_1x)}{d_T(g_1x, g_2x)}.$$

This choice clearly implies that (1.2) holds for $g_1$ and $g_2$. Pick $h \in G$. To prove (1) we need to show that $\lim\limits_{i\to\infty} d_{X_i}(t_i, hx_i) = d_T(t, hx)$.

Possibly after exchanging $g_1$ and $g_2$ we can assume that $t \in [g_2x, hx]$ as in Figure 1.2

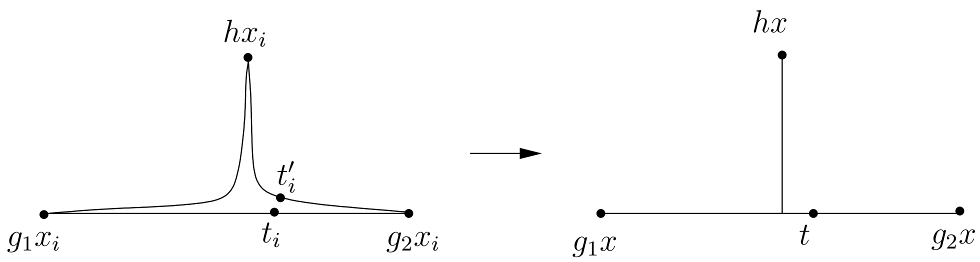



Figure 1.2: An approximating sequence $(t_i)$ of $t$.

Now choose $t'_i \in [g_2x_i, hx_i]$ such that

$$d_{X_i}(g_2x_i, t'_i) = d_{X_i}(g_2x_i, t_i).$$

It is easily verified that $\lim\limits_{i\to\infty} d_{X_i}(t_i, t'_i) = 0$. This implies that

$$\lim_{i\to\infty} d_{X_i}(t_i, hx_i) = \lim_{i\to\infty} d_{X_i}(t'_i, hx_i) =$$

$$\lim_{i\to\infty}(d_{X_i}(g_2x_i, hx_i) - d_{X_i}(t_i', g_2x_i)) =$$

$$= d_T(g_2x, hx) - d_T(t, g_2x) = d_T(t, hx).$$

Thus $(t_i)$ is an approximating sequence of $t$ and (1) is established.

To prove (2) note first that it suffices to deal with the case where $(t_i)$ is constructed as in the proof of (1), in particular $t_i \in [g_1x_i, g_2x_i]$ for all $i$ and some fixed $g_1, g_2 \in G$. As $(\bar{t}_i)$ is an approximating sequence for $t$ it follows that

$$\lim_{i\to\infty} d_{X_i}(g_kx_i, \bar{t}_i) = \lim_{i\to\infty} d_{X_i}(g_kx_i, t_i)$$

for $k = 1, 2$. As $X_i$ is $\delta_i$-hyperbolic with $\lim_{i\to\infty} \delta_i = 0$ this implies that $\lim_{i\to\infty} d_{X_i}(\bar{t}_i, t_i) = 0$.

Part (3) is trivial and part (4) follows from (2) and the fact that we can construct approximating sequences for $t$ and $y$ as in the proof of (1) by choosing $g_1, g_2$ such that both $t$ and $y$ lie on $[g_1x, g_2x]$. □

*Remark* 1.8. In the setup of Lemma 1.4, the pseudo-metrics $\frac{1}{|\varphi_i|}d_{\varphi_i}$ are induced by the action of $G$ on the scaled Cayley graph $X_i$ of $\Gamma$. Since the Cayley graph is geodesic, it follows from Lemma 1.7 that any point $t \in T$ has an approximating sequence in $(X_i)$. But this implies that each point has an approximating sequence in $\Gamma$, the vertex set of $X_i$, as for large enough $i$, due to the scaling, $t_i$ can be replaced by a nearby vertex.

We conclude the section with the following theorem.

**Theorem 1.9.** *Let $(\varphi_i)$ and $(T, x_0, \rho)$ be as in Lemma 1.4. If for all but finitely many $i \in \mathbb{N}$ and any $g \in \Gamma$,*

$$|c_g \circ \varphi_i| \geq |\varphi_i| \tag{1.3}$$

*(where $c_g$ denotes conjugation by $g$), then the limit action of $G$ on $T$ is non-trivial and minimal.*

*Proof.* It follows from the construction of $T$ that $T$ does not consist of a single point. Thus it suffices to show that the action is minimal as a minimal action on a non-degenerate tree is non-trivial. The proof of the minimality is by contradiction.

Assume that $T' \subset T$ is a proper $G$-invariant subtree. Recall that $T$ is spanned by the orbit ${}_\rho G x_0$ of the base point $x_0$. This implies that $x_0 \notin T'$ as otherwise ${}_\rho G x_0$ and therefore also $T$ would be contained in $T'$. Let $p_{x_0}$ be the nearest point projection of $x_0$ to $T'$.

The $G$-invariance of $T'$ implies that for any $g \in G$ either ${}_\rho g p_{x_0} = p_{x_0}$ or $[x_0, {}_\rho g x_0] = [x_0, p_{x_0}] \cup [p_{x_0}, {}_\rho g p_{x_0}] \cup [{}_\rho g p_{x_0}, {}_\rho g x_0]$. Moreover for some $s \in S_G$ we have ${}_\rho s x_0 \neq x_0$ as otherwise ${}_\rho G x_0 = \{x_0\}$ and therefore $T = \{x_0\}$ as $T$ is spanned by ${}_\rho G x_0$. Thus $\sum\limits_{s \in S_G} d_T(p_{x_0}, {}_\rho s p_{x_0}) < \sum\limits_{s \in S_G} d_T(x_0, {}_\rho s x_0)$.

Let $(p^i_{x_0})$ be an approximating sequence for $p_{x_0}$ such that all $p^i_{x_0}$ lie in $\Gamma$. In particular for any $s \in S_G$, the sequence $(s p^i_{x_0})$ approximates ${}_\rho s p_{x_0}$. Put $\hat{\varphi}_i = c_{p^i_{x_0}} \circ \varphi_i$, where $c_{p^i_{x_0}}$ denotes conjugation by $p^i_{x_0}$. This implies that

$$|\hat{\varphi}_i(g)| = d_X(1, \hat{\varphi}_i(g)) = d_X(1, (p^i_{x_0})^{-1} \varphi_i(g) p^i_{x_0}) = d_X(p^i_{x_0}, \varphi_i(g) p^i_{x_0})$$

for all $i$ and $g \in G$. Note that

$$\lim_{i\to\infty} |\hat{\varphi}_i| = \lim_{i\to\infty} \sum_{s \in S_G} |\hat{\varphi}_i(s)| = \sum_{s \in S_G} \lim_{i\to\infty} d_X(p^i_{x_0}, \varphi_i(s) p^i_{x_0}) = \sum_{s \in S_G} d_T(p_{x_0}, {}_\rho s p_{x_0}) <$$

$$< \sum_{s \in S_G} d_T(x_0, {}_\rho s x_0) = \sum_{s \in S_G} \lim_{i\to\infty} d_X(1, \varphi_i(s)) = \lim_{i\to\infty} \sum_{s \in S_G} |\varphi_i(s)| = |\varphi|.$$

It follows that for large $i$ we have $|\hat{\varphi}_i| < |\varphi_i|$ contradicting the minimality assumption for the $\varphi_i$. □

## 1.3 Stability of limit actions

Throughout this section let $G$ be a f.g. and $\Gamma$ be a hyperbolic group, equipped with word metrics relative fixed finite generating sets $S_G$ and $S_\Gamma$ respectively.

We have seen in section 1.2 that for a stable convergent sequence of homomorphisms $(\varphi_i) \subset \mathrm{Hom}(G, \Gamma)$ the group $G$ acts on a limit tree $T$ such that $\underrightarrow{\ker}(\varphi_i)$ acts trivially.

This induces an action of the associated $\Gamma$-limit group $G/\underrightarrow{\ker}(\varphi_i)$ on $T$. In this section we will study this action, in particular we study the stability properties of this action. All stabilizers considered are pointwise stabilizers.

We call a tree *degenerate* if it consists of a single point, otherwise *non-degenerate*. Moreover, a *tripod* is the convex hull of three points in a tree, and we call it non-degenerate if it is not an interval.

**Definition 1.10** ([BF1])**.** Let $T$ be a $G$-tree. A non-degenerate subtree $S \subset T$ is called *stable* if for every non-degenerate subtree $S' \subset S$, $\mathrm{stab}_G(S') = \mathrm{stab}_G(S)$. Otherwise $S$ is called *unstable*. The tree $T$ is stable if every non-degenerate subtree of $T$ contains a stable subtree.

The following theorem is the main result of this section.

**Theorem 1.11.** *Let $(\varphi_i) \subset Hom(G, \Gamma)$ be a convergent stable sequence with induced $\Gamma$-limit map $\varphi$, and $L = L_\varphi$. Suppose that $\{|\varphi_i| \,|\, i \in \mathbb{N}\}$ is unbounded, and let $T$ be the limit $L$-tree (cf. Remark 1.5). Then the following hold for the action of $L$ on $T$.*

1. *The stabilizer of any non-degenerate tripod is finite.*
2. *The stabilizer of any non-degenerate arc is finite-by-abelian.*
3. *Every subgroup of $L$ which leaves a line in $T$ invariant and fixes its ends is finite-by-abelian.*
4. *The stabilizer of any unstable arc is finite.*

*Moreover, if* $\ker T$ *denotes the kernel of the $G$-action, then* $\ker T/\underrightarrow{\ker}(\varphi_i)$ *is finite.*

Before we proceed with the proof of Theorem 1.11 we recall some useful facts about torsion subgroups of hyperbolic groups.

**Proposition 1.12.** *Let $\Gamma$ be a hyperbolic group. Then the following hold.*

1. *There exists a constant $N = N(\Gamma)$ such that every torsion subgroup of $\Gamma$ has at most $N$ elements.*

2. *There exists a constant $L = L(\Gamma)$ such that for every subgroup $H \leq \Gamma$, one of the following holds.*

   (a) *$H$ is a finite group (of order at most $N(\Gamma)$).*

   (b) *For any generating set $S \subset \Gamma$ of $H$, there exists an element $\gamma \in H$ of infinite order such that $|\gamma|_S \leq L$, where $|\cdot|_S$ denotes the word metric on $H$ relative $S$.*

*Proof.* Note first that torsion subgroups of hyperbolic groups are finite, see e.g. Corollaire 36, Chapitre 4 of [GdlH]. Thus (1) follows from the fact that for any hyperbolic groups $\Gamma$ there exists $N(\Gamma)$ such that any finite group is of order at most $N(\Gamma)$, see [Br, BG].

Part (2) is essentially due to M. Koubi [K]. Proposition 3.2 of [K] implies that there exists a finite set $\bar{S} \subset \Gamma$ such that any set $S \subset \Gamma$ is either conjugate to a subset of $\bar{S}$ or that there exists a word $w$ in $S \cup S^{-1}$ of length at most 2 such that $w$ represents a hyperbolic element of $\Gamma$. Now for each subset $S$ of $\bar{S}$ let $L(S) = 0$ if $\langle S \rangle$ is finite and let $L(S)$ be the length of shortest word in $S \cup S^{-1}$ that represents a hyperbolic element otherwise. Because of (1) such an element always exists. The conclusion now follows by putting $L(\Gamma) := \max(2, \max_{S \subset \bar{S}}(L(S)))$. □

A useful consequence of Proposition 1.12 is the following lemma.

**Lemma 1.13.** *Let $\Gamma$ be hyperbolic and $L$ be a $\Gamma$-limit group. Then the following hold.*

1. *Every torsion subgroup of $L$ has at most $N(\Gamma)$ elements.*

2. *A subgroup $A \leq L$ is finite-by-abelian iff all f.g. subgroups of $A$ are finite-by-abelian.*

*Proof.* Choose a stable sequence $(\varphi_i) \subset \text{Hom}(G,\Gamma)$ with induced $\Gamma$-limit map $\varphi : G \to L = L_\varphi$.

We prove (1) by contradiction. Thus assume there exists a torsion subgroup $E \leq L$ such that $E$ contains $N(\Gamma)+1$ pairwise distinct elements $g_0, \ldots, g_{N(\Gamma)}$. For each $k = 0, \ldots, N(\Gamma)$, pick $\tilde{g}_k \in G$ s.th. $\varphi(\tilde{g}_k) = g_k$.

This implies that $\varphi_i(\tilde{g}_m) \neq \varphi_i(\tilde{g}_n)$ for large $i$ and $0 \leq n \neq m \leq N(\Gamma)$.

Thus Proposition 1.12 (1) implies that $\langle \varphi_i(\tilde{g}_0), \ldots, \varphi_i(\tilde{g}_{N(\Gamma)})\rangle$ is infinite for large $i$.

Proposition 1.12 (2) then implies that for large $i$ there exists a word $w_i$ in $\tilde{g}_0, \ldots, \tilde{g}_{N(\Gamma)}$ of length at most $L(\Gamma)$ such that $\varphi_i(w_i)$ is of infinite order. Now there are only finitely many such words. Thus there exists a word $w$ such that $w = w_i$ for infinitely many $i$. As $E$ is assumed to be a torsion group it follows that $\varphi(w)^k = 1$ for some $k$, i.e. that $w^k \in \underrightarrow{\ker}(\varphi_i)$ and therefore $w^k \in \ker \varphi_i$ for almost all $i$, a contradiction. Thus (1) is proven.

We now show (2). Clearly, if $A$ is finite-by-abelian, so are all f.g. subgroups. Thus we need to show that if the commutator subgroup of $A$ is infinite, i.e. contains $N(\Gamma)+1$ distinct elements $g_0, \ldots, g_{N(\Gamma)}$, then the same is true for some f.g. subgroup of $A$. This however is obvious as any element of the commutator subgroup of some group is the product of finitely many commutators and therefore lies in the commutator subgroup of a finitely generated subgroup. □

The following lemma is the main step in the proof of Theorem 1.11 (2) and (3).

**Lemma 1.14.** *Let $G$ be a f.g. group and $\Gamma$ be a hyperbolic group with $\delta$-hyperbolic Cayley graph $X$. Let $(\varphi_i) \subset Hom(G,\Gamma)$.*

*Let $(x_i^1, x_i^2)_{i\in\mathbb{N}}$ be a sequence of pairs of points in $X$ and $\lim_{i\to\infty} d(x_i^1, x_i^2) = \infty$. Suppose further that $U \leq G$ is a f.g. subgroup such that for any $u \in U$,*

$$\lim_{i\to\infty} \frac{d(x_i^j, \varphi_i(u)x_i^j)}{d(x_i^1, x_i^2)} = 0 \textit{ for } j \in \{1,2\}.$$

*Then $\varphi_i(U)$ is either finite or 2-ended for sufficiently large $i$, and $|[\varphi_i(U), \varphi_i(U)]| < \infty$, hence $\leq N(\Gamma)$ by Proposition 1.12.*

*Proof.* Let $U = \langle h_1, \ldots, h_k \rangle$. If $\varphi_i(U)$ is finite and therefore of order at most $N(\Gamma)$ for infinitely many $i$ then $u^{N(\Gamma)!} \in \underrightarrow{\ker}(\varphi_i)$ for all $u \in U$. This implies that $\varphi(U)$ is a torsion group and therefore finite by Lemma 1.13 (1). We can therefore assume that $\varphi_i(U)$ is infinite for all $i$. It follows from Proposition 1.12 that for each $i$ there exists some $u_i \in U$ of length at most $L(\Gamma)$ (with respect to the word metric relative to the $h_i$) such that $w_i := \varphi_i(u_i)$ is hyperbolic. As there are only finitely many such $u_i$ it follows from the assumption that

$$\lim_{i\to\infty} \frac{d(x_i^j, \varphi_i(u_i)x_i^j)}{d(x_i^1, x_i^2)} = \lim_{i\to\infty} \frac{d(x_i^j, w_i x_i^j)}{d(x_i^1, x_i^2)} = 0 \text{ for } j \in \{1, 2\}.$$

Let $p_+^i, p_-^i \in \partial X$ be the fixed points of $w_i$ in $\partial X$. To prove the lemma it clearly suffices to show that for large $i$, $\varphi_i(h_j)$ fixes $p_+^i$ and $p_-^i$ for each $j = 1, \ldots, k$ as this would imply that $\varphi_i(U)$ fixes $p_+^i$ and $p_-^i$ and is therefore finite or finite-by-$\mathbb{Z}$.

Note further that to show that $\varphi_i(h_j)$ fixes $p_+^i$ and $p_-^i$ we only need to show that for large $i$ the commutator $[\varphi_i(h_j), w_i]$ fixes $p_+^i$ and $p_-^i$ for all $j \in \{1, \ldots, k\}$. Indeed this implies that $\varphi_i(h_j) w_i \varphi_i(h_j)^{-1} = [\varphi_i(h_j), w_i] w_i$ fixes $p_+^i$ and $p_-^i$ which in turn implies that $\varphi_i(h_j)$ preserves the set $\{p_+^i, p_-^i\}$. It follows easily from the hypothesis that for large $i$, $\varphi_i(h_j)$ does not interchange $p_+^i$ and $p_-^i$. Thus $\varphi_i(h_j)$ fixes $p_+^i$ and $p_-^i$, which proves the lemma.

Let now $v_1, \ldots, v_{2p}$ be elements of $U$ such that $v_{2l-1}$ is hyperbolic for $1 \leq l \leq p$. We show that for large $i$ the set

$$\{\varphi_i([v_{2j-1}, v_{2j}]) | 1 \leq j \leq p\}$$

is conjugate into the ball of radius $20\delta$ around the identity, denote the cardinality of this set by $M$. The proof relies on the fact that for a hyperbolic element $\gamma$ with axis $A_\gamma$ we have

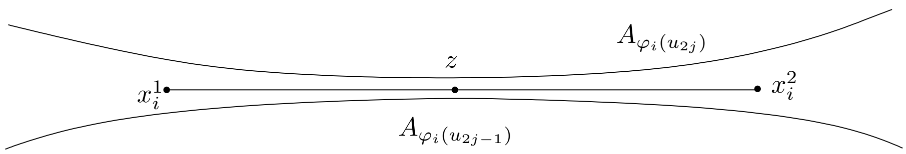
$$d_X(x, \gamma x) \geq 2 d_X(x, A_\gamma) - C \tag{1.4}$$

for any $x \in X$ where $C$ is a constant depending only on $\Gamma$ and the generating set. This holds as there is a lower bound on the stable translation length of hyperbolic elements.

Note that for sufficiently large $i$ the hypothesis of the lemma implies that

$$d(x_i^j, \varphi_i(v_{2l-1})x_i^j), d(x_i^j, \varphi_i(v_{2l}v_{2l-1}^{-1}v_{2l}^{-1})x_i^j) \leq \frac{d(x_i^1, x_i^2)}{1000}$$

for $1 \leq l \leq p$ and $j \in \{1, 2\}$. Thus (1.4) implies that $x_i^1$ and $x_i^2$ lie in the $\frac{d(x_i^1, y_i^1)}{100}$-neighbourhood of the axes $A_{\varphi_i(v_{2l-1})}$ and $A_{\varphi_i(v_{2l}v_{2l-1}^{-1}v_{2l}^{-1})}$ for $1 \leq l \leq p$. In particular a long part of $[x_i^1, x_i^2]$ around its midpoint $z_m$ lies in the $2\delta$-neighborhood of the axes. It now follows easily that the commutators $\varphi_i([v_{2l-1}, v_{2l}]) = \varphi_i(v_{2l-1}) \cdot \varphi_i(v_{2l}v_{2l-1}^{-1}v_{2l}^{-1})$ do not move $z_m$ by more than $20\delta$, the claim follows.




As $w_i$ is hyperbolic the above implies that the commutators

$$[w_i, \varphi_i(h_j)], w_i[w_i, \varphi_i(h_j)]w_i^{-1}, \ldots, w_i^{M+1}[w_i, \varphi_i(h_j)]w_i^{-(M+1)}$$

are not pairwise distinct. Thus

$$[w_i, \varphi_i(h_j)] = w_i^{l_j}[w_i, \varphi_i(h_j)]w_i^{-l_j}$$

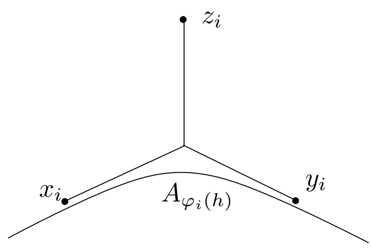
for some $l_j \leq M + 1$ This implies that $w_i^{l_j}$ and $w_i^{l_j}[w_i, \varphi_i(h_j)]w_i^{-l_j}$ commute and so $w_i^{l_j}[w_i, \varphi_i(h_j)]w_i^{-l_j}$ fixes $p_+$ and $p_-$. Since $U$ is f.g., for large enough $i$ the above argument holds for all $j$, which concludes the proof. □

We can now prove Theorem 1.11.

*Proof of Theorem 1.11.* To prove (1) it suffices to show that $H := \text{stab}_L(D)$ is a torsion group if $D$ is a non-degenerate tripod spanned by vertices $x$, $y$ and $z$. Let

$h \in H$ and pick $\tilde{h} \in G$ such that $\varphi(\tilde{h}) = h$. We need to show that $\varphi_i(\tilde{h})$ has finite order for large $i$ as this implies that $\tilde{h}^{N(\Gamma)!} \in \underrightarrow{\ker}(\varphi_i)$ and that $h$ is a torsion element.

We follow the argument from the proof of Lemma 4.1 in [RS]. Let $(x_i)$, $(y_i)$ and $(z_i)$ be approximating sequences of $x$, $y$ and $z$. It follows as in the proof of Lemma 1.14 that for large $i$, either the element $\varphi_i(\tilde{h})$ is of finite order or the segment $[x_i, y_i]$ is contained in a small neighbourhood of the axis $A_{\varphi_i(\tilde{h})}$ of $\varphi_i(\tilde{h})$. But the latter implies that $d(z_i, \varphi_i(\tilde{h})z_i) \geq 2d(A_{\varphi_i(\tilde{h})}, z_i) - C$ if $C$ is as in the proof of Lemma 1.14. This implies that $z$ is not fixed by $\varphi(\tilde{h})$. Thus $\varphi_i(\tilde{h})$ is elliptic. This proves (1).




To prove (2), assume that $H \leq G$ stabilizes a non-degenerate arc $[x^1, x^2]$ in $T$. Let $(x_i^1)$ and $(x_i^2)$ be approximating sequences of $x^1$ and $x^2$ respectively. Clearly $\lim_{i\to\infty} d_X(x_i^1, x_i^2) = \infty$ and $\lim_{i\to\infty} \frac{d_X(x_i^j, \varphi_i(h)x_i^j)}{d_X(x_i^1, x_i^2)} = 0$ for any $h \in H$ and $j \in \{1, 2\}$. It follows from Lemma 1.14 that for every f.g. subgroup $U$ of $H$ the group $\varphi_i([U, U]) = [\varphi_i(U), \varphi_i(U)]$ is of order at most $N(\Gamma)$ for large $i$. The stability of the sequence $(\varphi_i)$ now implies that $\varphi([U, U])$ is a torsion group and therefore finite by Lemma 1.13 (1). Thus $\varphi(U) \leq L$ is also finite-by-abelian. Thus by Lemma 1.13 (2), $\varphi(H)$ is finite-by-abelian.

The proof of (3) is similar to that of (2). Assume that $H \leq G$ acts orientation-preservingly on a line $Y \subset T$ with ends $x^1$ and $x^2$. Choose sequences $(x^{1,k})_{k\in\mathbb{N}}$ and $(x^{2,k})_{k\in\mathbb{N}}$ of points on $Y$ that converge to $x^1$ and $x^2$ respectively.

Clearly $\lim_{k\to\infty} d_T(x^{1,k}, x^{2,k}) = \infty$ and therefore

$$\lim_{k\to\infty} \frac{d_T(x^{j,k}, \varphi(h)x^{j,k})}{d_T(x^{1,k}, x^{2,k})} = 0$$

for all $h \in H$ and $j \in \{1, 2\}$ as $d_T(x^{j,k}, \varphi(h)x^{j,k})$ is just the translation length of $\varphi(h)$ on $Y$.

For each $k$ and $j \in \{1, 2\}$ choose approximating sequences $(x_i^{j,k})_{i\in\mathbb{N}}$ of $x^{j,k}$. Now fix $h \in H$ and $k \in \mathbb{N}$. It follows from the definition of approximating sequences that

$$\lim_{i\to\infty} \frac{d(x_i^{j,k}, \varphi_i(h)x_i^{j,k})}{d(x_i^{1,k}, dx_i^{2,k})} = \frac{d_T(x^{j,k}, \varphi(h)x^{j,k})}{d_T(x^{1,k}, x^{2,k})}$$

for $j \in \{1, 2\}$. As the last term tends to 0 as $k$ tends to $\infty$ if follows that for some subsequence $(\varphi_{m_i})$ we get

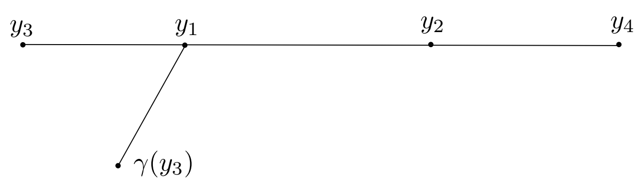
$$\lim_{i\to\infty} \frac{d(x_{m_i}^{j,i}, \varphi_{m_i}(h)x_{m_i}^{j,i})}{d(x_{m_i}^{1,i}, dx_{m_i}^{2,i})} = 0$$

for $j \in \{1, 2\}$. As $H$ is countable a diagonal argument shows that we can assume that this holds for all $h \in H$ after passing to a subsequence. Thus we argue as in the proof of (2).

To prove (4), let $[y_1, y_2] \subsetneq [y_3, y_4]$ and

$$\gamma \in \mathrm{stab}_L[y_1, y_2] \setminus \mathrm{stab}_L[y_3, y_4].$$

As $\gamma$ does not fix both $y_3$ and $y_4$, we may assume $\gamma(y_3) \neq y_3$.




Note that for each $\bar{\gamma} \in \mathrm{stab}_L[y_3, y_4]$ we have

$$\bar{\gamma}(\gamma(y_3)) = [\bar{\gamma}, \gamma](\gamma(\bar{\gamma}(y_3))) = [\bar{\gamma}, \gamma](\gamma(y_3)).$$

As the commutator subgroup of $\mathrm{stab}_L[y_1, y_2]$ is finite by (2) it follows that $\{[\bar{\gamma}, \gamma] | \bar{\gamma} \in \mathrm{stab}_L[y_3, y_4]\}$ and therefore the $\mathrm{stab}_L[y_3, y_4]$-orbit of $\gamma(y_3)$ is finite. It follows that a finite index subgroup $U$ of $\mathrm{stab}_L[y_3, y_4]$ fixes $\gamma(y_3)$ and therefore also the tripod spanned by $y_3$, $y_2$ and $\gamma(y_3)$. By (1) the subgroup $U$ is finite. Thus $\mathrm{stab}_L[y_3, y_4]$ is finite. □

## 1.4 Almost abelian subgroups of Γ-limit groups

Call a group *almost abelian* if it contains a finite-by-abelian subgroup of finite index. In the case of finitely generated groups almost abelian groups are precisely the virtually abelian groups. Note that subgroups of almost abelian groups are almost abelian and that almost abelian subgroups of hyperbolic groups are 2-ended.

Throughout this section $\Gamma$ is a hyperbolic group. We establish some basic facts about almost abelian subgroups of $\Gamma$-limit groups. A crucial fact is that each almost abelian subgroup of a $\Gamma$-limit group contains a finite-by-abelian subgroup of index at most 2. This allows us to prove as an analogue of Lemma 1.13 that almost abelian subgroups of $\Gamma$-limit groups are characterized by their f.g. subgroups being almost abelian.

**Lemma 1.15.** *Let $L$ be a $\Gamma$-limit group and $A \leq L$ be an infinite subgroup. Then the following hold.*

1. *If $A$ is almost abelian, it is either finite-by-abelian or contains a unique finite-by-abelian subgroup $U$ of index 2.*
2. *$A$ is almost abelian iff all f.g. subgroups of $A$ are almost abelian.*

*Proof.* Assume that $A = \langle a_0, a_1, \ldots \rangle$, as $A$ is infinite we can assume that $a_0$ is of infinite order. Choose a stable sequence $(\varphi_i) \subset \mathrm{Hom}(G, \Gamma)$ with induced $\Gamma$-limit map $\varphi : G \to L = L_\varphi$. For each $j \in \mathbb{N}$ pick a lift $\tilde{a}_j \in G$ such that $\varphi(\tilde{a}_j) = a_j$. For $k \in \mathbb{N}$

put $A_k := \langle a_0, \dots, a_k \rangle$ and $\tilde{A}_k := \langle \tilde{a}_0, \dots, \tilde{a}_k \rangle$, clearly $A = \bigcup_{k \in \mathbb{N}} A_k$. Note that all $A_k$ and $\tilde{A}_k$ are infinite.

We first prove (1), so let $A \leq L$ be almost abelian. Note first that all $A_k$ are finitely generated and almost abelian and therefore finitely presented. Choose relators $r_1, \dots, r_{m_k} \in F(a_0, \dots, a_k)$ such that

$$A_k = \langle a_0, \dots, a_k \,|\, r_1, \dots, r_{m_k} \rangle.$$

Let $\tilde{r}_l$ be the word obtained from $r_l$ by replacing occurences of $a_j^{\pm 1}$ by $\tilde{a}_j^{\pm 1}$.

As $\varphi(\tilde{r}_l) = 1$ for all $l$ it follows that $\tilde{r}_l \in \underrightarrow{\ker}(\varphi_i)$ and therefore $\varphi_i(\tilde{r}_l) = 1$ for all $l$ and large $i$. This implies that $\Gamma_i^k := \varphi_i(\tilde{A}_k)$ is a quotient of $A_k$ for large $i$ and therefore almost abelian. For large $i$ we further have that $\varphi_i(\tilde{a}_0)$ is of infinite order. Thus $\Gamma_i^k$ is an infinite almost abelian subgroup of some hyperbolic group and hence 2-ended.

Let $V_i^k$ be the subgroup of $\Gamma_i^k$ consisting of all elements that preserve the ends of $\Gamma_i^k$. Clearly, $|\Gamma_i^k : V_i^k| \leq 2$. Moreover put $\tilde{V}_i^k := \varphi_i^{-1}(V_i^k) \cap \tilde{A}_k$, again it follows that $|\tilde{A}_k : \tilde{V}_i^k| \leq 2$.

As $\tilde{A}_k$ is finitely generated it contains only finitely many subgroups of index 2. Thus after passing to a subsequence we can assume that for each $k$ there exists $\tilde{V}_k$ such that $\tilde{V}_k = \tilde{V}_i^k$ for all $i$. As the images $\varphi_i(\tilde{V}_k)$ act orientation preservingly on an axis of $\Gamma$, $\tilde{V}_k$ satisfies the assumptions of Lemma 1.14. It follows that $V_i^k = \varphi_i(\tilde{V}_k)$ is finite-by-abelian for large $i$. It follows as in the proof of Theorem 1.11 that also $U_k := \varphi(\tilde{V}_k) \leq L$ is finite-by-abelian.

Clearly, $U_k$ is of index at most 2 in $A_k$. It is further easily verified that $U_k \leq U_{k+1}$ as for large $i$ we have $V_i^k \leq V_i^{k+1}$ and therefore $\tilde{V}_i^k \leq \tilde{V}_i^{k+1}$. It follows that $U = \bigcup_{k \in \mathbb{N}} U_k$ is finite-by-abelian by Lemma 1.13 and a subgroup of $A = \bigcup_{k \in \mathbb{N}} A_k$ of index at most 2.

It remains to show the uniqueness of $U$ if $|A : U| = 2$. Let $U' \neq U$ be another almost abelian index 2 subgroup of $A$. Pick $k \in \mathbb{N}$ and put $\tilde{V}_k' := \varphi^{-1}(U' \cap A_k)$. Then $\tilde{V}_k'$ is of index 2 in $\tilde{A}_k$ and distinct from $\tilde{V}_k$ if $k$ is large enough. Therefore $\tilde{V}_k'$ contains an element $g \in \tilde{A}_k \setminus \tilde{V}_k$. Then for large $i$, $\varphi_i(g)$ swaps the ends of $\Gamma_i^k$. Thus $\varphi_i(\tilde{V}_k')$ contains a dihedral group and cannot be finite-by-abelian. As this holds for all (large enough) $i$, it follows easily that $U'$ is not finite-by-abelian, which is a contradiction. This proves (1).

We now prove (2). Clearly, if $A$ is almost abelian, so is every f.g. subgroup. Conversely, assume that all finitely generated subgroups of $A$ are almost abelian. This implies in particular that $A_k$ is almost abelian for all $k$. If infinitely many $A_k$ are finite-by-abelian, then each f.g. subgroup is finite-by-abelian as a subgroup of some $A_k$, and the claim follows from Lemma 1.13. So assume that (for large enough $k$) $A_k$ is not finite-by-abelian. By (1) $A_k$ contains a unique finite-by-abelian subgroup $U_k$ of index 2. The uniqueness of $U_k$ implies that $U_k \leq U_{k+1}$, as $U_{k+1} \cap A_k$ is a finite-by-abelian subgroup of $A_k$ of index 2 and therefore equal to $U_k$. It follows that $U = \bigcup_{k \in \mathbb{N}} U_k$ is of index 2 in $A$, and finite-by-abelian by Lemma 1.13. The assertion follows. $\square$

We are now able to establish the following properties of almost abelian subgroups which will be important later on.

**Lemma 1.16.** *Let $L$ be a $\Gamma$-limit group, $A \leq L$ almost abelian and $U \leq A$ finite-by-abelian of index at most 2. Then the following hold.*

1. *The subgroup $E := \langle \{g \in U \mid |g| < \infty\} \rangle \leq U$ is finite and therefore of order at most $N(\Gamma)$.*

2. *If $B \leq L$ is almost abelian and $|A \cap B| = \infty$ then $\langle A, B \rangle$ is almost abelian.*

*Proof.* To prove (1) let $\{g_0, g_1, \ldots\} \subset U$ be the set of torsion elements of $U$. For each $k \in \mathbb{N}$, pick $\tilde{g}_k \in G$ satisfying $\varphi(\tilde{g}_k) = g_k$. Note that for each $k$, $\varphi_i(\tilde{g}_k)$ is of finite order for large enough $i$. Put $\tilde{E}_k := \langle \{\tilde{g}_0, \ldots \tilde{g}_k\} \rangle$ for each $k \in \mathbb{N}$. We show that each $E_k := \varphi(\tilde{E}_k)$ is finite, hence of order at most $N(\Gamma)$. As $E_k \leq E_{k+1}$ for each $k$, this clearly implies that $E = \bigcup_{k \in \mathbb{N}} E_k$ is finite.

Fix $k \in \mathbb{N}$. By Lemma 1.14, $\varphi_i(\tilde{E}_k)$ is finite or 2-ended for sufficiently large $i$. Hence it acts invariantly on an axis in $\Gamma$ and clearly this action is orientation-preserving. If the image is infinite, it is therefore isomorphic to an HNN-extension

$$\varphi_i(E_k) \cong F_i *_{F_i} .$$

But this HNN-extension is not generated by torsion elements, which is a contradiction. Thus $\varphi_i(\tilde{E}_k)$ is finite for large $i$, hence of order at most $N(\Gamma)$. This implies that $\ker\varphi_i \cap \tilde{E}_k$ is of index at most $N(\Gamma)$ in $\tilde{E}_k$. As $\tilde{E}_k$ is f.g., there are only finitely many such kernels. The stability of $(\varphi_i)$ then implies that $\ker\varphi_i \cap \tilde{E}_k$ eventually stabilizes. It follows that $E_k = \varphi(\tilde{E}_k) = \varphi_i(\tilde{E}_k)$ (for large $i$) is finite, hence (1) is proven.

To prove (2) let $A = \langle a_0, a_1, \ldots \rangle$ and $B = \langle b_0, b_1, \ldots \rangle \leq L$ be almost abelian such that $A \cap B$ is infinite. As $L$ does not contain infinite torsion subgroups (cf. Lemma 1.13) it follows that $A \cap B$ contains an element of infinite order, so we assume w.l.o.g. that $a_0 = b_0$ is of infinite order. Now for each $k$ we choose $\tilde{a}_k, \tilde{b}_k \in G$ s.th. $\varphi(\tilde{a}_k) = a_k$ and $\varphi(\tilde{b}_k) = b_k$. Define $\tilde{A}_k := \langle \tilde{a}_0, \ldots, \tilde{a}_k \rangle$ and $\tilde{B}_k := \langle \tilde{b}_0, \ldots, \tilde{b}_k \rangle$. The same argument as in the proof of Lemma 1.15 shows for each $k$ and sufficiently large $i$ both $\varphi_i(\tilde{A}_k)$ and $\varphi_i(\tilde{B}_k)$ are almost abelian, hence 2-ended. Now for large $i$ the element $\varphi_i(\tilde{a}_0) = \varphi_i(\tilde{b}_0)$ is of infinite order, which implies that $\varphi_i(\langle \tilde{A}_k, \tilde{B}_k \rangle)$ lies in the unique maximal 2-ended subgroup of $\Gamma$ containing $\varphi_i(\tilde{a}_0)$. Hence $\varphi_i(\langle \tilde{A}_k, \tilde{B}_k \rangle)$ is 2-ended for large enough $i$. It follows easily that $\langle A_k, B_k \rangle$ is almost abelian. Now $\langle A, B \rangle = \bigcup_{k \in \mathbb{N}} \langle A_k, B_k \rangle$, so the result follows from Lemma 1.15 (2). □

We get the following immediate consequences.

**Corollary 1.17.** *Let $L$ be a $\Gamma$-limit group and $a \in L$ be an element of infinite order. Then*

$$A := \langle \{ a' \in L \mid \langle a, a' \rangle \text{ is almost abelian} \} \rangle$$

*is the unique maximal almost abelian subgroup of $L$ containing $a$.*

*Proof.* Let $\{a_0, a_1, \ldots\}$ be the set of those elements that satisfy that $\langle a, a_i \rangle$ is almost abelian. Applying Lemma 1.16 (2) repeatedly implies that for each $k$, $A_k :=$

$\langle a, a_0, \ldots, a_k \rangle$ is almost abelian. Thus $A$ is almost abelian by Lemma 1.15 (2). The uniqueness of $A$ is trivial. $\square$

**Corollary 1.18.** *Let $A$ be a maximal almost abelian subgroup of a $\Gamma$-limit group $L$ and $g \in L$. If $gAg^{-1} \cap A$ is infinite then $g \in A$.*

*Proof.* Suppose that $gAg^{-1} \cap A$ is infinite. It follows from Lemma 1.16 that $\langle A, gAg^{-1} \rangle$ is almost abelian and therefore equal to $A$ as $A$ is maximal. Choose an element $a \in A$ of infinite order. Then $\langle a, gag^{-1} \rangle \leq A$ is almost abelian. Pick lifts $\tilde{g}, \tilde{a}$ of $g, a$ in $G$. Then $\varphi_i(\langle \tilde{a}, \tilde{g}\tilde{a}\tilde{g}^{-1} \rangle)$ is almost abelian and therefore 2-ended for large $i$. Thus $\varphi_i(\tilde{g})$ preserves or exchanges the ends of $\langle \varphi_i(\tilde{a}) \rangle$. It follows that $\langle \varphi_i(\tilde{g}), \varphi_i(\tilde{a}) \rangle$ is 2-ended for large $i$, thus $\langle a, g \rangle$ is almost abelian. The statement follows now from Corollary 1.17. $\square$

# Chapter 2

# The structure of groups acting on real trees

Bass-Serre theory clarifies the algebraic structure of groups acting on simplicial trees. The structure of groups acting on real trees is more complicated but still fairly well understood provided that the action satisfies certain properties. This theory is mainly based on ideas of Rips who in turn applied ideas from the Makanin-Razborov rewriting process. Rips (unpublished) described the structure of finitely presented groups acting freely on real trees, see [GLP] for an account of his ideas. This was then generalized to stable actions by Bestvina and Feighn [BF0]. Sela [Sel1] then proved a version for finitely generated groups under stronger stability assumptions; the version we present is a generalization of Sela's result due to Guirardel [G].

We first fix notations for graphs of groups and recall the notion of a graph of actions. We then formulate the structure theorem of [G] in those terms.

## 2.1 Graphs of groups and the Bass-Serre tree

In this section we fix the notations for basic Bass-Serre theory as we will need precise language later on. For details see Serre's book [S] or [KMW] for slightly more similar notation.

A graph $A$ is understood to consist of a vertex set $VA$, a set of oriented edges $EA$, a fixed point free involution $^{-1} : EA \to EA$ and a map $\alpha : EA \to VA$ which assigns to each edge $e$ its initial vertex $\alpha(e)$. Moreover, we will denote $\alpha(e^{-1})$ alternatively by $\omega(e)$ and call $\omega(e)$ the terminal vertex of $e$.

A graph of groups $\mathbb{A}$ then consists of an underlying graph $A$ and the following data.

1. For each $v \in VA$, a vertex group $A_v$.
2. For each $e \in EA$, an edge group $A_e = A_{e^{-1}}$.
3. For each $e \in EA$, an embedding $\alpha_e : A_e \to A_{\alpha(e)}$.

Again, the embedding $\alpha_{e^{-1}}$ will alternatively be denoted by $\omega_e$. The maps $\alpha_e$ and $\omega_e$ are called the *boundary monomorphisms* of the edge $e$.

An $\mathbb{A}$-path from $v \in VA$ to $w \in VA$ is a sequence

$$a_0, e_1, a_1, \ldots, e_k, a_k$$

where $e_1, \ldots, e_k$ is an edge path in $A$ from $v$ to $w$, $a_0 \in A_v$ and $a_i \in A_{\omega(e_i)}$ for $i = 1, \ldots, k$. For two $\mathbb{A}$-paths $p = a_0, e_1, \ldots, e_{n_p}, a_{n_p}$ and $q = b_0, e'_1, \ldots, e'_{n_q}, b_{n_q}$ satisfying that $\omega(e_{n_p}) = \alpha(e'_1)$, we define a product $pq$ by

$$pq := a_0, e_1, \ldots, e_{n_p}, a_{n_p} b_0, e'_1, \ldots, e'_{n_q}, a'_{n_q}.$$

An equivalence relation on the set of $\mathbb{A}$-paths is defined as the relation generated by the elementary equivalences $a, e, b \sim a\alpha_e(c), e, \omega_e(c^{-1})b$ and $a, e, 1, e^{-1}, b \sim ab$. We denote the equivalence class of a an $\mathbb{A}$-path $p$ by $[p]$.

Given a base vertex $v_0 \in VA$, the fundamental group of $\mathbb{A}$ with respect to $v_0$, $\pi_1(\mathbb{A}, v_0)$, is the set of equivalence classes of $\mathbb{A}$-paths from $v_0$ to $v_0$, with the multiplication given by $[p][q] := [pq]$.

If $p$ is an $\mathbb{A}$-path from $v_0$ to $v$ then we denote by $[pA_v]$ the set of all $\mathbb{A}$-paths that are equivalent to $p$ after right multiplication with an element of $A_v$. Those sets are precisely the vertices of the Bass-Serre tree $\widetilde{(\mathbb{A}, v_0)}$. We will usually simply write $\tilde{\mathbb{A}}$ rather than $\widetilde{(\mathbb{A}, v_0)}$. For any vertex $\tilde{v}$ of $\tilde{\mathbb{A}}$, we will denote the projection of $\tilde{v}$ to $VA$ by $\downarrow\tilde{v}$.

If we choose for each $e \in EA$ a set $\mathcal{C}_e$ of left coset representatives of $\alpha_e(A_e)$ in $A_{\alpha(e)}$, then each $\mathbb{A}$-path $q$ is equivalent to a unique reduced $\mathbb{A}$-path $q' = a_0, e_1, \ldots, e_k, a_k$ such that $a_{i-1} \in \mathcal{C}_{e_i}$ for $1 \le i \le k$. We say that $q'$ is in *normal form* (relative to the set $\{\mathcal{C}_e | e \in EA\}$, which we usually don't mention explicitly).

Any vertex $\tilde{v} \in V\tilde{\mathbb{A}}$ is represented by a unique reduced $\mathbb{A}$-path

$$p_{\tilde{v}} = a_0, e_1, a_1, \ldots, a_{k-1}, e_k, 1$$

which is in normal form. We call $p_{\tilde{v}}$ the *representing path* of $\tilde{v}$. Note that any normal form $\mathbb{A}$-path $p$ representing $\tilde{v}$ is of the form $p = p_{\tilde{v}}a$ for a unique $a \in A_{\downarrow\tilde{v}}$.

The edge set $E\tilde{\mathbb{A}}$ is then the set of pairs $(\tilde{v}_1, \tilde{v}_2)$ of vertices satisfying

$$p_{\tilde{v}_1}a_1, e, 1 \sim p_{\tilde{v}_2}a_2 \tag{2.1}$$

for some $e \in EA$, $a_1 \in A_{\downarrow\tilde{v}_1}$ and $a_2 \in A_{\downarrow\tilde{v}_2}$.

Note that if $(\tilde{v}_1, \tilde{v}_2) \in E\tilde{\mathbb{A}}$, then also $(\tilde{v}_2, \tilde{v}_1) \in E\tilde{\mathbb{A}}$, thus the map

$$^{-1} : E\tilde{\mathbb{A}} \to E\tilde{\mathbb{A}}, \qquad (\tilde{v}_1, \tilde{v}_2) \mapsto (\tilde{v}_2, \tilde{v}_1)$$

is an involution on $E\tilde{\mathbb{A}}$, which is fixed point free as $\tilde{v}_1 \neq \tilde{v}_2$ if $(\tilde{v}_1, \tilde{v}_2) \in E\tilde{\mathbb{A}}$. For any $\tilde{e} = (\tilde{v}_1, \tilde{v}_2) \in E\tilde{\mathbb{A}}$ we put $\alpha(\tilde{e}) = \omega(\tilde{e}^{-1}) = \tilde{v}_1$. Moreover, for $\tilde{e} = (\tilde{v}_1, \tilde{v}_2)$ as above, we denote the edge $e \in EA$ (cf. (2.1)), by $\downarrow\tilde{e}$.

With the above notations, we obtain a natural action of $\pi_1(\mathbb{A}, v_0)$ on $\tilde{\mathbb{A}}$ in the following way. For $g = [q] \in \pi_1(\mathbb{A}, v_0)$ put

$$\begin{aligned} g\tilde{v} &:= [qp_{\tilde{v}}A_{\downarrow\tilde{v}}] \quad \text{for } \tilde{v} \in V\tilde{\mathbb{A}}, \\ g(\tilde{v}_1, \tilde{v}_2) &:= (g\tilde{v}_1, g\tilde{v}_2) \quad \text{for } (\tilde{v}_1, \tilde{v}_2) \in E\tilde{\mathbb{A}}. \end{aligned}$$

With this $G$-action on $\tilde{\mathbb{A}}$, for every $\tilde{v} \in V\tilde{\mathbb{A}}$ the map

$$\theta_{\tilde{v}} : A_{\downarrow\tilde{v}} \to \mathrm{stab}_{\tilde{\mathbb{A}}}(\tilde{v}), \qquad h \mapsto [p_{\tilde{v}}hp_{\tilde{v}}^{-1}]$$

is an isomorphism between the vertex group $A_{\downarrow\tilde{v}}$ and the stabilizer of the vertex $\tilde{v}$ in $\tilde{\mathbb{A}}$. Likewise, for an edge $\tilde{e} = (\tilde{v}_1, \tilde{v}_2) \in E\tilde{\mathbb{A}}$, the map

$$\theta_{\tilde{e}} := \theta_{\tilde{v}_1} \circ c_{a_1} \circ \alpha_e : A_{\downarrow\tilde{e}} \to \mathrm{stab}_{\tilde{\mathbb{A}}}(\tilde{e}), \tag{2.2}$$

where $a_1$ is as in (2.1) and $c_{a_1}$ denotes conjugation by $a_1$, is an isomorphism between the edge group $A_{\downarrow\tilde{e}}$ and the stabilizer of the edge $\tilde{e}$. It follows easily from (2.1) that $\theta_{\tilde{e}} = \theta_{\tilde{e}^{-1}}$.

Sometimes we will need to refine a given splitting of a group, i.e. increase the complexity of a graph of groups decomposition by splitting some vertex group in a way that is compatible with the existing splitting. The following is obvious.

**Definition & Lemma 2.1.** *Let $\mathbb{A}$ be a graph of groups, $v \in VA$, and $\mathbb{A}^v$ a graph of groups such that $A_v = \pi_1(\mathbb{A}^v, v_0^v)$ for some $v_0^v \in A^v$. Suppose that for each edge $e \in EA$ with $\alpha(e) = v$, $\alpha_e(A_e)$ is conjugate into a vertex group $A^v_{w_e}$ for some vertex $w_e \in VA^v$.*

*Then the graph of groups $\mathbb{A}'$ defined below is called the refinement of $\mathbb{A}$ by $\mathbb{A}^v$. The underlying graph $A'$ has vertex set $VA' = (VA \setminus \{v\}) \cup VA^v$ and edge set $EA' = EA \cup EA^v$. Moreover for each edge $e \in EA'$ the attaching map $\alpha'$ and boundary monomorphism $\alpha'_e$ are as follows.*

1. *If $e \in EA$ and $\alpha(e) \neq v$ then $\alpha'(e) = \alpha(e)$ and $\alpha'_e = \alpha_e$.*

*2. If $e \in EA^v$ then $\alpha'(e) = \alpha^v(e)$ and $\alpha'_e = \alpha^v_e$.*

*3. If $e \in EA$ and $\alpha(e) = v$ then $\alpha'(e) = w_e$ and $\alpha'_e : A_e \to A^v_{w_e}$ is such that $i_{w_e} \circ \alpha'_e$ is in $A_v$ conjugate to $\alpha_e$ where $i_{w_e}$ is the (up to cojugacy) unique inclusion of $A_{w_e}$ in $A_v$.*

*If $\mathbb{A}'$ is a refinement of $\mathbb{A}$ then $\pi_1(\mathbb{A}') \cong \pi_1(\mathbb{A})$. The operation inverse to a refinement is called a collapse.*

## 2.2 Graphs of actions

In this section we recall the notion of a graph of actions. This is a way of decomposing an action of a group on a real tree into pieces. In the structure theorem these pieces will be of very simple types.

**Definition 2.2.** A graph of actions is a tuple

$$\mathcal{G} = \mathcal{G}(\mathbb{A}) = (\mathbb{A}, (T_v)_{v \in VA}, (p^\alpha_e)_{e \in EA}, l)$$

where

- $\mathbb{A}$ is a graph of groups,
- for each $v \in VA$, $T_v = (T_v, d_v)$ is a real $A_v$-tree,
- for each $e \in EA$, $p^\alpha_e \in T_{\alpha(e)}$ is a point fixed by $\alpha_e(A_e)$,
- $l : EA \to \mathbb{R}_{\geq 0}$ is a function satisfying $l(e) = l(e^{-1})$ for all $e \in EA$.

If $l = 0$ then we omit $l$, i.e. we write $\mathcal{G} = \mathcal{G}(\mathbb{A}) = (\mathbb{A}, (T_v)_{v \in VA}, (p^\alpha_e)_{e \in EA})$.

The points $p^\alpha_e$ are called *attaching points*. In the following we will denote $p^\alpha_{e^{-1}}$ alternatively by $p^\omega_e$. Note that $p^\omega_e \in T_{\omega(e)}$ and that $p^\omega_e$ is fixed by $\omega_e(A_e)$.

Associated to any graph of actions $\mathcal{G}$ is a real tree $T_{\mathcal{G}}$ obtained by replacing the vertices of the Bass-Serre tree $\widetilde{(\mathbb{A}, v_0)}$ by copies of the trees $T_v$ and any lift $\tilde{e} \in E\tilde{\mathbb{A}}$ of $e \in EA$ by a segment of length $l(e)$. $T_{\mathcal{G}}$ comes with a natural $\pi_1(\mathbb{A}, v_0)$-action. In the remainder of this section we will give a detailed description of the construction of $T_{\mathcal{G}}$ and its natural metric.

Let $\mathcal{G}$ be a graph of actions as above. Choose a base vertex $v_0 \in VA$ and sets of left coset representatives $\mathcal{C}_e$ of $\alpha_e(A_e)$ in $A_{\alpha(e)}$ for all $e \in EA$.

For any $\tilde{v} \in V\tilde{\mathbb{A}}$, define $T_{\tilde{v}} := T_{\downarrow\tilde{v}} \times \{\tilde{v}\}$ to be a copy of $T_{\downarrow\tilde{v}}$ with an induced metric $d_{\tilde{v}}$ given by $d_{\tilde{v}}((x_1, \tilde{v}), (x_2, \tilde{v})) := d_{\downarrow\tilde{v}}(x_1, x_2)$. We further put

$$T_{\mathcal{G}}^V := \bigcup_{\tilde{v} \in V\tilde{\mathbb{A}}} T_{\tilde{v}}.$$

For any edge $\tilde{e} = (\tilde{v}_1, \tilde{v}_2) \in E\tilde{\mathbb{A}}$ and $a_1, a_2$ and $e$ as in (2.1), we then define

$$p_{\tilde{e}}^{\alpha} := (a_1 p_e^{\alpha}, \tilde{v}_1) \in T_{\tilde{v}_1}. \tag{2.3}$$

Again, for an edge $\tilde{e}$ we denote $p_{\tilde{e}^{-1}}^{\alpha}$ alternatively by $p_{\tilde{e}}^{\omega}$. We call $p_{\tilde{e}}^{\alpha}$ and $p_{\tilde{e}}^{\omega}$ the *attaching points* of the edge $\tilde{e}$.

For any $(x, \tilde{v}) \in T_{\mathcal{G}}^V$ and $g = [q] \in \pi_1(\mathbb{A}, v_0)$, put

$$g(x, \tilde{v}) := (ax, g\tilde{v})$$

if $qp_{\tilde{v}} \sim p_{g\tilde{v}}a$. It follows from the above definitions that this defines an action of $\pi_1(\mathbb{A}, v_0)$ on $T_{\mathcal{G}}^V$ with the following properties:

1. $d_{\tilde{v}}(x, y) = d_{g\tilde{v}}(gx, gy)$ for all $x, y \in T_{\tilde{v}}$ and $g \in \pi_1(\mathbb{A}, v_0)$.

2. $gp_{\tilde{e}}^{\alpha} = p_{g\tilde{e}}^{\alpha}$ for all $\tilde{e} \in E\tilde{\mathbb{A}}$ and $g \in \pi_1(\mathbb{A}, v_0)$.

For any $\tilde{e} \in E\mathbb{A}$ define $T_{\tilde{e}} := [0, l(\downarrow\tilde{e})] \times \{\tilde{e}\}$ to be a copy of the real interval $[0, l(\downarrow e)]$. Let $d_{\tilde{e}}$ be the standard metric on $T_{\tilde{e}}$. We then define

$$T_{\mathcal{G}}^E = \bigcup_{\tilde{e} \in E\mathbb{A}} T_{\tilde{e}}.$$

Note that $T_{\tilde{e}}$ consists of a single point if $l(\downarrow \tilde{e}) = 0$. Now for any $(x, \tilde{e}) \in T_{\mathcal{G}}^E$ and $g \in \pi_1(\mathbb{A}, v_0)$, put

$$g(x, \tilde{e}) := (x, g\tilde{e}).$$

This clearly defines an action of $\pi_1(\mathbb{A}, v_0)$ on $T_{\mathcal{G}}^E$, which satisfies $d_{\tilde{e}}(x, y) = d_{g\tilde{e}}(gx, gy)$ for all $x, y \in T_{\tilde{e}}$ and $g \in \pi_1(\mathbb{A}, v_0)$.

We can now define the tree $T_{\mathcal{G}}$. We put

$$T_{\mathcal{G}} := (T_{\mathcal{G}}^V \cup T_{\mathcal{G}}^E)/\sim$$

where $\sim$ is the equivalence relation generated by

1. $p_{\tilde{e}}^{\alpha} \sim (0, \tilde{e})$ for any $\tilde{e} \in E\tilde{\mathbb{A}}$,
2. $(k, \tilde{e}) \sim (l(\downarrow \tilde{e}) - k, \tilde{e}^{-1})$ for each $\tilde{e} \in E\tilde{\mathbb{A}}$ and $k \in [0, l(\downarrow \tilde{e})]$.

The first equivalences ensure that $T_{\alpha(\tilde{e})}$ and $T_{\omega(\tilde{e})}$ are joined by a segment of length $l(\downarrow \tilde{e})$ for any $\tilde{e} \in E\tilde{\mathbb{A}}$, see Figure 2.1, and the second part takes care of the fact that in $\tilde{\mathbb{A}}$ each geometric edge occurs with both orientations.

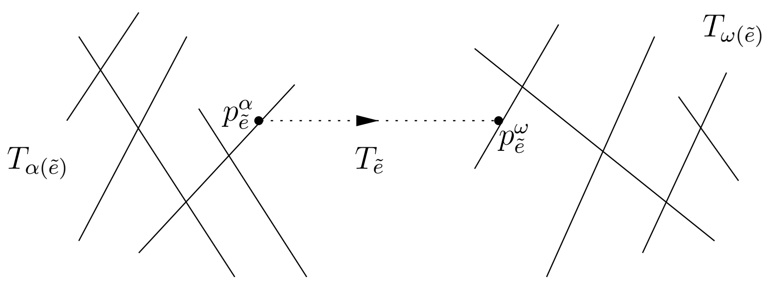



Figure 2.1: $T_{\alpha(\tilde{e})}$ and $T_{\omega(\tilde{e})}$ are joined by a segment of length $l(\downarrow \tilde{e})$.

It is clear from the above observations that this equivalence relation is preserved by the $\pi_1(\mathbb{A}, v_0)$-action on $T_{\mathcal{G}}^V \cup T_{\mathcal{G}}^E$, thus it induces a $\pi_1(\mathbb{A}, v_0)$-action on $T_{\mathcal{G}}$.

$T_{\mathcal{G}}$ has a natural $\pi_1(\mathbb{A}, v_0)$-invariant path metric $d_{\mathcal{G}}$ such that

$$d_{\mathcal{G}}(y_1, y_2) = \begin{cases} d_{\tilde{v}}(y_1, y_2) & \text{if } y_1, y_2 \in T_{\tilde{v}} \\ d_{\tilde{e}}(y_1, y_2) & \text{if } y_1, y_2 \in T_{\tilde{e}} \end{cases}$$

and otherwise, $d_{\mathcal{G}}(y_1, y_2)$ is computed as follows.

1. If $y_1 = (x_1, \tilde{v}_1), y_2 = (x_2, \tilde{v}_2) \in T_{\mathcal{G}}^V$ and $\tilde{e}_1, \ldots, \tilde{e}_k$ is a reduced path in $\tilde{\mathbb{A}}$ from $\tilde{v}_1$ to $\tilde{v}_2$ then
$$d_{\mathcal{G}}(y_1, y_2) = d_{\tilde{v}_1}(y_1, p^{\alpha}_{\tilde{e}_1}) + \sum_{i=1}^{k} l(\downarrow \tilde{e}_i) + \sum_{i=1}^{k-1} d_{\omega(\tilde{e}_i)}(p^{\omega}_{\tilde{e}_i}, p^{\alpha}_{\tilde{e}_{i+1}}) + d_{\tilde{v}_2}(p^{\omega}_{\tilde{e}_k}, y_2)$$

2. If $y_1 = (x_1, \tilde{e}_1) \in T_{\mathcal{G}}^E$, $y_2 = (x_2, \tilde{v}_2) \in T_{\mathcal{G}}^V$ and $\tilde{e}_1, \ldots, \tilde{e}_k$ is a reduced path in $\tilde{\mathbb{A}}$ with $\omega(\tilde{e}_k) = \tilde{v}_2$ then
$$d_{\mathcal{G}}(y_1, y_2) = (l(\downarrow \tilde{e}_1) - x_1) + \sum_{i=2}^{k} l(\downarrow \tilde{e}_i) + \sum_{i=1}^{k-1} d_{\omega(\tilde{e}_i)}(p^{\omega}_{\tilde{e}_i}, p^{\alpha}_{\tilde{e}_{i+1}}) + d_{\tilde{v}_2}(p^{\omega}_{\tilde{e}_k}, y_2)$$

3. If $y_1 = (x_1, \tilde{e}_1), y_2 = (x_2, \tilde{e}_k) \in T_{\mathcal{G}}^E$ and $\tilde{e}_1, \ldots, \tilde{e}_k$ is a reduced path in $\tilde{\mathbb{A}}$ then
$$d_{\mathcal{G}}(y_1, y_2) = (l(\tilde{e}_1) - x_1) + \sum_{i=2}^{k-1} l(\downarrow \tilde{e}_i) + \sum_{i=1}^{k-1} d_{\omega(\tilde{e}_i)}(p^{\omega}_{\tilde{e}_i}, p^{\alpha}_{\tilde{e}_{i+1}}) + x_2$$

Recall that the restriction of $d_{\mathcal{G}}$ to any vertex tree $T_{\tilde{v}}$, resp. edge segment $T_{\tilde{e}}$, equals $d_{\tilde{v}}$, resp. $d_{\tilde{e}}$. It therefore follows from (2.3) that the distance of two points in $T_{\mathcal{G}}$ can be computed entirely in terms of the metrics $d_v$ of the vertex trees of $\mathcal{G}$ and its length function $l$. The case we are mostly interested in is case 1 above, i.e. the case where $y_1$ and $y_2$ are contained in vertex trees $T_{\tilde{v}_1}$ and $T_{\tilde{v}_2}$. If $p = a_0, e_1, a_1, \ldots, a_k$ is a reduced $\mathbb{A}$-path equivalent to $p_{\tilde{v}_1}^{-1} p_{\tilde{v}_2}$, then $d_{\mathcal{G}}(y_1, y_2)$ can be computed as

$$
\begin{aligned}
d_{\mathcal{G}}(y_1, y_2) &= d_{\alpha(e_1)}(x_1, a_0 p^{\alpha}_{e_1}) \\
&+ \sum_{i=1}^{k} l(e_i) \\
&+ \sum_{i=1}^{k-1} d_{\omega(e_i)}(p^{\omega}_{e_i}, a_i p^{\alpha}_{e_{i+1}}) \\
&+ d_{\omega(e_k)}(p^{\omega}_{e_k}, a_k x_2). \qquad (2.4)
\end{aligned}
$$

We say that a $G$-tree $T$ *splits as a graph of actions* $\mathcal{G}(\mathbb{A})$ if $G \cong \pi_1(\mathbb{A})$ and there is a $G$-equivariant isometry from $T$ to $T_{\mathcal{G}}$.

*Remark* 2.3. Let $\mathcal{G}$ be a graph of actions, $e \in EA$ and $g \in A_{\alpha(e)}$. Assume that $\mathcal{G}'$ is the graph of actions obtained from $\mathcal{G}$ by replacing the attaching point $p^{\alpha}_e \in T_{\alpha(e)}$ by $gp^{\alpha}_e$ and the embedding $\alpha_e : A_e \to A_{\alpha(e)}$ by $i_g \circ \alpha_e$. Then $T_{\mathcal{G}}$ also splits as the graph of actions $\mathcal{G}'$.

It follows from the remark that in a graph of actions splitting of a tree $T$ we are free to alter the attaching points within their orbits of the vertex actions. In particular, if a vertex group $A_v$ acts with dense orbits on $T_v$, the attaching points in $A_v$ can be chosen to be arbitrarily close to each other. This will turn out useful in chapter 4.

## 2.3 The structure theorem

In this section we state the structure theorem for finitely generated groups acting on $\mathbb{R}$-trees as it appears in [G]. This theorem (and its relatives) are usually simply referred to as the Rips machine.

We recall from [G] that a $G$-tree $T$ *satisfies the ascending chain condition* if for any sequence of arcs $I_1 \supset I_2 \supset \dots$ in $T$ whose lengths converge to 0, the sequence of the stabilizers of the segments is eventually constant.

The following theorem is a generalization of Sela's version of the Rips machine for finitely generated groups [Sel1]. It extends Sela's original version to allow the group which acts on an $\mathbb{R}$-tree to have torsion.

**Theorem 2.4** (Main Theorem of [G]). *Consider a non-trivial action of a f.g. group $G$ on an $\mathbb{R}$-tree $T$ by isometries. Assume that*

- *$T$ satisfies the ascending chain condition,*
- *for any unstable arc $J \subset T$,*
  - $\mathrm{stab}(J)$ *is finitely generated*
  - $\mathrm{stab}(J)$ *is not a proper subgroup of any conjugate of itself, i.e.* $\forall g \in G, (\mathrm{stab}(J))^g \subset \mathrm{stab}(J) \Rightarrow (\mathrm{stab}(J))^g = \mathrm{stab}(J)$.

*Then either $G$ splits over the stabilizer of an unstable arc or over the stabilizer of an infinite tripod, or $T$ splits as a graph of actions*

$$\mathcal{G} = (\mathbb{A}, (T_v)_{v \in VA}, (p_e^{\alpha})_{e \in EA})$$

*where each vertex action of $A_v$ on the vertex tree $T_v$ is either*

- *simplicial: a simplicial action on a simplicial tree,*
- *of orbifold (or Seifert) type: the action of $A_v$ has kernel $N_v$ and the faithful action of $A_v/N_v$ is dual to an arational measured foliation on a closed 2-orbifold with boundary, or*
- *axial: $T_v$ is a line and the image of $A_v$ in $Isom(T_v)$ is a finitely generated group acting with dense orbits on $T_v$.*

For a detailed description of measured foliations on 2-orbifolds and orbifold type vertex groups see [RW].

Note that if the $G$-tree $T$ admits a splitting as a graph of actions $\mathcal{G}$ as in Theorem 2.4 and if $\mathbb{A}$ contains a nondegenerate simplicial vertex tree, we get a refined splitting of $T$ as a graph of actions

$$\mathcal{G}' = (\mathbb{A}', (T_v)_{v \in VA'}, (p_e^\alpha)_{e \in EA'}, l)$$

such that any vertex tree is either of axial type, or of orbifold type or is degenerate, i.e. consists of a single point. This is easily achieved by decomposing each simplicial vertex tree using Bass-Serre theory, possibly after subdiving some edges to ensure that the original attaching points are vertices. Note that if an edge $e$ of the refined graph of actions has non-zero length then both $T_{\alpha(e)}$ and $T_{\omega(e)}$ are degenerate.

# Chapter 3

# The almost abelian JSJ-decomposition of Γ-limit groups

In chapters 1 and 2 we have seen that Γ-limit groups admit natural actions on real trees, which give us decompositions of Γ-limit groups as fundamental groups of graphs of groups with almost abelian edge groups. In this chapter we first study basic properties of almost abelian splittings of Γ-limit groups and then discuss almost abelian JSJ-decompositions of Γ-limit groups, splittings that reveal all almost abelian splittings simultaneously. We closely follow Sela's construction of the JSJ-decomposition [Sel1]. Similar to the discussion of Bestvina and Feighn [BF1] we do not require the JSJ to be unfolded, although using the shortening argument of chapter 4, we will prove in chapter 5 that the JSJ can in fact be chosen unfolded.

## 3.1 Modifying splittings

Recall that a splitting is *minimal* if the corresponding Bass-Serre tree contains no invariant proper subtree. In the case of graphs of groups with finite underlying graph this is equivalent to no boundary monomorphism into a vertex group of a valence 1 vertex being surjective. A graph of groups is further called *reduced* if no boundary monomorphism into a vertex group of a vertex of valence greater than 1 is surjective. The JSJ-decompositions we construct reveal almost abelian splittings only up to certain modifications, which we introduce in the following definition. It is easy to verify that none of these modifications changes the fundamental group of a graph of groups.

**Definition 3.1.** Let $\mathbb{A}$ be a graph of groups. A splitting move on $\mathbb{A}$ is one of the following modifications of $\mathbb{A}$.

1. *Boundary slide:* Let $e \in EA$. A *boundary slide* (of the boundary monomorphism $\alpha_e$) is the replacement of $\alpha_e$ by $i_g \circ \alpha_e$ for an element $g \in A_{\alpha(e)}$.

2. *Edge slide:* Let $v_1, v_2 \in VA$ and $e_1 \neq e_2 \in EA$ such that $v_1 = \omega(e_1) = \alpha(e_2)$, $v_2 = \omega(e_2)$. Suppose that $\omega_{e_1}(A_{e_1})$ is in $A_{v_1}$ conjugate to a subgroup of $\alpha_{e_2}(A_{e_2})$. Then we first perform a boundary slide such that $\omega_{e_1}(A_{e_1}) \leq \alpha_{e_2}(A_{e_2})$ and then replace $e_1$ with an edge $e_1'$ such that

   (a) $A_{e_1'} = A_{e_1}$, $\alpha(e_1') = \alpha(e_1)$ and $\alpha_{e_1'} = \alpha_{e_1}$.

   (b) $\omega(e_1') = v_2$.

   (c) $\omega_{e_1'} = \omega_{e_2} \circ \alpha_{e_2}^{-1} \circ \omega_{e_1}$.

   The combination of the initial boundary slide and the subsequent modification is called an *edge slide* of $e_1$ over $e_2$.

3. *Folding/Unfolding:* Let $e \in EA$ and $\alpha_e(A_e) < C < A_{\alpha(e)}$. A *folding* along $e$ is the replacement of $A_e$ by $C$, $A_{\omega(e)}$ by $C *_{A_e} A_{\omega(e)}$ and the corresponding replacement of the boundary monomorphisms $\alpha_e$ and $\omega_e$. The inverse of a folding along $e$ is an *unfolding* along $e$.

Note that we usually only consider graphs of groups up to boundary slides. Thus when we say that a splitting $\mathbb{B}$ is obtained from a splitting $\mathbb{A}$ by certain operations we mean that $\mathbb{B}$ is the obtained splitting up to boundary slides.

On the level of the Bass-Serre tree an edge slide can be defined as follows. Given two non-equivalent edges $f_1$ and $f_2$ such that $\omega(f_1) = \alpha(f_2)$ and $\text{stab}(f_1) \leq \text{stab}(f_2)$ we slide $f_1$ over $f_2$ in an equivariant way. One can also think of it as first subdividing $f_1$ into $f_1^1$ and $f_1^2$ and then folding $f_1^2$ onto $f_2$. Note that the action on the vertex set of the tree is unchanged under this operation.

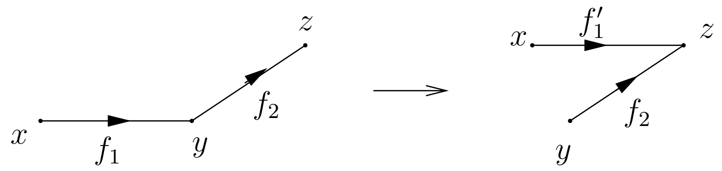



Figure 3.1: An edge slide as seen in the tree

The following is a trivial observation

**Lemma 3.2.** *Let $G$ be a group and $\mathbb{A}$ be a graph of groups decomposition of $G$. Assume that $\mathbb{A}'$ is obtained from $\mathbb{A}$ by boundary slides and edge slides. Then $\mathbb{A}$ is reduced and minimal iff $\mathbb{A}'$ is reduced and minimal.*

## 3.2 Almost abelian splittings of Γ-limit groups

Throughout this section let $\Gamma$ be a fixed hyperbolic group and $L$ a $\Gamma$-limit group. We study splittings of $L$ as fundamental groups of graphs of groups with almost abelian edge groups. We call such splittings *almost abelian splittings.*

A crucial observation in this chapter will be that any almost abelian splitting of $L$

can be modified by boundary slides and some further simple modifications such that all almost abelian subgroups are elliptic. In the following we call an almost abelian group *large* if it contains a one-ended subgroup. Note that if the group is f.g., large is clearly equivalent to one-ended.

Further, an almost abelian graph of groups $\mathbb{A}$ is *compatible* if all large almost abelian subgroups of $\pi_1(\mathbb{A})$ are elliptic, i.e. conjugate into a vertex group of $\mathbb{A}$.

**Lemma 3.3.** *Let $L$ be a $\Gamma$-limit group with almost abelian graph of groups decomposition $\mathbb{A}$. Let $M \leq L$ be a maximal large almost abelian subgroup which is not elliptic in $\mathbb{A}$. Then $M$ acts with an invariant line $T \subset \tilde{\mathbb{A}}$, satisfying*

1. *for each edge $e$ in $T$, $\mathrm{stab}_M(e) = \mathrm{stab}_L(e)$,*
2. *if $e_1$ and $e_2$ are edges in $T$ and $g \in L$ s.t. $ge_1 = e_2$, then $g \in M$.*

*Proof.* As $M$ does not contain a non-abelian free group it either acts with a fixed point or with an invariant line or parabolically, i.e. fixes a unique end of $T$. By assumption $M$ does not act with a fixed point.

Assume that $M$ acts parabolically on $\tilde{\mathbb{A}}$, i.e. preserves a unique end. Being almost abelian, $M$ can not be a strictly ascending HNN-extension, hence every $g \in M$ is elliptic. Let $e_1, e_2, \ldots$ be a ray in $\tilde{\mathbb{A}}$ representing the fixed end. We get an infinite ascending sequence of stabilizers

$$\mathrm{stab}_M(e_1) \leq \mathrm{stab}_M(e_2) \leq \ldots,$$

such that $M = \bigcup \mathrm{stab}_M(e_i)$. As $M$ is infinite and there is a uniform bound on the order of finite $\Gamma$-limit groups it follows that there exists $i_0$ such that $\mathrm{stab}_M(e_i)$ is infinite for $i \geq i_0$. This implies in particular that $\mathrm{stab}_M(e_i) = \mathrm{stab}_L(e_i)$ for $i \geq i_0$ as $M$ is the unique maximal almost abelian subgroup of $L$ containing $\mathrm{stab}_M(e_i)$. As $EA$

is finite there exists $i > j \geq i_0$ and $g \in L$ such that $e_i = ge_j$. Now

$$\begin{aligned} \operatorname{stab}_L(e_j) &= \operatorname{stab}_L(e_i) \cap \operatorname{stab}_L(e_j) \\ &= \operatorname{stab}_L(ge_j) \cap \operatorname{stab}_L(e_j) \\ &= g\operatorname{stab}_L(e_j)g^{-1} \cap \operatorname{stab}_L(e_j), \end{aligned}$$

thus by Corollary 1.18 $g$ is contained in the maximal almost abelian subgroup containing $\operatorname{stab}_L(e_j)$, i.e. $g \in M$. But $g$ acts without fixed point, which contradicts the assumption on the action of $M$.

It follows that $M$ preserves a line $T \subset \tilde{\mathbb{A}}$. As $M$ is large it follows that $\operatorname{stab}_M(e)$ is infinite for all $e \subset T$. As before we see that $\operatorname{stab}_L(e) = \operatorname{stab}_M(e)$ for all $e \subset T$. Also the same argument as before shows 2. □

**Proposition 3.4.** *Let $L$ be a $\Gamma$-limit group with almost abelian graph of groups decomposition $\mathbb{A}$. Then after finitely many edge slides we can assume for any large maximal almost abelian subgroup $M$ of $L$ one of the following holds.*

1. *$M$ is elliptic.*
2. *$M$ is the unique maximal almost abelian subgroup containing some edge group $A_e$ and is of type $A_1 *_{A_e} A_2$ where $A_1 \leq A_{\alpha(e)}$, $A_2 \leq A_{\omega(e)}$ and $|A_1 : \alpha_e(A_e)| = |A_2 : \omega_e(A_e)| = 2$.*
3. *$M$ is the unique maximal almost abelian subgroup containing the edge group $A_e$ of some loop edge $e$. Furthermore $M = A_e *_{A_e}$ where the stable letter is the element corresponding to the loop edge $e$, in particular $\alpha_e(A_e) = \omega_e(A_e) \leq A_{\alpha(e)}$.*

*Proof.* Assume that $M$ is not elliptic. By Lemma 3.3, there is a line $T \subset \tilde{\mathbb{A}}$ on which $M$ acts invariantly. Let $f_1, f_2, \ldots, f_k \subset EA$ be an edge path which lifts to a fundamental domain of the $M$-action on $T$.

If $k = 1$, i.e. if the edge path consists of a single edge, then there is nothing to show as we are either in situation (2) or in situation (3). Thus we can assume that $k \geq 2$.

It follows from Lemma 3.3 that $\mathrm{stab}_L(f_1) = \mathrm{stab}_L(f_2)$. Thus we can $L$-equivariantly slide $f_1$ over $f_2$. The new fundamental domain for $M$ has only $k-1$ edges. After finitely many slides the fundamental domain consists of a single edge and the conclusion follows for $M$.

The conclusion now follows from the observation that an edge is slid over another edge only if their edge groups are contained in the same maximal almost abelian subgroup. Thus the above process for one maximal almost abelian subgroup does not affect the validity of the conclusion of the proposition for the other. Thus we can use edge slides to obtain the desired conclusion for all maximal one-ended almost abelian subgroups simultaneously. □

Once the splitting is as in the conclusion of Proposition 3.4 we can easily modify the splitting such that afterwards all maximal almost abelian one-ended subgroups are elliptic.

Let $M$ be a large maximal almost abelian subgroup of $L$. Assume first that $M$ satisfies (2) of Proposition 3.4. Then we subdivide the edge $e$ into edge $e_1$ and $e_2$ such that $\alpha(e_1) = \alpha(e)$, $\omega(e_2) = \omega(e)$ and $\omega(e_1) = \alpha(e_2) = v'$ is a new vertex. Moreover $A_{e_1} = A_1$, $A_{e_2} = A_2$ and $A_{v'} = A_1 *_{A_e} A_2$ and the boundary monomorphisms are the natural ones, see Figure 3.2 for an illustration of both the case where $e$ is a non-loop edge, and where $e$ is a loop edge.

If $M$ satisfies (3) of Proposition 3.4 then we remove the edge $e$ and add a new edge $e'$ such that $\alpha(e') = \alpha(e)$, and $\omega(e') = w$ is a new vertex with vertex group $A_w = A_e *_{A_e}$. Moreover $A_{e'} = A_e$ and the boundary monomorphisms are the natural embeddings of $A_e$, see Figure 3.3.

We now argue that there exists an upper bound on the complexity of a minimal (and

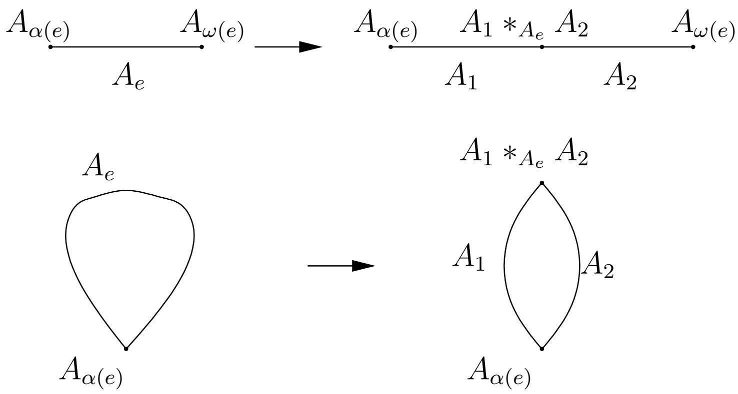



Figure 3.2: A new vertex with maximal almost abelian vertex group

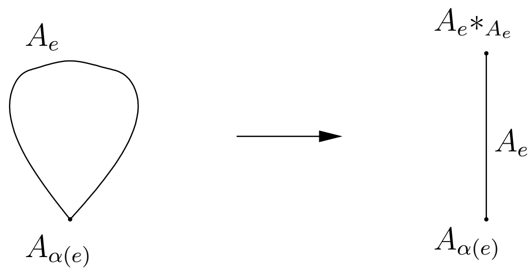



Figure 3.3: A new vertex with maximal almost abelian vertex group

reduced) almost abelian splitting of $L$, which only depends on the rank of $L$ and $N(\Gamma)$ (cf. Proposition 1.12). For a given graph of groups $\mathbb{A}$, we define its complexity $C(\mathbb{A})$ by

$$C(\mathbb{A}) := |EA| + \beta_1(A) \tag{3.1}$$

where $|EA|$ denotes the number of edges of the graph $A$ underlying $\mathbb{A}$, and $\beta_1(A)$ is the first Betti number of $A$. While the Betti number is bounded from above by the rank of $L$, a bound on $|EA|$ can be obtained from Theorem 3.5 and Lemma 3.6 below. Recall that a graph of groups is called $(k, C)$-*acylindrical* if the stabilizer of any segment $[v, w]$ in the Bass-Serre tree with $d(v, w) > k$ is of order at most $C$.

The following theorem from [W2] provides a bound on the complexity of $(k, C)$-acylindrical splittings. It is a generalization of Sela's acylindrical accessibility theorem [Sel1] which deals with the case $C = 1$, see also [W1].

**Theorem 3.5.** *Let $\mathbb{A}$ be a reduced and minimal $(k, C)$-acylindrical graph of groups with $k \geq 1$. Then*

$$|EA| \leq (2k+1) \cdot C \cdot (\text{rank}(\pi_1(\mathbb{A})) - 1).$$

While in general almost abelian compatible splittings of Γ-limit groups are not acylindrical it turns out that they can easiliy be modified to be so without increasing the complexity. Thus Theorem 3.5 together with the following lemma provide a bound on the complexity of almost abelian compatible splittings of Γ-limit groups.

For an infinite almost abelian subgroup $H \leq L$, in the following we denote by $[H]$ its conjugacy class in $L$. Moreover, $\text{MA}([H])$ denotes the conjugacy class of maximal almost abelian subgroups of $L$ which has a representative containing $H$. Note that we will use the notation $\text{MA}(A_e)$ correspondingly, regarding an edge group as a conjugacy class of subgroups of $L$.

**Lemma 3.6.** *Let $\mathbb{A}$ be a compatible almost abelian splitting of $L$. Then $\mathbb{A}$ can be modified by a finite sequence of edge slides to be $(2, N(\Gamma))$-acylindrical.*

*Proof.* For any conjugacy class of maximal almost abelian subgroups $[M]$ we choose a vertex $v_{[M]}$ such that $M$ is conjugate into $A_{v_{[M]}}$. By a finite sequence of edge slides any edge $e$ can be slid such that it is adjacent to $v_{\text{MA}(A_e)}$. The $(2, N(\Gamma))$-acylindricity of the obtained splitting is easily verified. □

The construction of the $(2, N(\Gamma))$-acylindrical graph of groups in the above proof depends on the choice of the vertices $v_{[M]}$ and the output is therefore not unique. We will establish the uniqueness (up to boundary slides) by performing the following *normalization process* for a given compatible almost abelian splitting $\mathbb{A}$ of $L$.

The normalization process is only a slight modification of the proof of Lemma 3.6. Let $[M_1], \ldots, [M_k]$ be the collection of those conjugacy classes of maximal almost abelian edge groups which appear as $\text{MA}(A_e)$ for some $e \in EA$. This collection is clearly finite as $EA$ is finite.

For each $i = 1, \dots, k$ choose a vertex $v_i$ such that $M_i$ is conjugate into $A_{v_i}$ and introduce a new vertex $v_{[M_i]}$ joined to $v_i$ by an edge $e_i$ such that the vertex group $A_{v_{[M_i]}}$ and the edge group $A_{e_i}$ are isomorphic to $M_i$ with the boundary monomorphisms being isomorphisms. Note that this produces a non-minimal splitting. We then slide as in the proof of Lemma 3.6, i.e. slide every edge $e$ such that it is adjacent to $v_{\mathrm{MA}(A_e)}$. Finally, minimize the obtained graph of groups by removing unnecessary valence 1 vertices and corresponding edges.

It is clear that, up to boundary slides, the obtained graph of groups does not depend on the choice of $v_i$ and is therefore unique.

We say that $\mathbb{A}$ *is in normal form* if it is the output of the normalization process for some graph of groups $\mathbb{A}'$ (or equivalently, if the normalization process of $\mathbb{A}$ reproduces $\mathbb{A}$). Note that in particular, a graph of groups in normal form is $(2, N(\Gamma))$-acylindrical and minimal. Moreover, although it may be non-reduced, any normal form graph of groups can be obtained by normalizing a reduced, minimal and $(2, N(\Gamma))$-acylindrical graph of groups $\mathbb{A}'$, and by construction $C(\mathbb{A}) \leq 2C(\mathbb{A}')$. It follows, using Theorem 3.5, that there is a global upper bound on the complexity of all normal form splittings of a given one-ended $\Gamma$-limit group $L$.

## 3.3 Morphisms of graphs of groups

For a $G$-tree $T$ with base point $\tilde{v}_0$ and an $H$-tree $Y$ with base point $\tilde{u}_0$ a morphism from $T$ to $Y$ is a pair $(\varphi, f)$ where $\varphi : G \to H$ is a homomorphism and $f : T \to Y$ is a simplicial map such that $f(\tilde{v}_0) = \tilde{u}_0$ and that

$$f(gx) = \varphi(g)f(x)$$

for all $x \in T$ and $g \in G$. This morphism can be encoded on the level of the associated graphs of groups. We will discuss such morphisms for graphs of groups and make some basic observations.

A *morphism* from a graph of groups $\mathbb{A}$ to a graph of groups $\mathbb{B}$ is a tuple

$$\mathfrak{f} = (f, \{\psi_v | v \in VA\}, \{\psi_e | e \in EA\}, \{o_e \,|\, e \in EB\}, \{t_e \,|\, e \in EB\})$$

where

1. $f : A \to B$ is a graph morphism.
2. $\psi_v$ is a homomorphism from $A_v$ to $B_{f(v)}$ for all $v \in VA$.
3. $\psi_e$ is a homomorphism from $A_e$ to $B_{f(e)}$ and $\psi_e = \psi_{e^{-1}}$ for all $e \in EA$.
4. $o_e \in B_{f(\alpha(e))}$, $t_e \in B_{f(\omega(e))}$ and $t_e^{-1} = o_{e^{-1}}$ for all $e \in EA$.
5. $\psi_{\alpha(e)} \circ \alpha_e = i_{o_e} \circ \alpha_{f(e)} \circ \psi_e$ for all $e \in EA$.

A morphism from $\mathbb{A}$ to $\mathbb{B}$ induces a homomorphism

$$\mathfrak{f}_* : \pi_1(\mathbb{A}, v_0) \to \pi_1(\mathbb{B}, f(v_0))$$

given by

$$[a_0, e_1, a_1, \ldots, a_{k-1}, e_k, a_k] \mapsto [b_0, f(e_1), b_1, \ldots, b_{k-1}, f(e_k), b_k]$$

where $b_0 = \psi_{\alpha(e_1)}(a_0)o_{e_1}$ and $b_i = t_{e_i}\psi_{\omega(e_i)}(a_i)o_{e_{i+1}}$ for $i = 1, \ldots, k$ (with $t_{e_{k+1}} = 1$).

We will write $\psi_v^{\mathfrak{f}}$ or $\psi_e^{\mathfrak{f}}$ instead of $\psi_v$ or $\psi_e$ if we want to make explicit that the maps come from the morphism $\mathfrak{f}$. We will further say that a morphism $\mathfrak{f}$ is surjective if $\mathfrak{f}_*$ is surjective.

The morphism $\mathfrak{f}$ further determines a morphism $\tilde{f} : \widetilde{(\mathbb{A}, v_0)} \to \widetilde{(\mathbb{B}, u_0)}$ (that maps the base point $\tilde{v}_0$ to $\tilde{u}_0$). The pair $(\mathfrak{f}_*, \tilde{f})$ is a morphism from the $\pi_1(\mathbb{A}, v_0)$-tree $\widetilde{(\mathbb{A}, v_0)}$ to the $\pi_1(\mathbb{B}, u_0)$-tree $\widetilde{(\mathbb{B}, u_0)}$, see [KMW] for details. Moreover any morphism from a $G$-tree $T$ to an $H$-tree $Y$ occurs this way.

In the subsequent sections we will need the following simple fact about morphisms. Note that in the statement of the proposition we identify the fundamental group of $\mathbb{A}$

and the fundamental groups of $\bar{\mathbb{A}}$ obtained from $\mathbb{A}$ by edge collapses and subdivision in the natural way.

**Proposition 3.7.** *Let $\mathbb{A}$ and $\mathbb{B}$ be finite graphs of groups and*

$$\eta : \pi_1(\mathbb{A}, v_0) \to \pi_1(\mathbb{B}, u_0)$$

*be an isomorphism. Suppose further that there exists a map $h : VA \to VB$ such that the following hold:*

1. *$h(v_0) = u_0$ and $\eta([A_{v_o}]) \subset [B_{u_0}]$*
2. *$\eta(A_v)$ is conjugate to a subgroup of $B_{h(v)}$ for all $v \in VA$.*

*Then there exists a graph of groups $\bar{\mathbb{A}}$ obtained from $\mathbb{A}$ by collapses of edges followed by subdivisions of edges and a morphism $\mathfrak{f} : \bar{\mathbb{A}} \to \mathbb{B}$ such that $f(v) = h(v)$ for all $v \in VA$ and $\mathfrak{f}_* = \eta$.*

*Proof.* Let $T_A = \widetilde{(\mathbb{A}, v_0)}$ and $T_B = \widetilde{(\mathbb{B}, u_0)}$ be the Bass-Serre trees with base points $\tilde{v}_0$ and $\tilde{u}_0$. It suffices to show that there exists a morphism $(\eta, f)$ from the $\pi_1(\mathbb{A}, v_0)$-tree $T_A$ to the $\pi_1(\mathbb{B}, u_0)$-tree $T_B$ such that $f(\tilde{v}_0) = \tilde{u}_0$ and that $\pi_B(f(v)) = h(\pi_A(v))$ for all $v \in VT_A$ where $\pi_A : T_A \to A$ and $\pi_B : T_B \to B$ are the canonical quotient maps.

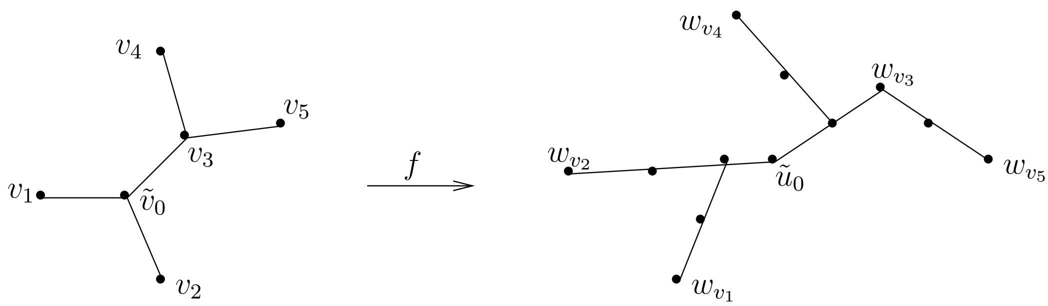



Figure 3.4: The restriction of $f$ to $\tilde{Y}_A$

Pick a maximal tree $Y_A$ in $A$ and a lift $\tilde{Y}_A$ to $T_A$ such that the lift of $v_0$ is $\tilde{v}_0$. By assumption we can choose for each vertex $v \in \tilde{Y}_A$ a vertex $w_v \in T_B$ such that $\eta(\text{stab}(v)) \le \text{stab}(w_v)$ and that $h(\pi_A(v)) = \pi(w_v)$; we can further assume that $w_{\tilde{v}_0} = \tilde{u}_0$. We now define a map $f : VT_A \to VT_B$ by $gv \mapsto \eta(g)w_v$ for all $v \in \tilde{Y}_A$ and $g \in \pi_1(\mathbb{A}, v_0)$. The map is easily extended to the edges of $T_A$ by mapping an edge $e = (v_1, v_2)$ to the reduced edge path from $f(v_1)$ to $f(v_2)$ after subdividing $e$ $d_{T_B}(f(v_1), f(v_2)) - 1$ times and then mapping the subdivided edge simplicially. If $f(v_1) = f(v_2)$ we map $e$ to $f(v_1)$ which corresponds to a collapse of $e$. □

*Remark* 3.8. It follows from the above proof that the number of subdivisions applied to an edge $e = (v_1, v_2) \in EA$ is bounded by $\text{diam}_{\tilde{\mathbb{B}}}(\text{Fix}\ \eta(A_e)) - 1$ as $\eta(A_e)$ fixes the segment $[f(v_1), f(v_2)]$.

## 3.4 The almost abelian JSJ-decomposition of a Γ-limit group

In this section we establish the existence of almost abelian JSJ-decompositions of Γ-limit groups. An almost abelian JSJ-decomposition of a group $G$ is a splitting of $G$ in which all compatible almost abelian splittings of $G$ are apparent.

In the following we say that a one-ended vertex group $A_v$ of a graph of groups $\mathbb{A}$ is a QH-vertex group if the following hold.

1. $A_v$ is finite-by-2-orbifold, i.e. there exists a 2-orbifold group $O = \pi_1(\mathcal{O})$, some finite group $E$ and a short exact sequence

$$1 \to E \to A_v \xrightarrow{\pi} O \to 1.$$

   We will not mention the dimension 2 from now on, all orbifolds are understood to be of dimension 2. For a more detailed description of the arising orbifolds see [RW].

2. For any edge $e \in EA$ s.t. $\alpha(e) = v$ there exists a *peripheral* subgroup $O_e$ of $O$, i.e. a subgroup corresponding to a boundary component of $\mathcal{O}$, such that $\alpha_e(A_e)$ is in $A_v$ conjugate to a finite index subgroup of $\pi^{-1}(O_e)$.

We will also say that a subgroup of $G$ is a QH-subgroup if it is conjugate to a QH-vertex group of some splitting $\mathbb{A}$.

It is a trivial but important observation that any essential simple closed curve or essential segment joining two points on the reflection boundary of $\mathcal{O}$ induces a splitting of $G = \pi_1(\mathbb{A})$ over a 2-ended group. We call such a splitting *geometric* with respect to the QH-subgroup $A_v$.

We can depict the splitting $\mathbb{A}$ by drawing the orbifold $\mathcal{O}$ for the vertex group $A_v$ and depicting all non-QH vertex groups as balls joined by edges.

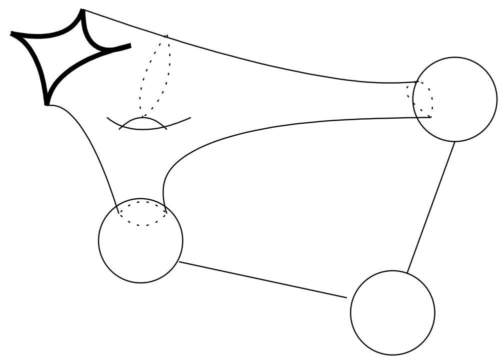


Figure 3.5: A QH-subgroup with a simple closed curve representing an HNN-extension of $\pi_1(\mathbb{A})$

Recall that a 1-edge splitting $\mathbb{A}_1$, i.e. a splitting as an amalgamated product or an HNN-extension, of a group $G$ is called elliptic with respect to another splitting $\mathbb{A}_2$ if the edge group of $\mathbb{A}_1$ is conjugate to a subgroup of a vertex group of $\mathbb{A}_2$. Otherwise $\mathbb{A}_1$ is called hyperbolic with respect to $\mathbb{A}_2$. It is an important observation in [RS2] that if $G$ is one-ended and $\mathbb{A}_1$ and $\mathbb{A}_2$ are 1-edge splittings of $G$ over 2-ended groups then the two splittings are either both elliptic or both hyperbolic with respect to each other. In the first case we say that $\mathbb{A}_1$ and $\mathbb{A}_2$ are elliptic-elliptic and in the latter that they are hyperbolic-hyperbolic.

The following theorem is the key step in the proof of the JSJ-decomposition, it in particular implies the existence of a splitting of a $\Gamma$-limit group that encodes all hyperbolic-hyperbolic splittings over 2-ended subgroups.

**Theorem 3.9.** *Let $G$ be a f.g. one-ended group. Assume that there exists $N$ such that any finite subgroup of $G$ has order at most $N$.*

*Then there exists a reduced, minimal $(2, N)$-acylindrical graph of groups decomposition $\mathbb{A}$ of $G$ where (possibly) some vertex groups are QH-vertex groups such that the following hold.*

1. *Any 1-edge splitting of $G$ over a 2-ended group that is hyperbolic-hyperbolic with respect to another 1-edge splitting over a 2-ended group is geometric with respect to some QH-vertex group.*
2. *If $\mathbb{B}$ is a compatible almost abelian splitting of $G$ such that no edge group of $\mathbb{B}$ is 2-ended and hyperbolic-hyperbolic with respect to another splitting over a 2-ended group then the QH-vertex groups of $\mathbb{A}$ are elliptic with respect to $\mathbb{B}$.*
3. *Any QH-subgroup of $G$ is conjugate to a subgroup of a QH-vertex group $A_v$, moreover this subgroup corresponds to a suborbifold of the orbifold of $v$. In particular the QH-vertex groups are the maximal QH-subgroups and are unique up to conjugacy.*

*Proof.* Note first that any splitting can be modified such that the new splitting is $(2, N)$-acylindrical while preserving the conjugacy classes of the $QH$-vertex groups. Indeed we first refine the splitting by replacing each QH-vertex group $A_v$ by a tree of groups consisting of a vertex $x_v$ with vertex group $A_v$ and an edge $e_P$ with edge group

$P$ and a vertex $x_P$ with vertex group $P$ for each peripheral subgroup $P$ such that $\alpha(e_P) = x_v$ and $\omega(e_P) = x_P$ with the boundary monomorphisms being the obvious ones. We refine such that all previous edges are adjacent to one the vertices of type $x_P$. We then collapse all edges not adjacent to a vertex of type $x_v$. The $(2, N)$-acylindricity of the obtained splitting follows from the fact that for any QH-vertex group $A_v$ and peripheral subgroups $P_1$ and $P_2$ corresponding to distinct boundary components the following hold.

1. $gP_1g^{-1} \cap P_1$ is finite for all $g \in A_v \backslash P_1$.
2. $gP_1g^{-1} \cap P_2$ is finite for all $g \in A_v$.

This observation implies in particular that the complexity is bounded in terms of $N$ and the rank of $G$ by Theorem 3.5. Note that also the complexity of the orbifolds is bounded as large orbifolds can be cut along essential simple closed curves or segments to produce a splitting with a higher number of QH-vertex groups.

The proof of part (1) of this statement follows essentially from the construction of the JSJ-decomposition of Dunwoody and Sageev [DS] and Fujiwara and Papasoglu [FP]. They construct splittings such that arbitrary finite collections of hyperbolic-hyperbolic splittings over 2-ended groups can be seen as geometric splittings in QH-vertex groups. In their proofs they then assume that $G$ is finitely presented so they can apply Bestvina-Feighn accessibility [BF3] to guarantee termination of the construction. In our case we can exploit the fact that the obtained splittings are $(2, N)$-acylindrical as discussed above. The same argument was applied in the construction of the quadratic decomposition in [RS2]. Part (3) also follows as in [RS2], [DS], [FP].

Part (2) follows from the same argument that shows that splittings over 2-ended groups are either elliptic-elliptic or hyperbolic-hyperbolic. Note first that all edge groups of $\mathbb{B}$ are elliptic in $\mathbb{A}$. Indeed if the edge group is not 2-ended and non-elliptic then it must act with an invariant line $Y$ and infinite pointwise stabilizer of $Y$ which contradicts the $(2, N)$-acylindricity of $\mathbb{A}$. If it is 2-ended and non-elliptic then it is hyperbolic-hyperbolic with respect to the splitting corresponding to some (2-ended) edge group of $\mathbb{A}$, contradicting our assumption on the edge groups of $\mathbb{B}$.

If now a QH-vertex group $A_v$ of $\mathbb{A}$ is non-elliptic in $\mathbb{B}$ then the edge group $A_e$ of a geometric splitting corresponding to some essential simple closed curve or essential segment on the orbifold of $A_v$ is non-elliptic in $\mathbb{B}$, see Corollary 4.12 of [FP]. Then the argument of the proof of Proposition 2.2 of [FP] shows that $G$ splits over an infinite index subgroup of $A_e$, i.e. that $G$ splits over a finite group contradicting the assumption that $G$ is one-ended. $\square$

Before we proceed with the almost abelian JSJ-decomposition, we introduce some notation concerning almost abelian vertex groups of splittings. As before if $M$ is an infinite almost abelian subgroup of a $\Gamma$-limit group $G$ (or infinite almost abelian vertex or edge group of a splitting of $G$) then $M^+$ denotes the maximal finite-by-abelian subgroup of $M$ which is of index at most 2.

**Definition 3.10.** Let $\mathbb{A}$ be an almost abelian splitting of a one-ended $\Gamma$-limit group $G$, $v \in VA$ and $A_v$ be almost abelian. Then $P_v \leq A_v^+$ denotes the subgroup generated by the $\alpha_e(A_e^+)$ for all $e$ with $\alpha(e) = v$. Further, $\bar{P}_v$ denotes the subgroup of $A_v^+$ that consists of all elements that lie in the kernel of all homomoprhisms $\eta : A_v^+ \to \mathbb{Z}$ such that $P_v \subset \ker \eta$.

A simple homology argument shows that $A_v^+/\bar{P}_v$ is a finitely generated free abelian group whose rank is bounded from above by $\text{rank}(L) - \text{val}(v)$, here $\text{val}(v)$ is the valence of $v$ in $A$. Together with Theorem 3.9 this implies in particular that JSJ-decompositions in the sense of the following definition exist.

**Definition 3.11.** Let $L$ be a one-ended $\Gamma$-limit group and $\mathbb{A}$ be an almost abelian compatible splitting of $L$. Then $\mathbb{A}$ is called an almost abelian JSJ-decomposition of $L$ if the following hold.

1. Every splitting over a 2-ended group that is hyperbolic-hyperbolic with respect to another splitting over a 2-ended group is geometric with respect to a QH-subgroup of $\mathbb{A}$.

2. Any edge group of $\mathbb{A}$ that can be unfolded to be finite-by-abelian is finite-by-abelian.

3. For any almost abelian vertex group $A_v$, the rank of $A_v^+/\bar{P}_v$ cannot be increased by unfoldings.

4. $\mathbb{A}$ is in normal form and of maximal complexity among all splittings which satisfy (1)-(3).

In the following we refer to vertex groups of an almost abelian JSJ-decomposition which are neither QH nor almost abelian as *rigid*. Note that any almost abelian JSJ-decomposition can be obtained from a splitting as in Theorem 3.9 by refinements of non-QH-subgroups, unfoldings and the normalization process. This is true as the maximal QH-subgroups must be elliptic by part (2) of Definition 3.11.

The following theorem describes basic properties of almost abelian JSJ-decompositions of Γ-limit groups. We say that a graph of groups $\mathbb{B}$ is *visible* in a graph of groups $\mathbb{A}$ if $\mathbb{B}$ can be obtaind from $\mathbb{A}$ by collapses of edges. In particular, this implies that $\mathbb{A}$ and $\mathbb{B}$ are splittings of the same group.

**Theorem 3.12.** *Let $L$ be a one-ended Γ-limit group and let $\mathbb{A}$ be an almost abelian JSJ-decomposition of $L$. Then the following hold.*

1. *Let $\mathbb{B}$ be an almost abelian compatible splitting such that all maximal QH-subgroups are elliptic. Assume further that $\mathbb{B}$ is either in normal form or a 1-edge splitting. Then $\mathbb{B}$ is visible in $\mathbb{A}$ after unfoldings, foldings and edge slides.*

2. *Any other JSJ-decomposition $\mathbb{B}$ of $L$ can be obtained from $\mathbb{A}$ by a sequence of unfoldings and foldings.*

Note that while part (2) of Theorem 3.12 implies that any finite collection of JSJ-decompositions has a common unfolding, the theorem does not imply that the JSJ can be chosen to be unfolded, i.e. such that no further unfolding is possible. This however will be proven in chapter 5.

*Proof.* We first prove part (1). Let $T_{\mathbb{A}}$ and $T_{\mathbb{B}}$ be the respective Bass-Serre-trees of $\mathbb{A}$ and $\mathbb{B}$. For each $v \in VA$, by restricting the $G$-action on $T_{\mathbb{B}}$ to $A_v$, we obtain a (possibly trivial) splitting $\mathbb{A}^v$ of $A_v$. By assumption these splittings are trivial for QH-vertex groups.

Denote by $\mathbb{A}'$ the graph of groups obtained by refining $\mathbb{A}$ in each vertex $v$ by $\mathbb{A}^v$, and normalizing this refined graph of groups. By construction, neither the Betti number nor the number of edges of $\mathbb{A}$ decrease by the refinement. As the complexity cannot increase by the maximality assumption, it follows that both the Betti number and the number of edges remain unchanged, in particular $C(\mathbb{A}') = C(\mathbb{A})$. We show that $\mathbb{A}'$ can be obtained from $\mathbb{A}$ by unfoldings and that $\mathbb{B}$ is visible in $\mathbb{A}'$ after edge slides and foldings.

By construction, all vertex groups of $\mathbb{A}'$ are elliptic in $\mathbb{A}$. Therefore, by Lemma 3.7, there is a graph of groups $\bar{\mathbb{A}}$ obtained from $\mathbb{A}'$ by collapses and subdivisions and a morphism $\mathfrak{f} : \bar{\mathbb{A}} \to \mathbb{A}$, which maps the characteristic vertices of $\bar{\mathbb{A}}$ to the characteristic vertices of $\mathbb{A}$. But as $\mathbb{A}$ is $(2, N(\Gamma))$-acylindrical, each edge will be subdivided at most once (cf. Remark 3.8). As $\mathbb{A}'$ is in normal form it follows that there are no subdivisions at all, as otherwise backtracking would occur.

Moreover, as $\mathbb{A}$ is minimal, $T_{\mathbb{A}}$ does not contain a proper $G$-invariant subtree, so $f$ is surjective. This implies that no edges are collapsed as $|EA| = |EA'|$. It follows that $\bar{A} = A'$, and we obtain a morphism, again called $\mathfrak{f}$, from $\mathbb{A}'$ to $\mathbb{A}$ whose underlying graph morphism $f$ is in fact a graph isomorphism.

For each $e \in EA'$, we have $A'_e \leq A_{f(e)}$. Assume that for some $e$, this inequality is proper, and assume w.l.o.g. that $\alpha(e)$ is the characteristic vertex of $\mathrm{MA}(A'_e)$. Then we can perform a folding along $e$, replacing $A'_e$ by $A_{f(e)}$ and $A'_{\omega(e)}$ by $A_{f(e)} *_{A'_e} A'_{\omega(e)}$. After applying finitely many such foldings we get that $A'_e \cong A_{f(e)}$ for all $e \in EA'$. As $\mathfrak{f}_* : \pi_1(\mathbb{A}') \to \pi_1(\mathbb{A})$ is an isomorphism, it follows that $\mathbb{A}'$, after the foldings, is isomorphic to $\mathbb{A}$. Conversely, $\mathbb{A}'$ can be obtained from $\mathbb{A}$ by unfoldings.

We now show that $\mathbb{B}$ is visible in a splitting obtained from $\mathbb{A}'$ by foldings and edge slides.

By construction all vertex groups of $\mathbb{A}'$ are elliptic in $\mathbb{B}$. Again there is a graph of groups $\bar{\mathbb{A}}$, which is obtained from $\mathbb{A}'$ by collapses of edges, and a morphism $\mathfrak{f} : \bar{\mathbb{A}} \to \mathbb{B}$. Indeed if $\mathbb{B}$ is in normal form this follows as before and if $\mathbb{B}$ is a 1-edge splitting it follows as all almost abelian compatible 1-edge splittings are $(2, N(\Gamma))$-acylindrical.

After foldings that replace the edge groups $\bar{A}_e$ with $B_{f(e)}$ we can assume that $\mathfrak{f}$ is bijective on edge groups. Assume now that $f(e_1) = f(e_2)$ for some $e_1, e_2 \in E\bar{A}$, in particular $\bar{A}_{e_1} = \bar{A}_{e_2} = B_{f(e_1)}$. Possibly after changing the orientation of $e_1$ and $e_2$ we can further assume that $\alpha(e_1) = \alpha(e_2) = v_{\mathrm{MA}(A_{e_1})}$.

We can now alter $\bar{\mathbb{A}}$ by identifying $e_1$ and $e_2$ by a fold of type IA or IIIA, see [BF3], clearly $\mathfrak{f}$ factors through this fold. Note that this fold can also be thought of as first sliding $e_1$ over $e_2$ and then collapsing $e_1$. After finitely many such operations we have a graph of groups $\hat{\mathbb{A}}$ and a morphism $\hat{\mathfrak{f}} : \hat{\mathbb{A}} \to \mathbb{B}$ such that $f$ is a graph isomorphism, that is bijective on edge groups and induces an isomorphism on the level of the fundamental group. Thus $\hat{\mathbb{A}}$ is isomorphic to $\mathbb{B}$. As $\hat{\mathbb{A}} \cong \mathbb{B}$ has been obtained from $\mathbb{A}'$ by edge collapses, foldings and edge slides it follows that $\mathbb{B}$ is visible in a splitting obtained from $\mathbb{A}'$ by foldings and edge slides. This concludes the proof of part (1).

If $\mathbb{B}$ is itself a JSJ-decomposition of $L$ then the same argument that shows that $\mathbb{A}$ can be obtained from $\mathbb{A}'$ by foldings shows that $\mathbb{B}$ can be obtained from $\mathbb{A}'$ by foldings. This proves (2). □

We conclude this section by defining the modular group of a Γ-limit group with respect to a given almost abelian splitting, using the following definition of natural extensions of vertex automorphisms.

**Definition 3.13.** Let $G$ be a group with a splitting $G = \pi_1(\mathbb{A}, v_0)$, and $v \in VA$. Assume that $\sigma_v \in \mathrm{Aut}(A_v)$ is an automorphism such that for each $e \in EA$ with $\alpha(e) = v$, the restriction $\sigma_v|_{\alpha_e(A_e)}$ is conjugation by an element $\gamma_e \in A_v$. Then the map

$$\phi : G \to G, [a_0, e_1, a_1, \ldots, e_n, a_n] \mapsto [\bar{a}_0, e_1, \bar{a}_1, \ldots, e_n, \bar{a}_n]$$

where

$$\bar{a}_k = \begin{cases} a_k & a_k \notin A_v \\ \gamma_{e_k^{-1}}^{-1} \sigma_v(a_k) \gamma_{e_{k+1}} & a_k \in A_v \end{cases}$$

($\gamma_{e_0^{-1}} = \gamma_{e_{n+1}} = 1$) is a well-defined automorphism of $G$. We call it a natural extension of $\sigma_v$ (with respect to the base vertex $v_0$), and say that $\sigma_v$ is naturally extendable.

Note that a natural extension of a vertex automorphism $\sigma_v$ is not unique as the $\gamma_e$ are not uniquely determined by $\sigma_v$. Moreover, the extension depends on the choice of the base-vertex $v_0$, as natural extensions with respect to distinct base vertices differ by inner automorphisms of $G$. This will be relevant in chapter 8.

**Definition 3.14.** Let $\mathbb{B}$ be an almost abelian splitting of a one-ended group $L$. Then $\mathrm{Mod}_{\mathbb{B}}(L) \leq \mathrm{Aut}(L)$ is the group generated by the following automorphisms.

1. inner automorhisms of $L$,
2. Dehn twists along an edge $e \in EB$ by an element of $Z(\mathrm{MA}(A_e))$ if $A_e$ is finite-by-abelian,
3. natural extensions of geometric automorphisms of a QH-subgroup,
4. natural extensions of automorphisms of a maximal almost abelian vertex group $A_v$ which restrict to the identity on $\bar{P}_v$ and to conjugation on each almost abelian subgroup $U \leq A_v$ with $U^+ = \bar{P}_v$.

It turns out that if $\mathbb{A}$ is an almost abelian JSJ-decomposition of $L$ then $\mathrm{Mod}_{\mathbb{A}}(L)$ contains all other modular groups.

**Lemma 3.15.** *Let $\mathbb{A}$ be an almost abelian splitting of a one-ended group $L$ and assume that $\mathbb{A}'$ is obtained from $\mathbb{A}$ by edge slides and boundary slides. Then $Mod_{\mathbb{A}}(L) = Mod_{\mathbb{A}'}(L)$.*

*Proof.* It is obvious that boundary slides preserve the modular group. So assume that $\mathbb{A}'$ is obtained from $\mathbb{A}$ by an edge slide of $e_1$ over $e_2$, i.e. $e_1$ is replaced by $e_1'$, using the notations of Definition 3.1. It is easy to see that all natural extensions of vertex automorphisms, as well as any Dehn twist along an edge distinct from $e_2$ are unaffected by the edge slide.

Now assume that $\alpha$ is a Dehn twist along $e_2$ by $g \in \mathbb{A}$. Then, in $\mathbb{A}'$, $\alpha$ appears as the product of the Dehn twists by $g$ along $e_2$ and by $\omega_{e_1'}^{-1} \circ \omega_{e_2}(g^{-1})$ along $e_1'$. □

**Proposition 3.16.** *Let $L$ be a one-ended $\Gamma$-limit group, $\mathbb{A}$ be an almost abelian JSJ-decomposition of $L$ and $\mathbb{B}$ be an almost abelian splitting of $L$. Then*

$$Mod_{\mathbb{B}}(L) \leq Mod_{\mathbb{A}}(L).$$

*Proof.* We first deal with the case where $\mathbb{B}$ is compatible. Let $\phi \in \mathrm{Mod}_{\mathbb{B}}(L)$. While there is nothing to prove if $\phi$ is an inner automorphism of $L$, we need to show that the Dehn twists and the natural extensions of vertex automorphisms arising in $\mathrm{Mod}_{\mathbb{B}}(L)$ are contained in $\mathrm{Mod}_{\mathbb{A}}(L)$.

First, assume that $\phi$ is a Dehn twist along an edge $e \in EB$. By Theorem 3.12, the induced 1-edge splitting of $L$ with edge $e$ and edge group $B_e$ is visible in $\mathbb{A}$ after unfoldings, foldings and edge slides. As Dehn twists in the modular group are in elements that centralize $\mathrm{MA}(A_e)$, they are preserved by foldings and unfoldings unless an edge is unfolded such that the corresponding edge group gets finite-by-abelian. But this cannot happen here as edge groups of the JSJ are finite-by-abelian if possible. Moreover, edge slides preserve the Dehn twist by Lemma 3.15.

Now assume that $\phi$ is a natural extension of an automorphism $\sigma \in \mathrm{Aut}(B_v)$ for some $v \in VB$. If $v$ is a QH-subgroup, there is nothing to show as these automorphisms lift to automorphisms of the QH-vertex group of $\mathbb{A}$ containing the QH-vertex group of $\mathbb{B}$. Thus we can assume that $B_v$ is almost abelian. It follows that $B_v$ is maximal almost abelian, as otherwise $\mathrm{MA}(A_v) > A_v$ would be conjugate into another vertex group $B_{v'}$, which implies that $\sigma$ does not restrict to conjugation on all edge groups and is therefore not naturally extendable. It follows that there is a vertex $u \in VA$ such that $A_u = B_v$. We show that $\phi$ arises as a natural extension of $\sigma$ in $\mathbb{A}$. It clearly suffices to show that $\bar{P}_u \subset \bar{P}_v$.

It follows from Theorem 3.12 that after unfolding $\mathbb{A}$ we get a graph of groups $\mathbb{A}'$ such that there exists a morphism from $\mathbb{A}'$ to $\mathbb{B}$. Denote the image of the vertex $u$ in $\mathbb{A}'$ by $u'$. The existence of this morphism implies that $\bar{P}_{u'} \subset \bar{P}_v$, thus it suffices to show that $\bar{P}_u = \bar{P}_{u'}$. Now both $A_u^+/\bar{P}_u$ and $A_u^+/\bar{P}_{u'}$ are finitely generated free abelian groups and as $\mathbb{A}$ is a JSJ-decomposition it follows that rank $A_u^+/\bar{P}_u \geq$ rank $A_u^+/\bar{P}_{u'}$. As the quotient map $\theta : A_u^+ \to A_u^+/\bar{P}_u$ factors through $A_u^+/\bar{P}_{u'}$ this implies that $\theta$ is an isomorphism as f.g. free abelian groups are hopfian. Thus $\bar{P}_u \subset \bar{P}_v$.

If $\mathbb{B}$ is not compatible, we can use edge slides to assure that $\mathbb{B}$ satisfies the conclusion of Proposition 3.4, by Lemma 3.15 the slides do not change $\mathrm{Mod}_{\mathbb{B}}(L)$. We will further modify $\mathbb{B}$ by performing the modifcations discussed following the proof of Proposition 3.4 to produce a compatible splitting. This modification possibly increases but does not decrease $\mathrm{Mod}_{\mathbb{B}}(L)$. Note that the Dehn twists along the edges that are being collapsed now occur as natural extensions of almost abelian vertex groups. Thus the claim follows from the case of $\mathbb{B}$ being compatible. $\square$

**Corollary 3.17.** *Let $L$ be a one-ended $\Gamma$-limit group and $\mathbb{A}$, $\mathbb{A}'$ almost abelian JSJ-decompositions of $L$. Then $Mod_{\mathbb{A}}(L) = Mod_{\mathbb{A}'}(L)$. In particular we can define*

$$Mod(L) := Mod_{\mathbb{A}}(L)$$

*where $\mathbb{A}$ is an arbitrary almost abelian JSJ-decomposition of $L$.*

# Chapter 4

# $\Gamma$-factor sets of one-ended groups

The term of *factor sets* was coined in [BF1] in the context of free groups. In this chapter we define $\Gamma$-factor sets, which are an obvious generalization of factor sets for an arbitrary group $\Gamma$. The goal of this chapter is then the proof of Theorem 4.2, which states that if $\Gamma$ is hyperbolic, then any finitely generated one-ended group $G$ admits a $\Gamma$-factor set. This is the main step in the construction of the Makanin-Razborov diagrams.

## 4.1 Factor sets

**Definition 4.1.** Let $G$ and $\Gamma$ be groups and $H \leq \mathrm{Aut}(G)$. A $\Gamma$-factor set of $G$ relative $H$ is a finite set of proper quotient maps $\{q_i : G \to \Gamma_i\}$ such that for each non-injective homomorphism $q : G \to \Gamma$, there exists some $\alpha \in H$ such that $q \circ \alpha$ factors through some $q_i$.

The following is the main theorem of chapter 4.

**Theorem 4.2.** *Let $G$ be a f.g. one-ended group and $\Gamma$ be a hyperbolic group. Then the following hold.*

1. *If $G$ is not fully residually $\Gamma$ then $G$ has a $\Gamma$-factor set relative $\{id\}$.*

2. *If $G$ is fully residually $\Gamma$ then $G$ has a $\Gamma$-factor set relative Mod$(G)$.*

The proof of the first part of Theorem 4.2 is trivial. Indeed, if $G$ is not fully residually $\Gamma$, then there is a finite set $S \subset G\setminus\{1\}$ such that for any $\varphi \in \mathrm{Hom}(G,\Gamma)$, $S \cap \ker\,\varphi \neq \emptyset$. Thus the set of quotient maps

$$\{q_s : G \to G/\langle\langle s\rangle\rangle \mid s \in S\}$$

is a $\Gamma$-factor set relative $\{id\}$.

If in turn $G$ is fully residually $\Gamma$, recall from Lemma 1.1 that $G$ is a $\Gamma$-limit group, thus Mod$(G)$ is defined. It turns out to be crucial to allow for precomposition with modular automorphisms as it allows us to only consider short homomorphisms in the sense of Definition 4.3 below. For the remainder of chapter 4, we fix a f.g. one-ended group $G$ and a hyperbolic group $\Gamma$ with fixed finite generating sets $S_G$ and $S_\Gamma$ respectively.

**Definition 4.3.** A homomorphism $\varphi : G \to \Gamma$ is called short relative $H \leq \mathrm{Aut}(G)$, if for every $\alpha \in H$ and $g \in \Gamma$,

$$|\varphi| \leq |i_g \circ \varphi \circ \alpha|$$

(where $i_g$ denotes conjugation by $g$ and $|\cdot|$ is as in Defintion 1.3).

*Remark* 4.4. In particular, a short homomorphism satisfies (1.3) of Theorem 1.9. Thus every convergent sequence of pairwise distinct short homomorphisms from $G$ to $\Gamma$ yields a non-trivial limit action of $G$ on a real tree.

For the remainder of this chapter we will not always explicitly mention Mod$(G)$, i.e. *short* will always mean *short relative Mod*$(G)$ and *factor set* will mean *factor set relative Mod*$(G)$. It will always be obvious that the constructed automorphisms are indeed modular automorphisms.

The proof of the second claim of Theorem 4.2 is by contradiction, i.e. we assume that $G$ has no $\Gamma$-factor set. This assumption implies the following.

**Lemma 4.5.** *Suppose that $G$ is fully residually $\Gamma$ and that $G$ has no $\Gamma$-factor set. Then there exists a stable convergent sequence $(\varphi_i) \subset Hom(G,\Gamma)$ of pairwise distinct non-injective short homomorphisms such that $\underrightarrow{\ker}(\varphi_i) = 1$.*

*Proof.* For $i \in \mathbb{N}$, let $B_i \subset G$ be the ball of radius $i$ with center 1 in $G$ (with respect to the word metric). For each $i$, there is a non-injective $\varphi_i \in \mathrm{Hom}(G,\Gamma)$ such that $B_i \cap \ker\, \varphi_i = \{1\}$, as otherwise the set of quotient maps

$$\{q_g : G \to G/\langle\langle g \rangle\rangle \mid g \in B_i,\ g \neq 1\}$$

would be a $\Gamma$-factor set of $G$. Moreover, since the definition of the factor set allows precomposition by a modular automorphism of $G$, each $\varphi_i$ can be chosen to be short.

Clearly, the sequence $(\varphi_i)$ of short homomorphisms is stable and $\underrightarrow{\ker}(\varphi_i) = 1$. Since each $\varphi_i$ is non-injective, it occurs only finitely many times in the sequence. Thus $(\varphi_i)$ has a convergent subsequence of pairwise distinct non-injective homomorphisms, see Lemma 1.4. $\square$

In the remainder of this chapter we prove the following proposition, which yields a contradiction to the conclusion of Lemma 4.5 and therefore implies Theorem 4.2.

**Proposition 4.6.** *Let $(\varphi_i)$ be a stable convergent sequence of pairwise distinct homomorphisms from $G$ to $\Gamma$. If $\underrightarrow{\ker}(\varphi_i) = 1$, then the $\varphi_i$ are eventually not short.*

*Remark* 4.7. Note that if we take a stable sequence $(\varphi_i) \subset \mathrm{Hom}(G,\Gamma)$ such that $\underrightarrow{\ker}(\varphi_i) = 1$ then we can associate to $(\varphi_i)$ a sequence $(\hat{\varphi}_i)$ such that each $\hat{\varphi}_i$ is short and equivalent to $\varphi_i$. After passing to a subsequence we can again assume that $(\hat{\varphi}_i)$ is stable. Proposition 4.6 implies that $Q := G/\underrightarrow{\ker}(\hat{\varphi}_i)$ is a proper quotient of $G$. This is an instance of a *shortening quotient* discussed in section 7.1.

Section 4.2 is entirely devoted to the proof of Proposition 4.6.

## 4.2 The shortening argument

Let $(\varphi_i)$ be as in Proposition 4.6. By Theorem 1.4, $(\varphi_i)$ converges to a non-trivial action of $G$ on an $\mathbb{R}$-tree $T$ with base point $x_0$.

Since $\underrightarrow{\ker}(\varphi_i) = 1$ it follows that the action of $G = G/\underrightarrow{\ker}(\varphi_i)$ on $T$ satisfies the stability assertions of Theorem 1.11. This implies that $T$ satisfies the assumptions of Theorem 2.4. As $G$ is assumed to be one-ended and the stabilizers of unstable arcs are finite, $G$ does not split over the stabilizer of an unstable arc. It follows that $T$ splits as a graph of actions. We will use this decomposition of the action of $G$ on $T$ to show that for large $i$, the homomorphisms $\varphi_i$ are not short. More precisely we will construct modular automorphisms $\alpha_i \in \text{Mod}(G)$ such that $|\varphi_i \circ \alpha_i| < |\varphi_i|$ for large $i$.

Let $\mathcal{G} = \mathcal{G}(\mathbb{A})$ be the graph of actions decomposition of $T$ given by Theorem 2.4. We identify $G$ with $\pi_1(\mathbb{A}, v_0)$ and assume that the basepoint $x_0$ is given by $x_0 = (\bar{x}_0, \tilde{v}_0)$, where $\tilde{v}_0 = [A_{v_0}]$ is the base vertex of $\tilde{\mathbb{A}} = \widetilde{(\mathbb{A}, v_0)}$. Moreover, we denote the metric $d_{\mathcal{G}}$ of $\mathcal{G}$ simply by $d$. Note that we can modify the graph of actions such that the following hold for any $e, f \in EA$, see Remark 2.3.

1. If $\alpha(e) = v_0$ and $\bar{x}_0$ is $A_{v_0}$-equivalent to $p_e^\alpha$ then $\bar{x}_0 = p_e^\alpha$.
2. If $\alpha(e) = \alpha(f)$ and $p_e^\alpha$ is $A_{\alpha(e)}$-equivalent to $p_f^\alpha$ then $p_e^\alpha = p_f^\alpha$.

In the construction of the shortening automorphisms we need to deal with each of the different types of vertex trees of $\mathcal{G}$, we will do this in the following three sections before plugging it all together to conclude. The shortening argument first appeared in [RS] but the underlying ideas are also implicit in the work of Razborov. In order to deal with torsion a number of additional issues need to be addressed.

### 4.2.1 Axial components

The purpose of this section is to prove the following.

**Theorem 4.8.** *Let $v_1 \in VA$ be an axial vertex. For any finite subset $S \subset G$ there exists some $\phi \in Mod(G)$ such that for any $g \in S$ the following hold.*

- *If $[x_0, gx_0]$ has a nondegenerate intersection with a vertex space $T_{\tilde{v}}$ and $\downarrow \tilde{v} = v_1$, then*
$$d(x_0, \phi(g)x_0) < d(x_0, gx_0),$$
- *otherwise,*
$$d(x_0, \phi(g)x_0) = d(x_0, gx_0).$$

The main step is to construct an automorphism of the axial vertex group that shortens the action on its vertex tree and that can be extended to an automorphism of $G$. We start by studying the algebraic structure of axial groups.

Let $G_A = A_{v_1}$ be a vertex group with an axial action on a tree $T_A = T_{v_1}$. We assume that the group $G_A$ does not preserve the ends of $T_A$, i.e. that $G_A$ contains elements that act by reflections. We denote the index 2 subgroup of $G_A$ which preserves the ends by $G_A^+$. The case where $G_A$ preserves the ends follows by considering only $G_A^+$. Note that $G_A^+$ is finite-by-abelian by Theorem 1.11.

Let $E := \langle \{g \in G_A^+ \mid |g| < \infty\} \rangle$. $E$ is normal in $G_A$ of order at most $N(\Gamma)$ by Lemma 1.16. Put $H := G_A/E$ and denote by $\pi$ the corresponding quotient map

$$\pi : G_A \to H = G_A/E.$$

As $E$ lies in the kernel of the action of $G_A$ on $T_A$ there is a natural action of $H$ on $T_A$. Let $H^+ = \pi(G_A^+)$. As $G_A^+$ is finite-by-abelian and any finite subgroup of $G_A^+$ lies in $E$, $H^+$ is abelian. Moreover, $H^+$ is torsion-free as for any $g \in G_A^+$ s.t. $g^k \in E$ for some $k \geq 1$ we have $g^{k \cdot |E|} = 1$, and so $g \in E$ by construction.

Since the image of $G_A^+$ (and therefore also the image of $H$) in $\mathrm{Isom}(T_A)$ is f.g. by Theorem 2.4, there is a decomposition

$$H^+ = A \oplus B$$

where $A$ is f.g. free abelian and $B$ is the torsion-free abelian kernel of the action of $H^+$ on $T_A$. Put $\tilde{A} = \pi^{-1}(A)$ and $\tilde{B} = \pi^{-1}(B)$.

**Lemma 4.9.** *$H$ is the semi-direct product $\mathbb{Z}_2 \ltimes H^+$. The action of $\mathbb{Z}_2 = \langle s | s^2 \rangle$ on $H^+$ is given by $shs^{-1} = h^{-1}$ for all $h \in H^+$.*

*Proof.* Let $s$ be an arbitrary element of $H \setminus H^+$ and let $\tilde{s}$ be a lift of $s$ to $G_A$, clearly $\tilde{s} \in G_A \setminus G_A^+$. It follows as in the proof of Lemma 1.16 that for large $i$ the element $\varphi_i(\tilde{s})$ is of finite order as it either lies in a finite group or in a 2-ended group exchanging the ends. Thus $\tilde{s}$ is of finite order, i.e. $\tilde{s}^2 \in E$. It follows that $s^2 = \pi(\tilde{s}^2) = 1$. This proves that $H = \mathbb{Z}_2 \ltimes H^+$.

We show that the action is as desired. Let $h \in H^+$. Choose $\tilde{h}$ such that $\pi(\tilde{h}) = h$. For large $i$ the group $\varphi_i(\langle \tilde{h}, \tilde{s} \rangle)$ is 2-ended with $\varphi_i(\tilde{s})$ exchanging ends and $\tilde{h}$ preserving ends. Here it is easily verified that $\varphi_i(\tilde{h}\tilde{s}\tilde{h}\tilde{s}^{-1})$ is of finite order for large $i$; thus $\tilde{h}\tilde{s}\tilde{h}\tilde{s}^{-1} \in E$, i.e. $hshs^{-1} = 1$. It follows that $shs^{-1} = h^{-1}$ as desired. □

As $\tilde{B}$ is the kernel of the action of $G_A$ on $T_A$ it follows that $\tilde{A}$ normalises $\tilde{B}$, i.e. $\tilde{A}$ acts on $\tilde{B}$ by conjugation.

**Lemma 4.10.** *The kernel $K_{\tilde{A}}^{\tilde{B}}$ of the action of $\tilde{A}$ on $\tilde{B}$ is of finite index in $\tilde{A}$.*

*Proof.* Note that for each $\tilde{b} \in \tilde{B}$ the group $\langle \tilde{b}, E \rangle$ is $E$-by-$\mathbb{Z}$. There are only finitely many isomorphism types of such groups all of which have finite automorphism groups. In particular there exists $N$ such that $\mathrm{Aut}(\langle \tilde{b}, E \rangle)$ is of order at most $N$ for all $\tilde{b} \in \tilde{B}$. Now $\tilde{A}$ acts on $\langle \tilde{b}, E \rangle$ with kernel of index at most $N$. It follows that the intersection of all subgroups of index at most $N$ of $\tilde{A}$ acts trivially on $\tilde{B}$. As $\tilde{A}$ is finitely generated there are only finitely many such subgroups; thus this intersection is also of finite index. Thus the kernel of the action of $\tilde{A}$ on $\tilde{B}$ contains a subgroup of finite index, the claim is immediate. $\square$

Let now $\tilde{s}$ be a lift of $s$ to $G_A$ and $\mathrm{Aut}_{\tilde{s}}(G_A) \leq \mathrm{Aut}(G_A)$ be the subgroup consisting of those automorphisms that restrict to the identity on $\langle \tilde{B}, \tilde{s} \rangle$ and preserve $\tilde{A}$. Further, let $\mathrm{Aut}^*_{\tilde{s}}(G_A) \leq \mathrm{Aut}_{\tilde{s}}(G_A)$ be the subgroup consisting of those automorphisms $\alpha$ that conjugate all point stabilizers pointwise. Thus for any $\alpha \in \mathrm{Aut}^*_{\tilde{s}}(G_A)$ and $x \in T_A$ there exists some $g_x$ such that $\alpha(g) = g_x g g_x^{-1}$ for all $g \in \mathrm{stab}(x)$.

**Lemma 4.11.** *$Aut^*_{\tilde{s}}(G_A)$ is a finite index subgroup of $Aut_{\tilde{s}}(G_A)$.*

*Proof.* Note first that for any $x \in T_A$ either $\mathrm{stab}(x) = \tilde{B}$ or $\mathrm{stab}(x) = \langle \tilde{B}, w\tilde{s} \rangle$ for some $w \in \tilde{A}$. As all elements of $\mathrm{Aut}_{\tilde{s}}(G_A)$ act trivially on $\tilde{B}$ by definition we can ignore the first case.

Note further that there are only finitely many conjugacy classes of stabilizers of type $\langle \tilde{B}, w\tilde{s} \rangle$; this follows from the fact that if $A$ is free abelian of rank $n$ then $\langle A, s \rangle$ has precisely $2^n$ conjugacy classes of reflections. As $\mathrm{Aut}_{\tilde{s}}(G_A)$ acts on conjugacy classes of stabilizers this implies that there exists a finite index subgroup $\mathrm{Aut}'_{\tilde{s}}(G_A)$ of $\mathrm{Aut}_{\tilde{s}}(G_A)$ that preserves conjugacy classes of stabilizers.

For any point stabilizer $C = \langle \tilde{B}, w\tilde{s} \rangle$ let $\mathrm{Aut}^C_{\tilde{s}}(G_A) \leq \mathrm{Aut}'_{\tilde{s}}(G_A)$ be the subgroup consisting of those automorphisms $\alpha$ that conjugate $C$, i.e. for which $\alpha(c) = gcg^{-1}$ for all $c \in C$ and some fixed $g$. Note that as $\alpha(gcg^{-1}) = \alpha(g)\alpha(c)\alpha(g)^{-1}$ it follows that $\alpha \in \mathrm{Aut}^C_{\tilde{s}}(G_A)$ if and only if $\alpha$ conjugates some conjugate of $C$.

To prove the claim of the lemma it suffices to show that for any such $C$ the group $\mathrm{Aut}^C_{\tilde{s}}(G_A)$ is of finite index in $\mathrm{Aut}'_{\tilde{s}}(G_A)$. Indeed as there are only finitely many conjugacy classes and the intersection of finitely many subgroups of finite index is of finite index, this proves the claim.

Let now $C = \langle \tilde{B}, w\tilde{s} \rangle$. Suppose that there exists a sequence $(\alpha_i)_{i \in \mathbb{N}} \subset \mathrm{Aut}'_{\tilde{s}}(G_A)$ such that $\alpha_i \mathrm{Aut}^C_{\tilde{s}}(G_A) \neq \alpha_j \mathrm{Aut}^C_{\tilde{s}}(G_A)$ for $i \neq j$. For each $i$ choose $f_i \in \tilde{A}$ and $e_i \in E$ such that

$$\alpha_i(w\tilde{s}) = f_i w\tilde{s}e_i f_i^{-1}.$$

Such elements $f_i$ and $e_i$ exist as by assumption $\alpha_i(w) \in \tilde{A}$ and $C$ is conjugate to $\alpha_i(C)$. After passing to a subsequence we can assume that $e_i = e$ for all $i \in \mathbb{N}$ and some fixed $e \in E$ and that $f_i K_{\tilde{A}}^{\tilde{B}} = f_j K_{\tilde{A}}^{\tilde{B}}$, i.e. that $f_j f_i^{-1} \in K_{\tilde{A}}^{\tilde{B}}$ for all $i, j \in \mathbb{N}$. The second claim follows from Lemma 4.10.

It follows that for all $i, j$ we have

$$\alpha_j(w\tilde{s}) = (f_j f_i^{-1})\alpha_i(w\tilde{s})(f_j f_i^{-1})^{-1}$$

which implies that the restriction of $\alpha_j \circ \alpha_i^{-1}$ to $\alpha_i(C)$ is conjugation by $f_j f_i^{-1}$. As $\alpha_i(C)$ is conjugate to $C$ this implies that $\alpha_j \circ \alpha_i^{-1} \in \mathrm{Aut}_{\tilde{s}}^{C}(G_A)$, a contradiction. □

Any $\alpha \in \mathrm{Aut}_{\tilde{s}}(G_A)$ restricts to an automorphism of $\tilde{A}$ and therefore induces an automorphism of $A = \tilde{A}/E$. Denote the subgroup of $\mathrm{Aut}(A)$ induced in this fashion by $K_{\tilde{s}}$. Moreover let $K_{\tilde{s}}^*$ be the subgroup of $\mathrm{Aut}(A)$ induced by $\mathrm{Aut}_{\tilde{s}}^*(G_A)$.

**Lemma 4.12.** *Let $\tilde{s}$ be as above. Then $K_{\tilde{s}}$ is of finite index in Aut($A$).*

*Proof.* Suppose that $A$ is free abelian of rank $n$ and let $a_1, \ldots, a_n$ be a basis of $A$. The proof is by contradiction, thus we assume that $K_{\tilde{s}}$ is of infinite index in $\mathrm{Aut}(A)$. Choose a sequence $(\alpha_i)$ of elements of $\mathrm{Aut}(A)$ that represent pairwise distinct cosets of $K_{\tilde{s}}$, i.e. that $\alpha_i \circ \alpha_j^{-1} \notin K_{\tilde{s}}$ for all $i \neq j$. For each $i \in \mathbb{N}$ let $P_i = (x_1^i, \ldots, x_n^i)$ where $x_k^i$ is a lift of $\alpha_i(a_k) \in A$ to $\tilde{A}$ for $1 \leq k \leq n$.

After passing to a subsequence we can assume that for all $i, j \in \mathbb{N}$ and $1 \leq k, l \leq n$ the following hold:

1. $[x_k^i, x_l^i] = [x_k^j, x_l^j]$.
2. The actions of $x_k^i$ and $x_k^j$ on $\tilde{B}$ coincide.
3. $\tilde{s}x_k^i\tilde{s}^{-1}x_k^i = \tilde{s}x_k^j\tilde{s}^{-1}x_k^j$.

This however implies that for $i, j$ the map $x_k^j \mapsto x_k^i$ for $1 \le k \le n$ extends to an automorphism $\alpha \in \mathrm{Aut}_{\tilde{s}}(G_A)$. Now this automorphism induces $\alpha_i \circ \alpha_j^{-1}$ on $A$, contradicting our assumption that $\alpha_i \circ \alpha_j^{-1} \notin K_{\tilde{s}}$. □

As an immediate consequence of Lemma 4.11 and Lemma 4.12 we get the following.

**Corollary 4.13.** *Let $\tilde{s}$ be as above. Then $K_{\tilde{s}}^*$ is of finite index in Aut($A$).*

The following proposition is the main technical result of this section.

**Proposition 4.14.** *Let $G_A$ be as above and $x, x_1, \dots, x_k \in T_A$. For each finite $S \subset G_A$ and $\epsilon > 0$, there exist elements $\gamma_1, \dots, \gamma_k \in G_A$ and an automorphism $\sigma$ of $G_A$ such that the following hold.*

1. *For each $g \in S$,*
$$d(x, \sigma(g)x) < \epsilon. \tag{4.1}$$
2. *$\sigma(g) = \gamma_i g \gamma_i^{-1}$ for $1 \le i \le k$ and $g \in \mathrm{stab}(x_i)$.*
3. *$d(x, \gamma_i x_i) < \epsilon$ ($i = 1, \dots, k$).*

*We can moreover assume that $\gamma_i = \gamma_j$ if $x_i = x_j$ and that $\gamma_i x_i = x_i$ if $x_i = x$.*

*Proof.* Possibly after choosing a different reflection $s$ we can assume that a lift $\tilde{s}$ of $s$ fixes a point $p_{\tilde{s}}$ such that $d(x, p_{\tilde{s}}) \le \epsilon/4$.

Let $a_1, \dots, a_n$ be a basis of the free abelian group $A$; recall that $A$ acts on $T_A$ by translations with dense orbits. The Euclidean algorithm guarantees the existence of a sequence $(\alpha_i) \subset \mathrm{Aut}(A)$ such that the translation length of $\alpha_i(a_k)$ $(1 \le k \le n)$ converges to 0 for $i \to \infty$. This implies that the translation lengths of $\alpha_i(a)$ converge to 0 for any $a \in A$. As $|\mathrm{Aut}(A) : K_{\tilde{s}}^*| < \infty$ we can choose $(\alpha_i) \subset K_{\tilde{s}}^*$. For any $i \in \mathbb{N}$ let $\tilde{\alpha}_i$ be a lift of $\alpha_i$ to $\mathrm{Aut}_{\tilde{s}}^*(G_A)$.

Now any element $g \in S$ can be written as $\tilde{a}_g \tilde{b}_g \tilde{s}^{\eta_g}$ where $\tilde{a}_g \in \tilde{A}$, $\tilde{b}_g \in \tilde{B}$ and $\eta_g \in \{0, 1\}$. As $\tilde{\alpha}_i \in \mathrm{Aut}_{\tilde{s}}(G_A)$ it follows that $\tilde{\alpha}_i(g) = \tilde{\alpha}_i(\tilde{a}_g)\tilde{b}_g \tilde{s}^{\tilde{\eta}_g}$ for all $i \in \mathbb{N}$ and $g \in S$. Moreover as $\tilde{\alpha}_i(\tilde{a}_g)$ is a lift of $\alpha_i(\pi(\tilde{a}_g))$ it follows that the translation length of $\tilde{\alpha}_i(\tilde{a}_g)$ converges to 0. Thus we get

$$\lim_{i \to \infty} d(x, \tilde{\alpha}_i(g)x) = \lim_{i \to \infty} d(x, \tilde{\alpha}_i(\tilde{a}_g)\tilde{b}_g \tilde{s}^{\eta_g} x) \leq$$
$$\lim_{i \to \infty} d(x, \tilde{s}^{\eta_g} x) + \lim_{i \to \infty} d(\tilde{s}^{\eta_g} x, \tilde{b}_g \tilde{s}^{\eta_g} x) + \lim_{i \to \infty} d(\tilde{b}_g \tilde{s}^{\eta_g} x, \tilde{\alpha}_i(\tilde{a}_g)\tilde{b}_g \tilde{s}^{\eta_g} x)$$
$$\leq \epsilon/2 + 0 + 0 = \epsilon/2$$

for all $g \in S$. This implies that for sufficiently large $i$ assertions (1) and (2) are satisfied for $\sigma = \tilde{\alpha}_i$.

If $\mathrm{stab}(x_i) = \tilde{B}$ then $\gamma_i$ can be replaced by $\gamma_i h$ with $h \in K_{\tilde{A}}^{\tilde{B}}$ while preserving (2). As $K_{\tilde{A}}^{\tilde{B}}$ acts on $T_A$ with dense orbits this ensures the existence of some $\gamma_i$ such that both (2) and (3) are satisfied.

If $\mathrm{stab}(x_i)$ is of type $\langle \tilde{B}, \tilde{a}\tilde{s} \rangle$ for some $\tilde{a} \in \tilde{A}$ then the fixed point of $\tilde{\alpha}_i(\mathrm{stab}(x_i)) = \langle \tilde{B}, \tilde{\alpha}_i(\tilde{a})\tilde{s} \rangle$ converges to $p_{\tilde{s}}$ as the translation length of $\tilde{\alpha}_i(\tilde{a})$ converges to 0. As this fixed point equals $\gamma_i x_i$ and as $d(x, p_{\tilde{s}}) \leq \epsilon/4$ assertion (3) follows for large $i$.

To conclude note first that it is trivial that we can choose $\gamma_i = \gamma_j$ if $x_i = x_j$. Moreover if $x = x_i$ then either $\mathrm{stab}(x) = \tilde{B}$ or we can choose $\tilde{s}$ such that $\mathrm{stab}(x) = \langle \tilde{B}, \tilde{s} \rangle$. As in both cases the $\tilde{\alpha}_i$ restrict to the identity on $\mathrm{stab}(x) = \mathrm{stab}(x_i)$ we can choose $\gamma_i = 1$ and the claim follows. □

*Proof of Theorem 4.8.* For any $g \in S$ and $\tilde{v}$ with $\downarrow \tilde{v} = v_1$ the intersection $T_{\tilde{v}} \cap [x_0, gx_0]$ is either empty or a (possibly degenerate) segment. If all such intersections are degenerate for all $g \in S$ then the conclusion of the theorem holds for $\phi = \mathrm{id}_G$. Thus we can assume that at least one such intersection is non-degenerate. Let $r > 0$ be the length of the shortest non-degenerate segment that occurs this way.

Recall from (2.4) that if $q$ is a normal form $\mathbb{A}$-path

$$q = a_0, e_1, a_1, e_2, \ldots, e_k, a_k \tag{4.2}$$

and $g = [q]$ then

$$d(x_0, gx_0) = d_{v_0}(\bar{x}_0, a_0 p^{\alpha}_{e_1}) + \sum_{j=1}^{k-1} d_{\omega(e_j)}(p^{\omega}_{e_j}, a_j p^{\alpha}_{e_{j+1}}) + d_{v_0}(p^{\omega}_{e_k}, a_k \bar{x}_0). \tag{4.3}$$

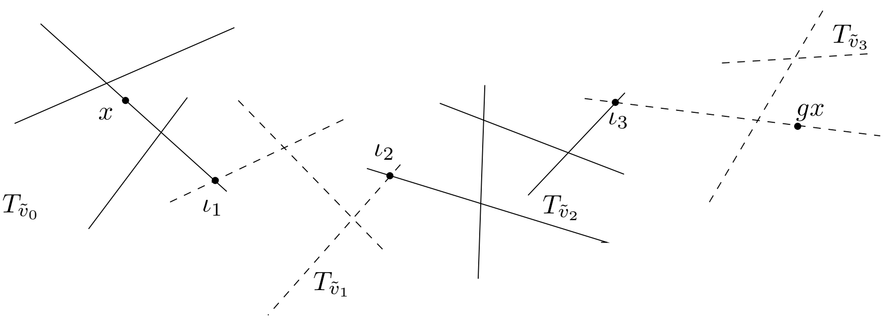



Figure 4.1: The path for an element $g = [a_0, e_1, \ldots, e_3, a_3]$

Choose a point $p \in T_{v_1}$, if $v_0 = v_1$ then we choose $p := \bar{x}_0$. Let $S_{v_1}$ be set of elements of $A_{v_1}$ that occur in the normal forms of the elements in $S$. By Proposition 4.14, there is an automorphism $\sigma$ of $A_{v_1}$ and for each $e \in EA$ with $\alpha(e) = v_1$ an element $\gamma_e \in A_{v_1}$ such that the following hold.

- $d_{v_1}(p, \sigma(a)p) < \frac{r}{6}$ for all $a \in S_{v_1}$.
- For $e \in EA$ with $\alpha(e) = v_1$, the restriction of $\sigma$ to $\alpha_e(A_e) \leq \mathrm{stab}(p^{\alpha}_e)$ is conjugation by $\gamma_e$, and
$$d_{v_1}(p, \gamma_e p^{\alpha}_e) < \frac{r}{6}. \tag{4.4}$$
- $\gamma_e = \gamma_f$ if $p^{\alpha}_e = p^{\alpha}_f$ and $\gamma_e p^{\alpha}_e = p^{\alpha}_e$ if $p^{\alpha}_e = p$.

Fix such an automorphism $\sigma$ and let $\phi \in \text{Aut}(G)$ be a natural extension of $\sigma$ (cf. Definition 3.13). Thus if $g = [a_0, e_1, \ldots, e_k, a_k]$ as before we get

$$\phi(g) = [\bar{a}_0, e_1, \bar{a}_1, e_2, \ldots, e_k, \bar{a}_k]$$

where $\bar{a}_i = \gamma_{e_i^{-1}}^{-1} \sigma(a_i) \gamma_{e_{i+1}}$ if $a_i \in A_{v_1}$ (and $\gamma_{e_0^{-1}} = \gamma_{e_{k+1}} = 1$) and $\bar{a}_i = a_i$ otherwise. In particular we have

$$d(x_0, \phi(g)x_0) = d_{v_0}(\bar{x}_0, \bar{a}_0 p_{e_1}^{\alpha}) + \sum_{j=1}^{k-1} d_{\omega(e_j)}(p_{e_j}^{\omega}, \bar{a}_j p_{e_{j+1}}^{\alpha}) + d_{v_0}(p_{e_k}^{\omega}, \bar{a}_k \bar{x}_0). \tag{4.5}$$

In the following we compare the summands occuring in (4.3) to those occuring in (4.5). If $a_i = \bar{a}_i$ the corresponding summands clearly coincide. Thus we can assume that $a_i \neq \bar{a}_i \in A_{v_0}$. We distinguish two cases.

**Case 1:** If $i \in \{1, \ldots, k-1\}$ then we get

$$\begin{aligned} d_{\omega(e_i)}(p_{e_i}^{\omega}, \bar{a}_i p_{e_{i+1}}^{\alpha}) = \\ = d_{\omega(e_i)}(p_{e_i}^{\omega}, \gamma_{e_i^{-1}}^{-1} \sigma(a_i) \gamma_{e_{i+1}} p_{e_{i+1}}^{\alpha}) = d_{\omega(e_i)}(\gamma_{e_i^{-1}} p_{e_i}^{\omega}, \sigma(a_i) \gamma_{e_{i+1}} p_{e_{i+1}}^{\alpha}) \\ \leq d_{\omega(e_i)}(\gamma_{e_i^{-1}} p_{e_i}^{\omega}, p) + d_{\omega(e_i)}(p, \sigma(a_i)p) + d_{\omega(e_i)}(\sigma(a_i)p, \sigma(a_i)\gamma_{e_{i+1}} p_{e_{i+1}}^{\alpha}) \\ \leq \frac{r}{6} + \frac{r}{6} + d_{\omega(e_i)}(p, \gamma_{e_{i+1}} p_{e_{i+1}}^{\alpha}) \leq \frac{2r}{6} + \frac{r}{6} = \frac{r}{2}. \end{aligned}$$

If moreover $d_{\omega(e_i)}(p_{e_i}^{\omega}, a_i p_{e_{i+1}}^{\alpha}) = 0$ then $p_{e_i}^{\omega}$ and $p_{e_{i+1}}^{\alpha}$ are $A_{v_1}$-equivalent and therefore $p_{e_i}^{\omega} = p_{e_{i+1}}^{\alpha}$ by assumption, thus $a_i \in \text{stab}(p_{e_i}^{\omega})$ and $\gamma_{e_i^{-1}} = \gamma_{e_{i+1}}$, by assumption this implies that $\sigma(a_i) = \gamma_{e_{i+1}} a_i \gamma_{e_{i+1}}^{-1}$. The above computation therefore implies that

$$d_{\omega(e_i)}(p_{e_i}^{\omega}, \bar{a}_i p_{e_{i+1}}^{\alpha}) = d_{\omega(e_i)}(p_{e_i}^{\omega}, p_{e_{i+1}}^{\alpha}) = 0.$$

**Case 2:** If $i = 0$ (the case $i = k$ is analogous) then we get

$$\begin{aligned} d_{v_0}(\bar{x}_0, \bar{a}_0 p_{e_1}^{\alpha}) = d_{v_0}(p, a_0 \gamma_{e_1} p_{e_1}^{\alpha}) \leq d_{v_0}(p, a_0 p) + d_{v_0}(a_0 p, a_0 \gamma_{e_1} p_{e_1}^{\alpha}) \leq \\ \leq \frac{r}{6} + d_{v_1}(p, \gamma_{e_1} p_{e_1}^{\alpha}) \leq \frac{r}{6} + \frac{r}{6} = \frac{r}{3}. \end{aligned}$$

If moreover $d_{v_1}(p, a_0 p^{\alpha}_{e_1}) = 0$ then $p$ and $p^{\alpha}_{e_1}$ are $A_{v_1}$ equivalent and therefore $p = p^{\alpha}_{e_1}$ by assumption. Thus $a_0 \in \mathrm{stab}(p)$ and therefore $\gamma_{e_1} \in \mathrm{stab}(p)$. Thus

$$d_{v_1}(p, \bar{a}_0 p^{\alpha}_{e_1}) = d_{v_1}(p, a_0 \gamma_{e_1} p) = d_{v_1}(p, p) = 0.$$

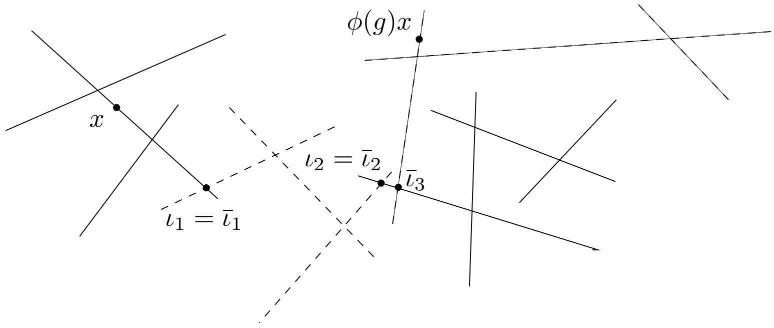



Figure 4.2: The path for $\phi(g)$ if precisely $\tilde{v}_2$ is of type $v_1$

Comparing the summands in (4.3) to those occuring in (4.5) shows that each summand is preserved unless it corresponds to a non-degenerate intersection with some vertex tree $T_{\tilde{v}}$ with $\downarrow \tilde{v} = v_1$ (of length at least $r$) in which case it is replaced by at most $\frac{r}{2}$. This proves the claim. □

### 4.2.2 Orbifold components

Analogous to Theorem 4.8, we prove the existence of a shortening automorphism provided there exists a vertex of orbifold type.

**Theorem 4.15.** *Let $v_1 \in VA$ be an orbifold type vertex. For any finite subset $S \subset G$ there exists some $\phi \in Mod(G)$ such that the following hold for any $g \in S$.*

- *If $[x_0, gx_0]$ has a nondegenerate intersection with a vertex space $T_{\tilde{v}_1}$ and $\downarrow \tilde{v}_1 = v_1$, then*

$$d(x_0, \phi(g)x_0) < d(x_0, gx_0),$$

- *otherwise,*

$$d(x_0, \phi(g)x_0) = d(x_0, gx_0).$$

The proof of Theorem 4.15 follows from the following proposition in exactly the same way as Theorem 4.8 follows from Proposition 4.14.

**Proposition 4.16.** *Let $v \in VA$ be an orbifold type vertex and $x, x_1, \ldots, x_k \in T_v$. For each finite $S \subset A_v$ and $\epsilon > 0$, there exist elements $\gamma_1, \ldots, \gamma_k \in A_v$ and an automorphism $\sigma \in Aut(A_v)$ such that the following hold.*

1. *For each $g \in S$,*
$$d(x, \sigma(g)x) < \epsilon. \tag{4.6}$$
2. *$\sigma(g) = \gamma_i g \gamma_i^{-1}$ for $1 \leq i \leq k$ and $g \in \mathrm{stab}(x_i)$.*
3. *$d(x, \gamma_i x_i) < \epsilon$ $(i = 1, \ldots, k)$.*

*We can moreover assume that $\gamma_i = \gamma_j$ if $x_i = x_j$ and that $\gamma_i x_i = x_i$ if $x_i = x$.*

The remainder of this section is decicated to the proof of Proposition 4.16. The argument in this case is essentially due to Rips and Sela [RS] who give a proof of Proposition 4.16 in the case where the action of $A_v$ on $T_v$ has trivial kernel. Thus we only need to address the case where this kernel is non-trivial.

If $\mathcal{H}$ is a family of subgroups of $G$ then we will denote by $\mathrm{Aut}_{\mathcal{H}}(G)$ the subgroup of $\mathrm{Aut}(G)$ consisting of those automorphisms that act on each $H \in \mathcal{H}$ by conjugation with an element of $G$.

**Lemma 4.17.** *Let $G$ be a f.p. group, $\mathcal{H} := \{H_1, \ldots, H_k\}$ a finite collection of cyclic, malnormal subgroups of $G$. Suppose that $\tilde{G}$ is an extension of some finite group $E$ by $G$, i.e. that there is the short exact sequence*

$$1 \longrightarrow E \longrightarrow \tilde{G} \xrightarrow{\pi} G \longrightarrow 1.$$

*Put* $\tilde{\mathcal{H}} := \{\tilde{H}_i := \pi^{-1}(H_i) | 1 \leq i \leq k\}$. *Let* $S$ *be the group of those automorphisms* $\sigma \in Aut_{\mathcal{H}}(G)$ *that lift to* $Aut_{\tilde{\mathcal{H}}}(\tilde{G})$, *i.e. for which there exists* $\tilde{\sigma} \in Aut_{\tilde{\mathcal{H}}}(\tilde{G})$ *such that* $\pi \circ \tilde{\sigma} = \sigma \circ \pi$.

*Then* $S$ *has finite index in* $Aut_{\mathcal{H}}(G)$.

*Proof.* The proof is in two steps. We first prove that the subgroup $S_1$ of $\mathrm{Aut}_{\mathcal{H}}(G)$ consisting of automorphisms that lift to automorphisms of $\tilde{G}$ is of finite index in $\mathrm{Aut}_{\mathcal{H}}(G)$. We then show that $S$ is a finite index subgroup of $S_1$.

Assume that $|\mathrm{Aut}_{\mathcal{H}}(G) : S_1| = \infty$. Then there exists a sequence $(\alpha_i) \subset \mathrm{Aut}(G)$ of automorphisms such that $\alpha_i S_1 \neq \alpha_j S_1$ for $i \neq j$. Let $\langle S_E | R_E \rangle$ be a presentation of $E$ and

$$\langle s_1, \ldots, s_m | r_1, \ldots, r_g \rangle$$

be a finite presentation of $G$. Every automorphism $\alpha_i$ of $G$ gives rise to a (not unique) presentation of $\tilde{G}$ as

$$\mathcal{P}_i(\tilde{G}) = \left\langle S_E, s_1, \ldots, s_m | R_E, r_1 e^1, \ldots r_h e^h, \{s_i e s_i^{-1} = f_{i,e}\} \right\rangle,$$

where the $e^i$ and $f_{i,e}$ lie in $E$ and the generators $s_i$ correspond to chosen lifts of the images of the generators of $G$ under $\alpha_i$. Since there are only finitely many such presentations, there are $i, j \in \mathbb{N}$ s.t. $i \neq j$ and $\mathcal{P}_i = \mathcal{P}_j$. It follows that $\alpha_i^{-1}\alpha_j \in S_1$ and therefore $\alpha_i S_1 = \alpha_j S_1$ contradicting the above assumption. Thus $|\mathrm{Aut}_{\mathcal{H}}(G) : S_1| < \infty$.

Let now $S_2$ be the subgroup of $\mathrm{Aut}(\tilde{G})$ consisting of lifts of automorphisms of $S_1$. It clearly suffices to show that $S_2 \cap \mathrm{Aut}_{\tilde{\mathcal{H}}}(\tilde{G})$ is of finite index in $S_2$.

Note that for any $\tilde{\alpha} \in S_2$ and $i = 1, \ldots, k$ we have $\alpha(\tilde{H}_i) = c_i \tilde{H}_i c_i^{-1}$ for some $c_i \in \tilde{G}$ (but the conjugation is not pointwise in general). Since $H_i$ is malnormal, $c_i$ is unique up to elements of $\tilde{H}_i$. It follows that $\alpha$ induces a well-defined outer automorphism $\sigma_i(\alpha)$ of $\tilde{H}_i$ represented by the automorphism

$$g \mapsto c_i^{-1}\alpha(g)c_i \qquad \text{for all } g \in \tilde{H}_i.$$

Now $S_2 \cap \mathrm{Aut}_{\tilde{\mathcal{H}}}(\tilde{G})$ is the kernel of the homomorphism

$$S_2 \to \prod_{i=1}^{k} \mathrm{Out}(\tilde{H}_i),\ \alpha \mapsto (\sigma_1(\alpha), \ldots, \sigma_k(\alpha)).$$

This proves the assertion as $\mathrm{Out}(H_i)$ is finite for all $i$. □

We can now proceed with the proof of Proposition 4.16.

*Proof of Proposition 4.16.* Denote by $E$ the kernel of the action of $A_v$ on $T_v$. Let $x$ be an arbitrary point of $T_v$. Since $T_v$ is not a line, $E$ stabilizes a tripod, and therefore is finite by Theorem 1.11. By Theorem 2.4, the quotient $P := A_v/E$, which acts faithfully on $T_v$, is the fundamental group of a compact 2-orbifold with boundary $\Sigma$. In particular, $P$ is finitely presented. Applying Proposition 5.2 of [RS] we get an infinite sequence $(\alpha_i)$ of automorphisms of $P$ satisfying the following.

- For each $g \in S$, $\lim\limits_{i\to\infty} d(x, \alpha_i(\pi(g))x) = 0$.
- For each (infinite cyclic) peripheral subgroup $Z$ of $P$ the restriction $\alpha_i|_Z$ is conjugation with an element $c_i^{(e)} \in P$.
- For any peripheral subgroup $Z$ of $P$ the distance between $x$ and the (unique) fixed point of $\alpha_i(Z)$ tends to 0.

Let $F \subset EA$ be the set of edges whose initial vertex is $v$. For each $e \in F$, the image $Z_e$ of the $\alpha_e(A_e)$ under $\pi$ in $P = \pi(A_v) = \pi(\Sigma)$ correspond to a loop in $\Sigma$ that is homotopic to a boundary component. This implies that $\pi(\alpha_e(A_e))$ is malnormal in $P$.

Define $S \le \mathrm{Aut}_{\mathcal{H}}(P)$ as in Lemma 4.17 with respect to the collection of subgroups $\mathcal{H} := \{Z_e \,|\, e \in F\}$ of $P$ and $\tilde{H} := \{\tilde{Z}_e := \pi^{-1}(Z_e) \,|\, e \in F\}$.

Then by Lemma 4.17, $|\mathrm{Aut}_{\mathcal{H}}(P) : S| < \infty$. It follows that there is a subsequence $(\alpha_{i_j}) \subset (\alpha_i)$ such that all $\alpha_{i_j}$ are in the same left coset $C$ of $S$. Fix a representative $\gamma \in C$. Then the sequence $(\alpha'_i)$ given by

$$\alpha'_j := \gamma^{-1}\alpha_{i_j}$$

is in $S$ and $\lim_{i\to\infty} |\alpha'_i| = 0$. Choosing $i$ large enough and extending $\alpha'_i$ to $A_v$, gives the desired automorphism. □

### 4.2.3 Simplicial components

Assume now that $\mathcal{G}$ has a nondegenerate simplicial vertex tree. Thus $\mathcal{G}$ can be refined in a simplicial type vertex yielding a (refined) graph of actions with non-zero length function $l$ such that all vertices that are adjacent to edges of non-zero length have degenerate vertex trees. We denote this graph of actions again by $\mathcal{G}$. Note that we can still assume that the base point $x_0$ is contained in a vertex tree $\tilde{v}_0 = [A_{v_0}]$, i.e. that $x_0 = [\bar{x}_0, \tilde{v}_0]$. Indeed, if $x_0$ is contained in the interior of an edge segment $T_{\tilde{e}}$, we can split the corresponding edge $\downarrow\tilde{e} \in EA$ by introducing a valence 2 vertex with vertex group $A_{\downarrow\tilde{e}}$ and degenerate vertex tree such that $x_0$ is precisely a lift of this vertex tree.

We will construct a Dehn twist automorphism on an edge with non-zero length such that powers of this Dehn twist shorten the action of $G$ on $X$ induced by $\varphi_i$ (for large enough $i$). Note that other than in the axial and orbifold cases these automorphisms do not shorten the action on the limit tree.

**Lemma 4.18.** *For an edge $e \in EA$ with non-zero length, the center $Z(A_e)$ of $A_e$ contains an element $c$ of infinite order. In particular $\varphi_i(c)$ is of infinite order and therefore hyperbolic for sufficiently large $i$.*

*Proof.* By Corollary 1.11, $A_e$ is finite-by-abelian. Let $E \lhd A_e$ be finite such that $A_e/E$ is abelian. $A_e$ acts on $E$ by conjugation. Let $H$ be the kernel of this action. $H$ has finite index in $A_e$, hence $H$ is infinite. Since $G$ is a $\Gamma$-limit group, it does not contain infinite torsion subgroups. Thus there is an element $g \in H$ of infinite order. For any $a \in A_e$, we have

$$gag^{-1} = ae$$

for some $e \in E$. Since $g$ centralizes $E$,

$$g^k a g^{-k} = g^{k-1}(gag^{-1})g^{-(k-1)} = g^{k-1} a g^{-(k-1)} e,$$

thus $g^k a g^{-k} = ae^k$. If $k = |E|$ then $e^k = 1$ for all $e \in E$; hence $c := g^k \in Z(A_e)$. □

The following proposition is the key observation needed for the construction of the shortening automorphisms. In the following $d_i := \frac{d_X}{|\varphi_i|}$ is the scaled metric on the Cayley graph $X$. Thus

$$\lim_{i\to\infty} d_i(\varphi_i(g), \varphi_i(h)) = d_\varphi(g, h)$$

for all $g, h \in G$. Recall that $(X, d_i)$ is $\delta_i$-hyperbolic with $\lim \delta_i = 0$. Also recall from (2.2) that for any $c \in A_e$ and lift $\tilde{e}$ of $e$ the element $\theta_{\tilde{e}}(c)$ is a natural lift of $c$ to the stabilizer of $\tilde{e}$.

**Proposition 4.19.** *Let $e \in EA$ be an edge with positive length and $c \in Z(A_e)$ of infinite order. There exists a sequence $(m_i) \subset \mathbb{Z}$ such that for any lift $\tilde{e}$ of $e^\varepsilon$, $\varepsilon \in \{-1, 1\}$, the following holds.*

*If $(y_i)$, $(z_i) \subset X$ are approximating sequences of $T_{\alpha(\tilde{e})}$ and $T_{\omega(\tilde{e})}$ (which are single points) respectively, then*

$$\lim_{i\to\infty} d_i(y_i, \varphi_i(\theta_{\tilde{e}}(c)^{\varepsilon\cdot m_i}) z_i) = 0. \tag{4.7}$$

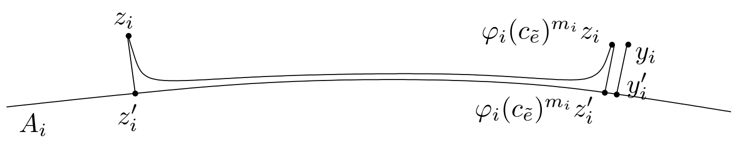
*Proof.* We assume that $\varepsilon = 1$, the case where $\varepsilon = -1$ is an immediate consequence due to the equivariance of the action. Fix some lift $\tilde{e}$ of $e$ and put $c_{\tilde{e}} := \theta_{\tilde{e}}(c)$. For large $i$ the element $\varphi_i(c_{\tilde{e}})$ is hyperbolic and we define $A_i$ to be the axis of $\varphi_i(c_{\tilde{e}})$ in $X$, i.e. the union of all geodesics joining the ends fixed by $\varphi_i(c_{\tilde{e}})$. $A_i$ is easily seen to be in the $4\delta_i$-neighbourhood of any of these geodesics with respect to the metric $d_i$.

For each $i$ let $y_i'$ and $z_i'$ be points on $A_i$ closest to $y_i$ and $z_i$ respectively. It is clear that $\lim_{i\to\infty} d_i(y_i, y_i') = 0$ and $\lim_{i\to\infty} d_i(z_i, z_i') = 0$ by (1.4) and the fact that $c_{\tilde{e}}$ fixes $T_{\tilde{v}_1}$ and $T_{\tilde{v}_2}$ in the limit action.

Moreover, there are integers $m_i$ such that

$$d(y_i', \varphi_i(c_{\tilde{e}}))^{m_i} z_i') \leq l(\varphi_i(c_{\tilde{e}})) + 8\delta_i$$

where $l(\varphi_i(c_{\tilde{e}}))$ denotes the translation length of $\varphi_i(c_{\tilde{e}})$. It follows in particular that $\lim_{i\to\infty} d_i(y_i', \varphi_i(c_{\tilde{e}})^{m_i} z_i') = 0$ and therefore also $\lim_{i\to\infty} d_i(y_i, \varphi_i(c_{\tilde{e}})^{m_i} z_i) = 0$.




To conclude it suffices to show that the choice of the $m_i$ does not depend on the choice of the lift of $e$. Indeed this follows immediately from the fact that if $\tilde{e}' = h\tilde{e}$ is another lift of $e$ then $(\varphi_i(h)y_i), (\varphi_i(h)z_i) \subset X$ are approximating sequences of $T_{\alpha(\tilde{e}')} = hT_{\alpha(\tilde{e})}$ and $T_{\omega(\tilde{e}')} = hT_{\omega(\tilde{e})}$ respectively, and $c_{\tilde{e}'} = hc_{\tilde{e}}h^{-1}$. $\square$

From now on let $e \in EA$ be a fixed edge with positive length. Let $c \in Z(A_e)$ be of infinite order and $(m_i)_{i\in\mathbb{N}}$ as in Proposition 4.19. Define $\sigma : G \to G$ to be the Dehn twist automorphism given by

$$[a_0, e_1, a_1, \ldots, e_n, a_n] \mapsto [\bar{a}_0, e_1, \bar{a}_1, \ldots, e_n, \bar{a}_n] \tag{4.8}$$

where

$$\bar{a}_k = \begin{cases} \omega_e(c^{-1})a_k & e_k = e, \\ \omega_e(c)a_k & e_k = e^{-1} \\ a_k & e_k \neq e^{\pm 1} \end{cases} .$$

**Proposition 4.20.** *Let $g = [q] \in G$. If $q$ is reduced and contains an edge $e^{\pm 1}$, then*

$$d_i(1, \varphi_i \circ \sigma^{m_i}(g)) < d_i(1, \varphi_i(g))$$

*for sufficiently large $i$. Otherwise, $d_i(1, \varphi_i \circ \sigma^{m_i}(g)) = d_i(1, \varphi_i(g))$.*

As the tree $T_{\mathcal{G}}$ is minimal it follows that for any generating set $S$ of $G$ the normal form of at least one element contains an edge $e^{\pm 1}$. Thus we obtain the following immediate corollary.

**Corollary 4.21.** *$|\varphi_i \circ \sigma^{m_i}| < |\varphi_i|$ for sufficiently large $i$.*

*Proof of Proposition 4.20.* Let $q = a_0, e_1, a_1, e_2, \ldots, a_n$ be a normal form $\mathbb{A}$-path s.th. $g = [q]$, and $\tilde{e}_1, \ldots, \tilde{e}_n$ be the reduced edge path in $\tilde{\mathbb{A}}$ from $\tilde{v}_0$ to $g\tilde{v}_0$. Then

$$\begin{aligned} d_{\mathcal{G}}(x_0, gx_0) &= d_{\tilde{v}_0}(\bar{x}_0, p^{\alpha}_{\tilde{e}_1}) \\ &+ \sum_{k=1}^{n-1} d_{\omega(\tilde{e}_k)}(p^{\omega}_{\tilde{e}_k}, p^{\alpha}_{\tilde{e}_{k+1}}) \\ &+ \sum_{k=1}^{n} d_{\tilde{e}_k}(p^{\alpha}_{\tilde{e}_k}, p^{\omega}_{\tilde{e}_k}) \\ &+ d_{g\tilde{v}_0}(p^{\omega}_{\tilde{e}_n}, g\bar{x}_0). \end{aligned}$$

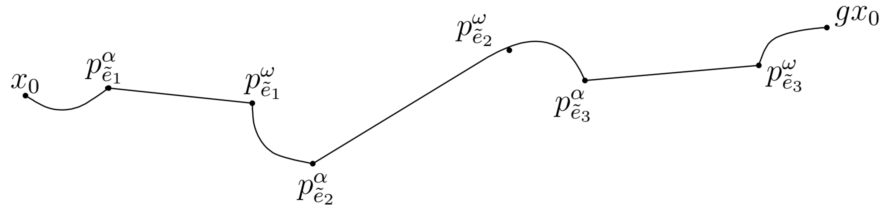



Figure 4.3: The segment $[x_0, gx_0] \subset T$ with $g = [a_0, e_1, \ldots, e_3, a_3]$

Now for each $k$, let $(p^{\alpha}_{k,i})_{i \in \mathbb{N}}$ and $(p^{\omega}_{k,i})_{i \in \mathbb{N}}$ be approximating sequences of $p^{\alpha}_{\tilde{e}_k}$ and $p^{\omega}_{\tilde{e}_k}$ respectively. Recall that (1) and $(\varphi_i(g))$ are approximating sequences of $x_0$ and $gx_0$

respectively. Thus by Lemma 1.7, for any $\epsilon > 0$ and large enough $i$, we have

$$\begin{aligned} d_X(1, \varphi_i(g)) &\geq d_i(1, p^\alpha_{1,i}) + \sum_{k=1}^{n-1} d_i(p^\omega_{k,i}, p^\alpha_{k+1,i}) \\ &+ \sum_{k=1}^{n} d_i(p^\alpha_{k,i}, p^\omega_{k,i}) + d_i(p^\omega_{n,i}, \varphi_i(g)) - \epsilon. \end{aligned} \tag{4.9}$$

For each $i \in \mathbb{N}$ we put $\bar{a}^i_0 := a_0$ and for $k \in \{1, \ldots, n\}$ put

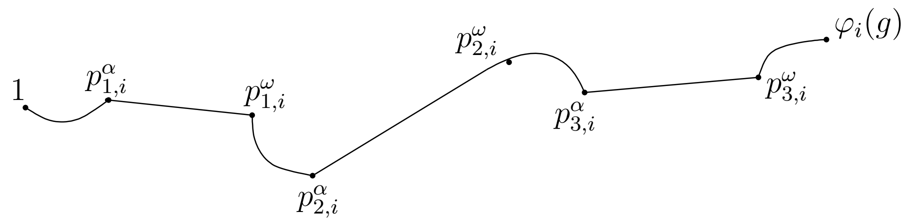



Figure 4.4: The segment $[1, \varphi_i(g)] \subset X$ with $g = [a_0, e_1, \ldots, e_3, a_3]$

$$\bar{a}^i_k = \begin{cases} \omega_e(c^{-m_i})a_k & \text{if } e_k = e \\ \omega_e(c^{m_i})a_k & \text{if } e_k = e^{-1} \\ a_k & \text{if } e_k \neq e^{\pm 1} \end{cases}.$$

Moreover for $0 \leq k \leq n$ and $i \in \mathbb{N}$ we define

$$q_k := a_0, e_1, a_1, \ldots, a_k,$$

$$\bar{q}^i_k := \bar{a}^i_0, e_1, \bar{a}^i_1, \ldots, \bar{a}^i_k.$$

Note that this implies that

$$[\bar{q}^i_n] = [\bar{a}^i_0, e_1, \bar{a}^i_1, \ldots, e_n, \bar{a}^i_n] = \sigma^{m_i}(g).$$

Further, for each $k$ and $i$, put

$$\bar{p}^\alpha_{k,i} := \varphi_i([\bar{q}^i_{k-1} q^{-1}_{k-1}]) p^\alpha_{k,i}$$

and

$$\bar{p}^{\omega}_{k,i} := \varphi_i([\bar{q}^i_k q_k^{-1}]) p^{\omega}_{k,i}.$$

Note that

$$\varphi_i([\bar{q}^i_n q_n^{-1}]) \cdot \varphi_i(g) = \varphi_i(\sigma^{m_i}(g) \cdot g^{-1}) \cdot \varphi_i(g) = \varphi_i \circ \sigma_i^{m_i}(g).$$

Using the triangle inequality, this implies that for large $i$ we get

$$\begin{aligned} d_i(1, \varphi_i \circ \sigma^{m_i}(g)) &\leq d_i(1, \bar{p}^{\alpha}_{1,i}) \\ &+ \sum_{k=1}^{n-1} d_i(\bar{p}^{\omega}_{k,i}, \bar{p}^{\alpha}_{k+1,i}) \\ &+ \sum_{k=1}^{n} d_i(\bar{p}^{\alpha}_{k,i}, \bar{p}^{\omega}_{k,i}) \\ &+ d_i(\bar{p}^{\omega}_{n,i}, \varphi_i \circ \sigma^{m_i}(g)) + \epsilon. \end{aligned}$$

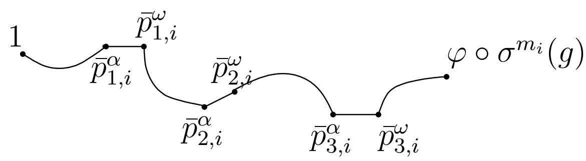



Figure 4.5: The segment $[1, \varphi_i \circ \sigma^{m_i}(g)] \subset X$ with $g = [a_0, e_1, \ldots, e_e, a_3]$

The $G$-equivariance of the metric $d_i$ immediately implies

1. $d_i(1, p^{\alpha}_{1,i}) = d_i(1, \bar{p}^{\alpha}_{1,i})$,

2. $d_i(p^{\omega}_{n,i}, \varphi_i(g)) = d_i(\bar{p}^{\omega}_{n,i}, \varphi_i \circ \sigma^{m_i}(g))$,

3. $d_i(\bar{p}^{\omega}_{k,i}, \bar{p}^{\alpha}_{k+1,i}) = d_i(p^{\omega}_{k,i}, p^{\alpha}_{k+1,i})$ for any $k$,

4. $d_i(\bar{p}^{\alpha}_{k,i}, \bar{p}^{\omega}_{k,i}) = d_i(p^{\alpha}_{k,i}, p^{\omega}_{k,i})$ whenever $\downarrow \tilde{e}_k \neq e^{\pm 1}$.

Assume that for some $k$, $\downarrow \tilde{e}_k = e^{\varepsilon}$ with $\varepsilon \in \{-1, 1\}$. Then $\theta_{\tilde{e}^k}(c)$ can be written as

$$\theta_{\tilde{e}}(c) = a_0, e_1, \ldots, a_{k-1}, e_k, \omega_{e_k}(c), e_k^{-1}, a_{k-1}^{-1}, \ldots, e_1^{-1}, a_0^{-1},$$

and it is easy to verify that $\bar{q}_k^i q_k^{-1} \sim \bar{q}_{k-1}^i q_{k-1}^{-1} \theta_{\tilde{e}_k}(c)^{\varepsilon \cdot m_i}$. Therefore by Proposition 4.19

$$\begin{aligned}
\lim_{i\to\infty} d_i(\bar{p}_{k,i}^{\alpha}, \bar{p}_{k,i}^{\omega}) &= \lim_{i\to\infty} d_i(\varphi_i([\bar{q}_{k-1}^i q_{k-1}^{-1}]) p_{k,i}^{\alpha}, \varphi_i([\bar{q}_k^i q_k^{-1}]) p_{k,i}^{\omega}) \\
&= \lim_{i\to\infty} d_i(\varphi_i([\bar{q}_{k-1}^i q_{k-1}^{-1}]) p_{k,i}^{\alpha}, \varphi_i(\bar{q}_{k-1}^i q_{k-1}^{-1} \theta_{\tilde{e}_k}(c)^{\varepsilon \cdot m_i}) p_{k,i}^{\omega}) \\
&= \lim_{i\to\infty} d_i(p_{k,i}^{\alpha}, \varphi_i(\theta_{\tilde{e}_k}(c)^{\varepsilon \cdot m_i}) p_{k,i}^{\omega}) \\
&= 0
\end{aligned}$$

Comparing this to (4.9) we obtain

$$\lim_{i\to\infty} (d_i(1, \varphi_i \circ \sigma^{m_i}(g))) = d(x_0, gx_0) - s \cdot l(e).$$

□

### 4.2.4 The shortening automorphism

In view of the previous sections, we are now able to conclude the proof of Proposition 4.6. Let $(\varphi_i) \subset \mathrm{Hom}(G, \Gamma)$ a converging stable sequence of pairwise distinct homomorphisms with associated $\Gamma$-limit map $\varphi$. Assume that all $\varphi_i$ are short (with respect to fixed finite generating sets of $G$ and $\Gamma$). Then, by Theorems 1.4 and 1.9 we obtain a non-trivial limit $G$-tree $T$, which splits as a graph of actions $\mathcal{G}$ by Theorem 2.4.

If $\mathcal{G}$ contains an axial vertex space or an orbifold type vertex space it follows from Theorem 4.8, respectively Theorem 4.15, that we can shorten the action on $T$ by precomposing with an automorphism $\alpha \in \mathrm{Aut}(G)$. As this action is being approximated by the action on $X$ via the $\varphi_i$ if follows that for large $i$ these actions can be shortened likewise by precomposing with $\alpha$. This proves that for large $i$ the $\varphi_i$ are not short, which is the claim of the theorem.

In the remaining case $\mathcal{G}$ contains a simplicial vertex space and the Proposition follows immediately from Corollary 4.21.

# Chapter 5

# Unfolding JSJ-decompositions

The aim of this chapter is to use the shortening argument of chapter 4 to prove that the almost abelian JSJ-decomposition constructed in section 3.4 can be chosen unfolded. This fact is not imperative for our applications as the construction in section 3.4 already ensures that the modular group of a JSJ does not increase by further unfoldings. Nevertheless, it adds a natural universality to the construction. An alternative proof of this result has been given independently by Guirardel and Levitt.

## 5.1 Acylindrical Accessibility of f.g. groups

With the help of the results of chapter 4, in this section we prove that for a one-ended f.g. group $G$, there exists a global upper bound on the length of short $(k, c)$-acylindrical $G$-actions for fixed $k, c \in \mathbb{N}$. In analogy to the length of a homomorphism (cf. Definition 1.3), we define the length of a group action as follows.

**Definition 5.1.** Let $G$ be a group with finite generating set $S$. The length of a $G$-action $\rho : G \to \mathrm{Isom}(X)$ on a based metric space $X = (X, x_0)$ with respect to the generating set $S$ is given by

$$|\rho|_S := \sum_{s\in S} d_X(x_0, {}_\rho s x_0).$$

Moreover, $\rho$ is called *short* (with respect to $S$) if $|i_g \circ \rho \circ \alpha|_S \geq |\rho|_S$ for all $\alpha \in \mathrm{Aut}(G)$ and $g \in \mathrm{Isom}(T)$.

Note that postcomposing by a conjugation is equivalent to altering the basepoint within its orbit. The goal of this section is the proof of the following theorem.

**Theorem 5.2.** *Let $G$ be a 1-ended group with finite generating set $S$, and $k, c \in \mathbb{N}$. There exists a constant $\lambda = \lambda(G, S, k, c)$ such that for each $(k, c)$-acylindrical $G$-action $\rho : G \to Isom(T)$ on a simplicial tree $T$, there exists an automorphism $\varphi \in Aut(G)$ and a choice of basepoint of $T$ such that*

$$|\rho \circ \varphi|_S \leq \lambda.$$

*If moreover $G$ is a $\Gamma$-limit group for some hyperbolic group $\Gamma$, then $\varphi$ can be chosen in $Mod(G)$.*

As a consequence of Theorem 5.2, we obtain an upper bound on the complexity (cf. (3.1)) of $(k, c)$-acylindrical splittings of $G$ as follows. Suppose that $(T, x_0, \rho)$ is a $(k, c)$-acylindrical $G$-tree and denote by $\pi : T \to A := T/G$ the projection to the quotient graph of the action. If $T$ is minimal and $S$ is a generating set, then every edge $e \in EA$ is contained in an edge path $\pi([x_0, sx_0])$ for some $s \in S$, which implies that $|EA| \leq \lambda$. As moreover the Betti number is bounded from above by the rank of $G$, we obtain the following consequence.

**Corollary 5.3.** *Let $G$ be a 1-ended group with finite generating set $S$, and $k, c \in \mathbb{N}$. There exists a constant $\mu = \mu(G, S, k, c)$ such that each minimal $(k, c)$-acylindrical splitting $\mathbb{A}$ of $G$ is of complexity $C(\mathbb{A}) \leq \mu$.* □

Corollary 5.3 is a weak version of Richard Weidmann's Acylindrical Accessibility below, which in particular provides an explicit value of $\mu$; however, the methods used in [W2] are not able to prove Theorem 5.2.

**Theorem 5.4** ([W2]). *Let $\mathbb{A}$ be a reduced and minimal $(k,c)$-acylindrical graph of groups with $k \geq 1$ such that $\pi_1(\mathbb{A})$ is f.g. Then*

$$|EA| \leq (2k+1) \cdot c \cdot (\text{rank}(\pi_1(\mathbb{A})) - 1).$$

The remainder of this section deals with the proof of Theorem 5.2. We fix a finite generating set $S_G$ of $G$, all lengths of actions and shortness are understood with respect to $S_G$.

The proof is by contradiction, so assume that Theorem 5.2 does not hold. Then there exists a sequence $((T_i, t_i, \rho_i))$ of based $G$-trees such that each action $\rho_i$ is short and

$$\lim_{i \to \infty} |\rho_i| = \infty.$$

By Lemma 1.2, the induced sequence $\left(\frac{1}{|\rho_i|} d_{\rho_i}\right)$ of scaled pseudo-metrics on $G$ converges in $\mathcal{A}(G)$. We denote the limit pseudo-metric by $d_\infty$. Moreover, as all pseudo-metrics $d_{\rho_i}$ are 0-hyperbolic (with respect to the Gromov product), so is $d_\infty$. It follows that there exists a based real $G$-tree $T = (T, t, \rho)$ such that $d_\infty = d_\rho$. The following theorem proves important stability properties of the $G$-action $\rho$ on $T$.

**Theorem 5.5.** *Let $k, c \in \mathbb{N}$ and $((T_i, t_i, \rho_i))_{i \in \mathbb{N}}$ be a sequence of $(k,c)$-acylindrical simplicial $G$-trees s.th.*

$$\lim_{i \to \infty} |\rho_i| = \infty.$$

*Assume moreover that $(T, t, \rho)$ is a $G$-tree such that*

$$\lim_{i \to \infty} \frac{1}{|\rho_i|} d_{\rho_i} = d_\rho.$$

*Then the $G$-action $\rho$ on $T$ satisfies the following.*

1. *The stabilizer of any non-degenerate interval is finite-by-cyclic.*
2. *The stabilizer of any non-degenerate tripod is finite.*
3. *Every subgroup of $G$ which acts invariantly and without reflections on a line in $T$ is finite-by-cyclic.*

*4. If $[x,y] \subset T$ is an interval and $|\mathrm{stab}[x,y]| = \infty$, then for any non-degenerate subinterval $[x',y'] \subset [x,y]$, $\mathrm{stab}[x',y'] = \mathrm{stab}[x,y]$.*

Before we proceed with the proof of Theorem 5.5, we establish two geometric facts which will be useful in the following.

**Lemma 5.6.** *Let $T$ be a $G$-tree, $x_1, x_2 \in T$ and $c \in [x_1, x_2]$. If for some $g \in G$,*

$$\max_{i=1,2} d(x_i, gx_i) \leq \min_{i=1,2} d(x_i, c), \tag{5.1}$$

*then $gc \in [x_1, x_2]$.*

*Proof.* Assume that $gc \notin [x_1, x_2]$ and let $p \in [x_1, x_2]$ be the projection of $gc$ to $[x_1, x_2]$. As $g$ acts isometrically, $g[x_1, x_2]$ is an interval (i.e., does not backtrack), hence $g[x_1, x_2] = [gx_1, gx_2]$. Thus for at least one $i \in \{1, 2\}$, $[p, gc] \cap [gc, gx_i] = \{gc\}$. This implies that

$$\begin{aligned} d(x_i, gx_i) &= d(x_i, gc) + d(gc, gx_i) \\ &= d(x_i, gc) + d(c, x_i) \\ &> d(c, x_i), \end{aligned}$$

contradicting (5.1). □

**Lemma 5.7.** *Let $T$ be a $(k,c)$-acylindrical $G$-tree and assume that $g_1, g_2 \in G$ act hyperbolically on $T$ with invariant axes $T_{g_1}$ and $T_{g_2}$ respectively. If $T_{g_1} \neq T_{g_2}$, then $T_{g_1} \cap T_{g_2}$ is either empty or compact of diameter at most $k + 2(c+1)\,(l(g_1) + l(g_2))$. In particular, $g_1$ and $g_2$ have either no or two common fixed ends.*

*Proof.* Assume that $T_{g_1} \neq T_{g_2}$ and $T_{g_1} \cap T_{g_2}$ is of diameter greater than $k + 2(c+1)\,(l(g_1) + l(g_2))$. Then there exists a segment $S \subset T_{g_1} \cap T_{g_2}$ of length $k$ which is of distance greater than $(c+1)\,(l(g_1) + l(g_2))$ from each of the endpoints of $T_{g_1} \cap T_{g_2}$. If $m \leq c+1$, it follows by iterative application of Lemma 5.6 that $[g_1^m, g_2]$ stabilizes $S$ as $g_1^m S$, $g_1^m g_2 S$ and $g_1^m g_2 g_1^{-m} S$ are contained in $T_{g_1} \cap T_{g_2}$.

By the acylindricity assumption on $T$, it follows that there exist $m \neq n \in \{1, \ldots, c+1\}$ such that $[g_1^m, g_2] = [g_1^n, g_2]$. This implies that $[g_1^{m-n}, g_2] = 1$, hence $g_1$ and $g_2$ have the same invariant axis, a contradiction. □

**Corollary 5.8.** *Let $T$ be a $(k, c)$-acylindrical $G$-tree and $H \leq G$ a non-elliptic subgroup. Then $H$ acts invariantly on a line of $T$ or contains a non-abelian free subgroup.*

*Proof.* It is easy to see that if $H$ is non-elliptic and does not act invariantly on a line, then there exist two hyperbolic elements $g_1, g_2 \in H$ with distinct axes. By Lemma 5.7, the axes have a compact intersection, thus the Ping-pong lemma provides a non-abelian free subgroup of $H$. □

With this in hand, we can prove Theorem 5.5.

*Proof of Theorem 5.5.* Let $[x, y] \subset T$ be a non-degenerate interval and put $S := \text{stab}[x, y]$. To prove (1), we show that for large $i$, $S$ acts invariantly and without reflections on a line $L_i \subset T_i$. As the action $\rho_i$ is discrete, this implies that the quotient of $S$ by the kernel of the $S$-action on $L_i$ is cyclic. As the kernel is finite by the acylindricity assumption on $T_i$, this implies the assertion.

Pick a non-degenerate interval $[x', y']$ in the interior of $[x, y]$. Let $(x_i)$, $(y_i)$ be approximating sequences of $x$ and $y$ respectively. Now pick approximating sequences $(x'_i)$, $(y'_i)$ of $x'$ and $y'$ where $x'_i, y'_i \in [x_i, y_i]$; the existence of these approximating sequences is obvious from the definition. If $g, h \in \text{stab}_T[x, y]$, it follows by iteratively applying Lemma 5.6 that for large enough $i$, the commutator $[g, h]$ fixes $[x'_i, y'_i]$. But as $\left(\frac{d_{T_i}(x'_i, y'_i)}{|\rho_i|}\right)$ converges to $d_T(x', y')$, it follows that $d_{T_i}(x'_i, y'_i) > k$ for large $i$, thus the commutator subgroup of $\text{stab}_T[x, y]$ is finite by the acylindricity assumption on $T_i$, i.e. $S$ is finite-by-abelian. As $S$ does not contain a non-cyclic free subgroup, it follows that $S$ acts invariantly on a line (cf. Corollary 5.8). This action is clearly without reflections as $S$ is finite-by-abelian, hence (1) is proven.

Now let $Y \subset T$ be a non-degenerate tripod with endpoints $x, y, z$ and branching point $c$. Choose approximating sequences $(x_i), (y_i), (z_i)$ of $x$, $y$ and $z$ respectively. For each $i$, let $c_i$ be the midpoint of the tripod spanned by $x_i, y_i$ and $z_i$ in $T_i$. If $g \in \text{stab}(Y)$, it follows from Lemma 5.6 that ${}_{\rho_i}gc_i \in [x_i, y_i] \cap [x_i, z_i] \cap [z_i, x_i]$ for large $i$, thus ${}_{\rho_i}gc_i = c_i$. As

$$\lim_{i \to \infty} d_{T_i}(x_i, y_i) = \infty,$$

it follows from Lemma 5.1 that for large $i$, a large neighbourhood of $c_i$ in $[x_i, y_i]$ is fixed by $\text{stab}_T(Y)$, and assertion (2) follows immediately from the acylindricity assumption.

Assertion (3) is proven similarly to (1), see the proof of Theorem 1.11 for details.

Let $[x, y] \subset T$ be an interval. Assume that for some $x' \in [x, y]$ we have that $\text{stab}_T[x', y] > \text{stab}_T[x, y]$. Pick $g \in \text{stab}_T[x, y]$ and $g' \in \text{stab}_T[x', y] \setminus \text{stab}_T[x, y]$. It follows that

$$g(g'x) = [g, g']g'(gx) = [g, g']g'x.$$

But as $g, g' \in \text{stab}_T[x, y]$, it follows from (1) that there are at most $c$ possible commutators $[g, g']$, i.e. the $\text{stab}_T[x, y]$-orbit of $g'x$ is of cardinality at most $c$. Thus a subgroup of $\text{stab}_T[x, y]$ of index at most $c$ fixes the tripod spanned by $x$, $y$ and $g'x$. This subgroup is of oder at most $c$ by (2). Thus $\text{stab}_T[x, y]$ is finite, which proves (4). □

Theorem 5.5 implies that we can apply the main theorem of [G] to the limit $G$-tree $T$. As $G$ is assumed 1-ended, it does not split over the stabilizer of an unstable arc. It follows that $T$ splits as a graph of actions $\mathcal{G} = \mathcal{G}(\mathbb{A})$. $\mathbb{A}$ is an almost abelian splitting of $G$, and we can construct a shortening automorphism $\alpha \in \text{Mod}(\mathbb{A})$ as in section 4.2, which implies that the actions $\rho_i$ are not short. Moreover, if $G$ is a $\Gamma$-limit group for some hyperbolic group $\Gamma$, then $\mathbb{A}$ is visible (after folds) in any almost abelian $JSJ$-decomposition of $G$, hence $\text{Mod}(\mathbb{A}) \leq \text{Mod}(G)$, and therefore $\alpha \in \text{Mod}(G)$. This concludes the proof of Theorem 5.2.

## 5.2 Foldings and unfoldings of $G$-trees

In this section we briefly introduce foldings and unfoldings of $G$-trees and establish notations which we will use in the following section.

**Definition 5.9.** Let $G$ be a group and $(T_1, \rho_1), (T_2, \rho_2)$ be minimal simplicial $G$-trees. A morphism $\mathfrak{f} = (\varphi, f) : T_1 \to T_2$ is called a *folding* if the following hold.

1. $\varphi : G \to G$ is an isomorphism.
2. Whenever $e_1, e_2 \in VT_1$ and $f(e_1) = f(e_2)$, there exists $g \in G$ such that ${}_{\rho_1}ge_1 = e_2$.

Analogously, if $\mathbb{A}_1$ and $\mathbb{A}_2$ are two splittings of a group $G$, we say that a morphism $\mathfrak{f} : \mathbb{A}_1 \to \mathbb{A}_2$ is a folding if the induced morphism $\tilde{\mathfrak{f}} : \tilde{\mathbb{A}}_1 \to \tilde{\mathbb{A}}_2$ is a folding.

Note that the terminology differs slightly from the one in literature. In [D3], a folding in the sense of Definition 5.9 would be referred to as a finite sequence of type II foldings.

If $(\varphi, f) : T_1 \to T_2$ is a folding, then condition (2) of Definition 5.9 ensures that the quotient map $T_1 \to T_1/G$ factors through $f$. Moreover, $f$ is surjective as $\varphi$ is surjective and $T_2$ is minimal. This implies that if a graph of groups $\mathbb{A}_2$ is a folding of $\mathbb{A}_1$ (i.e., there exists a folding $\mathfrak{f} : \mathbb{A}_1 \to \mathbb{A}_2$), then $\mathbb{A}_1$ and $\mathbb{A}_2$ have the same underlying graphs $A_1 = A_2$, and therefore, in particular, the same complexity.

Conversely, we say that a tree $T_1$ is obtained from $T_2$ by an *unfolding* if there exists a folding from $T_1$ to $T_2$. Note however that an unfolding is not a morphism as foldings are not invertible in general.

In the following we state a fact about foldings which will be of use in this section.

**Lemma 5.10.** *Let $T_i = (T_i, x_i, \rho_i)$ be based $G_i$-trees for $i = 1, 2, 3,$ and $\mathfrak{f}_i = (\varphi_i, f_i)$ :*

*$T_i \to T_{i+1}$ a morphism for $i = 1, 2$. Then $\mathfrak{f}_2 \circ \mathfrak{f}_1$ is a folding if and only if $\mathfrak{f}_1$ and $\mathfrak{f}_2$ are foldings.*

*Proof.* Assume that $\mathfrak{f}_1$ and $\mathfrak{f}_2$ are foldings, and let $x, y \in ET_1$ s.th. $f_2 \circ f_1(x) = f_2 \circ f_1(y)$. We need to show that $x \sim_{\rho_1} y$ in $T_1$. As $\mathfrak{f}_2$ is a folding, $f_1(x) \sim_{\rho_2} f_1(y)$, i.e. there exists $h \in G_2$ s.th. ${}_{\rho_2}hf_1(x) = f_1(y)$. Putting $g := \varphi_1^{-1}(h)$, it follows that

$$f_1({}_{\rho_1}gx) = {}_{\rho_2}\varphi_1(g)f_1(x) = {}_{\rho_2}hf_1(x) = f_1(y).$$

As $\mathfrak{f}_1$ is a folding, it follows that ${}_{\rho_1}gx \sim y$, and therefore also $x \sim y$.

Conversely, assume that $\mathfrak{f}_2 \circ \mathfrak{f}_1$ is a folding. Let $x, y \in ET_1$ s.th. $f_1(x) = f_1(y)$. Then $f_2 \circ f_1(x) = f_2 \circ f_1(y)$, hence $x \sim y$ as $\mathfrak{f}_2 \circ \mathfrak{f}_1$ is a folding. This shows that $\mathfrak{f}_1$ is a folding.

Now let $x', y' \in T_2$ s.th. $f_2(x') = f_2(y')$. As $f_1$ is surjective, there exist $x, y \in T_1$ s.th. $f_1(x) = x'$, $f_1(y) = y'$. Thus $f_2 \circ f_1(x) = f_2 \circ f_1(y)$, hence there exists $g \in G_1$ s.th. ${}_{\rho_1}gx = y$. It follows that ${}_{\rho_2}\varphi_1(g)x' = y'$. Hence $x' \sim y'$, which shows that $\mathfrak{f}_2$ is a folding. □

## 5.3 Unfolding JSJ-decompositions

The goal of this section is the proof of the following lemma.

**Lemma 5.11.** *Let $\Gamma$ be a hyperbolic group and $G$ a 1-ended $\Gamma$-limit group. Let $(\mathbb{A}_i)_{i\in\mathbb{N}_0}$ be a sequence of almost abelian JSJ-decompositions of $G$ and $\mathfrak{f}_i : \mathbb{A}_i \to \mathbb{A}_{i-1}$ a folding for each $i \in \mathbb{N}$. Then there exists an almost abelian JSJ-decomposition $\mathbb{A}$ of $G$, together with a folding $\mathfrak{h}_i : \mathbb{A} \to \mathbb{A}_i$ for each $i$.*

As a consequence of Lemma 5.11, it follows from Zorn's Lemma that there exists an almost abelian JSJ-decomposition $\mathbb{A}$ of $G$ which cannot be unfolded. Therefore, using Theorem 3.12, the proof of Lemma 5.11 immediately establishes the following theorem.

**Theorem 5.12.** *Let $\Gamma$ be a hyperbolic group and $G$ a 1-ended $\Gamma$-limit group. Then there exists an almost abelian JSJ-decomposition $\mathbb{A}$ of $G$ such that every almost abelian JSJ-decompositon of $G$ can be obtained from $\mathbb{A}$ by foldings. In particular, $\mathbb{A}$ does not admit any unfoldings.*

Again, we fix a finite generating set $S_G$ of $G$ and understand all lengths of actions and shortness with respect to $S_G$. To prove Lemma 5.11, recall that each $\mathbb{A}_i$ is $(2, N(\Gamma))$-acylindrical. From now on, denote by $\rho_i$ the standard action of $G$ on $\tilde{\mathbb{A}}_i$. Then by Theorem 5.2, there exists $\lambda \in \mathbb{N}$ and for each $i \in \mathbb{N}$ a modular automorphism $\alpha_i \in \mathrm{Mod}(G)$ and a choice of base vertex of $v_0^i \in V\tilde{\mathbb{A}}_i$ such that

$$|\rho_i \circ \alpha_i| \leq \lambda.$$

Denote by $d_i$ the pseudo-metric on $G$ induced by the action $\rho_i \circ \alpha_i$. By Lemma 1.2, the sequence $(d_i)$ has a subsequence which converges in $\mathcal{A}(G)$ to a pseudo-metric $d$ on $G$, and there is a based real $G$-tree $T = (T, x_0, \rho)$ such that $d = d_\rho$. We denote this converging subsequence again by $(d_i)$. The aim of this section is to show that for any $i$ there exists a folding $\mathfrak{h}_i : T \to \tilde{\mathbb{A}}_i$. It is then easy to verify that the splitting induced by the $G$-action on $T$ is again a JSJ-decomposition, which proves Theorem 5.11.

Let $\mathbb{A}$ be a JSJ-decomposition of $G$ and denote by $\rho$ the standard $G$-action on $\tilde{\mathbb{A}}$. To a modular automorphism $\alpha \in \mathrm{Mod}(G)$ we associate a graph map $f^\alpha_{\mathbb{A}} : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ as follows.

1. If $\alpha = c_g$ for $g \in G$, put $f^\alpha_{\mathbb{A}} = \rho(g)$.
2. If $\alpha$ is a natural extension of a vertex group automorphism or a Dehn twist and $v = [a_0, e_1, a_1, \ldots, e_k, A_{\omega(e_k)}]$ then put

$$f^\alpha_{\mathbb{A}}(v) := [\bar{a}_0, e_1, \bar{a}_1, \ldots, e_k, A_{\omega(e_k)}]$$

   (with the notation as in Definition 3.13 and (4.8) respectively) for $v \in V\tilde{\mathbb{A}}$, and $f^\alpha_{\mathbb{A}}(v_1, v_2) = (f^\alpha_{\mathbb{A}}(v_1), f^\alpha_{\mathbb{A}}(v_2))$ for $(v_1, v_2) \in E\tilde{\mathbb{A}}$.

3. Put $f_{\alpha_2 \circ \alpha_1} = f_{\alpha_2} \circ f_{\alpha_1}$.

It is not hard to verify that the above yields a well-defined isometry on $\tilde{\mathbb{A}}_i$ for every $\alpha \in \mathrm{Mod}(G)$, satisfying

$$A_{f^{\alpha}_{\mathbb{A}}(v)} = \alpha(A_v) \text{ for all } v \in V\tilde{\mathbb{A}}. \tag{5.2}$$

Indeed this holds by construction in the first two cases, and as (5.2) determines $\alpha$, it follows that $f_{\alpha_2 \circ \alpha_1}$ is well-defined and likewise satisfies (5.2) for all $\alpha_1$, $\alpha_2 \in \mathrm{Mod}(G)$.

The above implies that $\mathfrak{f}^{\alpha}_{\mathbb{A}} := (\alpha, f^{\alpha}_{\mathbb{A}}) : (\tilde{\mathbb{A}}, \rho) \to (\tilde{\mathbb{A}}, \rho \circ \alpha)$ is a tree morphism.

In the following, we put $\mathfrak{f}_{\alpha_i} := \mathfrak{f}^{\alpha_i}_{\mathbb{A}_i}$ and denote the $G$-action $\rho_i \circ \alpha_i$ by $\rho_i'$. Further, we denote by $T_i$ the $G$-tree $(\tilde{\mathbb{A}}_i, \rho_i')$ (whereas $\tilde{\mathbb{A}}_i$ is understood to denote the Bass-Serre tree with the standard action $\rho_i$).

As in particular the morphisms $\mathfrak{f}_{\alpha_i} : \tilde{\mathbb{A}}_i \to T_i$ and $\mathfrak{f}_{\alpha_i^{-1}}$ are foldings, by Lemma 5.10 the morphism $\mathfrak{f}_i' = (\varphi_i', f_i') := \mathfrak{f}_{\alpha_i^{-1}} \circ \mathfrak{f}_i \circ \mathfrak{f}_{\alpha_{i-1}} : T_i \to T_{i-1}$ is a folding.

In the remainder of the section we construct for any $i$ a folding $\mathfrak{h}_i' : T \to T_i$. Then $\mathfrak{h}_i := \mathfrak{f}_{\alpha_i^{-1}} \circ \mathfrak{h}_i'$ is the desired folding from $T$ to $\tilde{\mathbb{A}}_i$.

**Lemma 5.13.** *Let $(T_i)_{i \in \mathbb{N}}$ and $T$ be as above.*

1. *Any branching point of $T$ is at an integer distance from the base point $x_0$. In particular, any two branching points have integer distance from each other.*
2. *$T$ is $(2, N(\Gamma))$-acylindrical.*
3. *Any subgroup $H \leq G$ is elliptic in $T$ iff $H$ is elliptic in every $T_i$.*
4. *Every branching point of $T$ has an approximating sequence which consists of verices of the $T_i$.*

*Proof.* Let $x \in T$ be a branching point. As $T$ is spanned by ${}_{\rho}Gx_0$, there exist $g, h \in G$ such that $x$ is the branching point of the tripod spanned by $x_0$, ${}_{\rho}gx_0$ and ${}_{\rho}hx_0$.

This implies that $d_T(x_0, x) = ({}_{\rho'_i} g x_0 | {}_{\rho'_i} h x_0)^T_{x_0}$. But this distance is the limit of the sequence $\left(({}_{\rho'_i} g v_0^i | {}_{\rho'_i} h v_0^i)^{T_i}_{v_0^i}\right)_{i \in \mathbb{N}}$ which is a sequence of integers. Thus $d_T(x, x_0) \in \mathbb{Z}$, which immediately implies (1).

To prove (2), note first that every $T_i$ is $(2, N(\Gamma))$-acylindrical as $\tilde{\mathbb{A}}_i$ is. Let $v_1, v_2 \in T$ such that $d_T(v_1, v_2) > 2$. Let $(v_1^i)$ and $(v_2^i)$ be approximating sequences of $v_1$ and $v_2$ respectively. Then for large enough $i$, $d_i(v_1^i, v_2^i) > 2$.

Note that if $x \in T$ and $l \in \mathbb{R}$ is the minimal distance from $x$ to a branching point, then for any $g \in G$, either ${}_\rho g x = x$ or $d_T(x, {}_\rho g x) \geq 2l$. It follows that if $(x^i)$ is an approximating sequence of $x$ and $g \in G$ s.th. ${}_\rho g x = x$, then ${}_{\rho'_i} g x^i = x^i$ for large enough $i$ (if $x$ is not a branching point, which we may assume in view of (1)).

It follows that for any f.g. subgroup $H \leq \mathrm{stab}[v_1, v_2]$ there exists an $i_0$ such that $H$ fixes $[v_1^i, v_2^i]$ for $i \geq i_0$. But all $T_i$ are $(2, N(\Gamma))$-acylindrical, hence $|H| \leq N(\Gamma)$. As this holds for any f.g. subgroup $H \leq \mathrm{stab}[v_1, v_2]$, it follows that $|\mathrm{stab}[v_1, v_2]| \leq N(\Gamma)$. This proves (2).

To prove assertion (3) note first that $H$ is elliptic in $T_i$ (resp. $T$) iff every f.g. subgroup of $H$ is elliptic. Indeed if $H = \langle g_1, g_2, \ldots \rangle$ and for every $H_k := \langle g_1, \ldots, g_k \rangle$ there exists $v_k \in VT_i$ (resp. $T$) s.th. $H_k$ stabilizes $v_k$, it follows from the acylindricity of $T_i$ (resp. $T$, cf. (2)) that the sequence $(v_k)_{k \in \mathbb{N}}$ has bounded diameter and thus allows a constant subsequence, hence contains a vertex stabilized by $H$.

As assertion (3) trivially holds for f.g. subgroups, the above claim proves it for $H$.

To show (4), let $x \in T$ be a branching point. Note first that the existance of an approximating sequence follows from Lemma 1.7. Now, by (1), we have that $n := d_T(x, x_0) \in \mathbb{Z}$. Therefore, if $(x_i)$ is an approximating sequence of $x$, then

$$\lim_{x \to \infty} d_i(v_0^i, x_i) = n \in \mathbb{Z}.$$

This implies that for large $i$, $x_i$ is in $\tilde{\mathbb{A}}_i$ arbitrarily close to the (unique) closest vertex. Replacing each $x_i$ by this vertex, yields an approximating sequence of the desired type. $\square$

From now on, we regard $T$ as a simplicial tree with vertex set

$$VT := \{x \in T \mid d_T(x, x_0) \in \mathbb{Z}\}.$$

Lemma 5.13 (1) ensures that $VT$ contains all branching points of $T$, so indeed this gives $T$ the structure of a (not necessarily reduced) simplicial tree in the obvious way.

Choose approximating sequences of all vertices of $T$ equivariantly, i.e. in the following way: For any G-orbit ${}_{\rho}Gv$ of $VT$, pick a representative $v$ and an approximating sequence $(v^i)$ such that $v^i \in VT_i$ for each $i$. Now for any other vertex $w \in {}_{\rho}Gv$, pick $g \in G$ s.th. ${}_{\rho}gv = w$ and put $w^i := {}_{\rho_i'}gv^i$ for each $i \in \mathbb{N}$. Clearly $(w^i)$ is an approximating sequence of $w$. Note however that the sequence $(w^i)$ is not canonical as it depends on the choice of $g$.

From now on, for a vertex $v^i \in VT_i$ we will simply denote by $A_{v^i}$ its stabilizer in $T$, i.e. with respect to the action $\rho_i'$. The following observation about almost abelian vertex groups will be crucial.

**Lemma 5.14.** *There exists $i_0$ such that for every $i \geq i_0$ and $v \in VT$, the following hold.*

1. *$A_v$ is almost abelian if and only if $A_{v^i}$ is almost abelian.*
2. *$A_{v^i} \leq A_{v^{i-1}}$.*
3. *$A_v \leq A_{v^i}$.*

*Proof.* We fix a vertex $v$ and prove the existence of $i_0$ such that the above holds for $v$. The fact that $i_0$ can be chosen to satisfy the above for all vertices then follows from the equivariant choice of approximating sequences and the fact that there are only finitely many orbits of vertices.

Let $v \in VT$ and suppose first that $A_v$ is non-almost abelian. Then $A_v$ contains a f.g. non-almost abelian subgroup $H \leq A_v$ by Lemma 1.15. As $H \leq A_{v^i}$ for large enough $i$, $A_{v^i}$ is non-almost abelian.

It follows from the above that for any approximating sequence $(w^i)$ of a vertex $w \in VT$, $A_{w^i}$ is eventually always almost abelian or always non-almost abelian. Indeed, if $v$ is as above, i.e. $A_v$ non-almost abelian, then $d_i(v^i, w^i)$ is evantually constant (i.e. equal to $d(v, w)$). Recall from the definition of the JSJ-decomposition that every edge joins an almost abelian vertex with a non-almost abelian vertex (in particular, there are no loop edges). As $A_{v^i}$ is evantually always non-almost abelian, $A_{w^i}$ being almost abelian or not is determined by the parity of the distance $d_i(v^i, w^i)$.

It remains to show that if $A_v$ is almost abelian then $A_{v^i}$ is almost abelian for large $i$. So assume that $A_v$ is almost abelian and $A_{v^i}$ is non-almost abelian for large $i$. Let $\{w_1, w_2, \ldots\} \subset VT$ be the set of vertices satisfying $d(w_k, v) = 1$. Then the above argument shows that for every $w_k$ with approximating sequence $(w_k^i)$, $A_{w_k^i}$ is almost abelian for large $i$ as $d(v^i, w_k^i) = 1$, and therefore $A_w$ is almost abelian. Moreover, every $w_k$ lies in the same maximal almost abelian subgroup as $A_v$, hence for large $i$, $w_k^i \in VT_i$ is the unique vertex stabilized by this maximal almost abelian subgroup. This implies that there exists only one vertex $w_1$ of distance 1 from $v$, which implies that $T$ is not minimal, a contradiction. This proves (1).

We now prove (2). Assume that $A_{v^i} \not\leq A_{v^{i-1}}$ for infinitely many $i$. Let $A_v = \langle g_1, g_2, \ldots \rangle$ and put $H_k := \langle g_1, \ldots, g_k \rangle$ for each $k \in \mathbb{N}$. Clearly, for any $k$ there exists $i_k$ s.th. $H_k \leq A_{v^i}$ for all $i \geq i_k$. On the other hand, $H_k \leq A_{v^{i+1}} \leq A_{f_i'(v^i)}$ for any $i \geq i_k$. Now by assumption, for any $k$ there exists $i \geq i_k$ such that $f_i'(v^i) \neq v^{i-1}$. This implies that $H_k$ stabilizes the non-degenerate segment $[v^{i-1}, f_i'(v^i)] \subset T_{i-1}$, thus each $H_k$ is almost abelian. It follows now by Lemma 1.15 that $A_v$ is almost abelian.

If $A_v$ is almost abelian, then by (1) $A_{v^i}$ and $A_{v^{i+1}}$ are almost abelian for large $i$. This however implies that they are maximal almost abelian as all almost abelian vertex groups of JSJ-decompositions are maximal almost abelian. As the interesection of distinct maximal almost abelian subgroups is finite, it follows that $A_{v^i} = A_{v^{i+1}}$ for large $i$. Hence in particular, (2) holds.

To show (3), put $i_0$ such that (2) holds for $i \geq i_0$, and fix $i \geq i_0$. If $g \in A_v$, then there exists $j \geq i$ such that $g \in A_{v^j}$. By iterative use of (2) it follows that $g \in A_{v^i}$. □

**Corollary 5.15.** *All almost abelian vertex stabilizers in $T$ are maximal almost abelian.*

*Proof.* Let $v \in VT$ and $A_v$ almost abelian. Then $A_{v^i}$ is almost abelian by Lemma 5.14, hence maximal almost abelian. Therefore, $A_{v^i}$ is constant for large enough $i$, so $A_v = A_{v^i}$. □

**Lemma 5.16.** *For large $i$, the graph map $h_i$ induced by the map $\bar{h}_i : VT \to VT_i$, $v \mapsto v^i$, extends to a tree morphism $\mathfrak{h}'_i = (h_i, \mathrm{id}) : T \to T_i$, which is a folding.*

*Proof.* It follows straight from Lemma 5.14 (3) that $\mathfrak{h}'_i = (h_i, \mathrm{id})$ is a tree morphism. So it remains to prove that $\mathfrak{h}'_i$ is a folding. Let $i_0$ s.th. Lemma 5.14 holds and fix $i \geq i_0$. Let $e_1 \neq e_2 \in ET$ s.th. $h_i(e_1) = h_i(e_2) =: e \in ET_i$.

W.l.o.g. (up to orientation), we have that $A_{\alpha(e)}$ is almost abelian and $A_{\omega(e)}$ is non-almost abelian. It then follows from Lemma 5.14 (1) that $A_{\alpha(e_1)}$ and $A_{\alpha(e_2)}$ are both almost abelian. If $\alpha(e_1) \neq \alpha(e_2)$ then $A_{\alpha(e_1)^i} \cap A_{\alpha(e_2)^i}$ is finite for large $i$, it follows that $\alpha(e_1)^i \neq \alpha(e_2)^i$, a contradiction. Therefore, $\alpha(e_1) = \alpha(e_2)$.

Now $A_{\omega(e)}$ is non-almost abelian, which implies that $A_{\omega(e_1)}$ and $A_{\omega(e_2)}$ are non-almost abelian. Put $v := \omega(e)$, and let $T_v$ be the ${}_\rho A_v$-minimal subtree of $T$ and $\mathbb{A}^v$ be the splitting of $A_v$ obtained from the action $\rho|_{A_v}$ of $A_v$ on $T_v$.

We put $v_1 := \omega(e_1)$, $v_2 := \omega(e_2)$, and first show that $v_1, v_2 \in T_v$. Let $p_1$ be the nearest point projection of $v_1$ on $T_v$. As $A_{v_1} \leq A_v$ leaves $T_v$ invariant and fixes $v_1$, $A_{v_1}$ also fixes $p_1$. But as $A_{v_1}$ is non-almost abelian it does not fix a non-degenerate segment, hence $v_1 = p_1 \in T_v$. The same holds for $v_2$.

Moreover, it is clear that every subgroup $H \leq A_v$ which stabilizes an edge in $T_i$ is elliptic in $T_v$. Therefore, the JSJ-decompositon $\mathbb{A}_i$ allows a refinement in $\downarrow v$ by the splitting $\mathbb{A}^{\downarrow v} \cong \mathbb{A}^v$. As $\mathbb{A}_i$ is of maximal complexity, it follows that $\mathbb{A}^v$ contains at most one vertex $w$ with non-abelian vertex group as otherwise the complexity of $\mathbb{A}_i$ would increase by refining followed by the normalization process. Therefore, $\downarrow v_1 = \downarrow v_2 = w$ (where $\downarrow \cdot$ denotes the projection $T_v \mapsto \mathbb{A}^v$), which implies that there exists $g \in A_v$ such that ${}_\rho g v_1 = v_2$. As $g$ clearly fixes $\alpha(e_1) = \alpha(e_2)$ it follows that ${}_\rho g e_1 = e_2$. This concludes the proof. $\square$

Lemma 5.16 provides the desired foldings from $T$ to $T_i$ for each $i$, which implies that for each $i$ there exists a folding $\mathfrak{h}_i : T \to \tilde{\mathbb{A}}_i$ as seen earlier. Let $\mathbb{A}$ be the splitting of $G$ induced by the $G$-tree $T$. As $\mathbb{A}$ is obtained from a JSJ-decomposition of $G$ by an unfolding which leaves all large almost abelian subgroups of $G$ elliptic by Lemma 5.13 (3), it is easily checked that $\mathbb{A}$ is again a JSJ-decomposition of $G$. This concludes the proof of Lemma 5.11.

# Chapter 6

# Makanin-Razborov diagrams

In this chapter we use the existence of Γ-factor sets proven in chapter 4 to give a complete description of the set of all homomorphisms from a finitely generated group $G$ to a weakly equationally Noetherian hyperbolic group Γ. In section 7.2 we will see that this assumption was vacuous as all hyperbolic groups have this property.

## 6.1 Equationally Noetherian groups

Let Γ be a group and $F(x_1, \ldots, x_n)$ be a free group of rank $n$. We then define

$$\Gamma[x_1, \ldots, x_n] := \Gamma * F(x_1, \ldots, x_n).$$

For any $\eta \in \Gamma[x_1, \ldots, x_n]$ and $(\gamma_1, \ldots, \gamma_n) \in \Gamma^n$ we define $\eta(\gamma_1, \ldots, \gamma_n)$ to be the element of Γ obtained from $\eta$ by substituting any occurence of $x_i$ by $\gamma_i$. We say that $(\gamma_1, \ldots, \gamma_n)$ satisfies the equation $\eta$ if $\eta(\gamma_1, \ldots, \gamma_n) = 1$.

For any set $S \subset \Gamma[x_1, \ldots, x_n]$ the *radical* of $S$ is defined as

$$\text{rad}(S) := \{(\gamma_1, \ldots, \gamma_n) \in \Gamma^n, \mid \eta(\gamma_1, \ldots, \gamma_n) = 1 \text{ for all } \eta \in S\},$$

thus $\mathrm{rad}(S)$ is the set of all $n$-tuples of elements of $\Gamma$ that satisfy all equations of $S$ simultaneously.

$\Gamma$ is called *equationally Noetherian* if for every $n \in \mathbb{N}$ and any subset $S \subset \Gamma[x_1, \dots, x_n]$ there exists a finite subset $S_0 \subset S$ such that $\mathrm{rad}(S) = \mathrm{rad}(S_0)$.

It was shown by Guba [Gu] that f.g. free groups are equationally Noetherian. His proof exploited the fact that free groups are linear which allows him to appeal to some classical algebraic geometry. This fact was used in [BMR] to show that large classes of linear groups are equationally Noetherian. Note however that hyperbolic groups are not necessarily linear [Ka]. Thus we can in general not appeal to linearity to establish that hyperbolic groups are equationally Noetherian.

For our purposes we need a property that is slightly weaker than being equationally Noetherian, namely that for any set $S \subset F(x_1, \dots, x_n)$ there exists a finite subset $S_0 \subset S$ such that $\mathrm{rad}(S) = \mathrm{rad}(S_0)$, we call this property *weakly equationally Noetherian*. This corresponds to restricting to equations without constants.

**Lemma 6.1.** *If $\Gamma$ is weakly equationally Noetherian then for any sequence*

$$G_1 \to G_2 \to G_3 \to \dots$$

*of epimorphisms of finitely generated groups the associated embeddings*

$$Hom(G_1, \Gamma) \leftarrow Hom(G_2, \Gamma) \leftarrow Hom(G_3, \Gamma) \leftarrow \dots$$

*eventually become bijections.*

*Proof.* Given a finitely generated group $G = \langle x_1, \dots, x_n \,|\, R \rangle$ and a group $\Gamma$ there is a one-to-one correspondence between $\mathrm{Hom}(G, \Gamma)$ and $\mathrm{rad}(R)$ as if $\phi \in \mathrm{Hom}(G, \Gamma)$ then $(\phi(x_1), \dots, \phi(x_n)) \in \mathrm{rad}(R)$ and for each tuple $(\gamma_1, \dots, \gamma_n) \in \mathrm{rad}(R)$ the map $x_i \mapsto \gamma_i$ for $i = 1, \dots, n$ extends to a homomorphism $G \to \Gamma$.

Choose presentations $\langle x_1, \dots, x_n \,|\, R_i \rangle$ of $G_i$ such that $R_i \subset R_{i+1}$. Put

$$R_\infty = \bigcup_{i\in\mathbb{N}} R_i$$

and $G_\infty := \langle x_1, \ldots, x_n \,|\, R_\infty\rangle$, i.e. $G_\infty$ is the direct limit of the $G_i$. As $\Gamma$ is weakly equationally Noetherian it follows that $\mathrm{rad}(R_\infty) = \mathrm{rad}(R_i)$ for some $i$. The claim now follows. □

**Corollary 6.2.** *If $\Gamma$ is weakly equationally Noetherian then any sequence*

$$G_1 \to G_2 \to G_3 \to \ldots$$

*of epimorphisms of finitely generated groups that are residually $\Gamma$ eventually stabilizes.*

*Proof.* Because of Lemma 6.1 it clearly suffices to show that if $G$ and $G'$ are residually $\Gamma$ and $\pi : G \to G'$ is an epimorphism such that $K = \ker \pi$ is non-trivial then $\pi_* : \mathrm{Hom}(G', \Gamma) \to \mathrm{Hom}(G, \Gamma)$ is not surjective. This however is obvious as for $k \in K \setminus \{1\}$ there is a homomorphism $\phi : G \to \Gamma$ such that $\phi(k) \neq 1$ as $G$ is residually $\Gamma$. But $\phi$ can clearly not lie in the image of $\pi_*$. □

**Corollary 6.3.** *Suppose that $\Gamma$ is weakly equationally Noetherian. Then any $\Gamma$-limit group is fully residually $\Gamma$.*

*Proof.* Let $L = F_k/\underrightarrow{\ker}(\varphi_i)$ be a $\Gamma$-limit group. Choose a sequence

$$G_0 = F_k \to G_1 \to G_2 \to \ldots$$

of finitely presented groups such that $L$ is its direct limit. By Corollary 6.2 there exists some $G_{i_0}$ such that any homomorphism $\varphi : G_{i_0} \to \Gamma$ factors through $L$. After passing to a subsequence we can further assume that any $\varphi_i$ factors through $G_{i_0}$ as $G_{i_0}$ is finitely presented. Thus the sequence factors in fact through $L$, i.e. there exists a stable sequence $(\eta_i) \subset \mathrm{Hom}(L, \Gamma)$ such that $\underrightarrow{\ker}(\eta_i) = 1$, i.e. that $L = L/\underrightarrow{\ker}(\eta_i)$. This clearly implies that for any finite set $M \subset L$, $\eta_i|_M$ is injective for sufficiently large $i$. □

We will need the following simple lemma, its proof is identical to that in the case of a free group, see [BF1].

**Lemma 6.4.** *Let* $\Gamma$ *be a weakly equationally Noetherian group and* $G$ *be a finitely generated group. Then there exist finitely many groups* $L_1, \ldots, L_k$ *and epimorphisms* $\pi_i : G \to L_i$ *such that the following hold.*

1. $L_i$ *is fully residually* $\Gamma$ *for* $i = 1, \ldots, k$.
2. *For any homomorphism* $\phi : G \to \Gamma$ *there exists* $\alpha \in Aut(G)$ *such that* $\phi \circ \alpha$ *factors through some* $\pi_i$.

*Proof.* Let $\hat{G}$ be the universal residually $\Gamma$ quotient of $G$, i.e. $\hat{G} = G/N$ where $N$ is the intersection of all kernels of homomorphisms from $G$ to $\Gamma$. As any homomorphism $\phi : G \to \Gamma$ factors through $\hat{G}$ we have that $\mathrm{Hom}(\hat{G}, \Gamma) \to \mathrm{Hom}(G, \Gamma)$ is a bijection. In particular it suffices to prove existence of a factor set for $\hat{G}$ as precomposing its epimorphism with the quotient map $G \to \hat{G}$ provides a factor set for $G$. If $\hat{G}$ is fully residually $\Gamma$ then there is nothing to show.

Thus we can assume that $\hat{G}$ is not fully residually $\Gamma$. It follows that there exists a finite set $M = \{g_1, \ldots, g_k\}$ such that $M \cap \ker \phi \neq \emptyset$ for any homomorphism $\phi : \hat{G} \to \Gamma$. It follows that any $\phi : G \to \Gamma$ factors through one of the epimorphisms $\pi_1 : G \to L_i$ where $L_i$ is the universal residually $\Gamma$ quotient of $G/\langle\langle g_i \rangle\rangle$ for $1 \leq i \leq k$ and $\pi_i$ is the canonical quotient map.

If $L_i$ is not fully residually $\Gamma$ we repeat this construction for $L_i$, after finitely many iterations this must teminate by Lemma 6.1. Thus we get a finite directed tree of epimorphisms such that any homomorphism factors through one branch, the assertion follows by choosing as epimorphisms the composition along maximal (directed) branches of this tree. □

## 6.2 Dunwoody decompositions

Recall that a group $G$ is called accessible if there exists a reduced graph of groups $\mathbb{A}$ such that the following hold.

1. $\pi_1(\mathbb{A}) \cong G$.
2. Any edge group of $\mathbb{A}$ is finite.
3. Any vertex group of $\mathbb{A}$ is finite or one-ended.

We call any such $\mathbb{A}$ a Dunwoody decompositon of $G$. Note that the graph of groups $\mathbb{A}$ is far from being unique for a given accessible group $G$. However the maximal vertex groups are unique up to conjugacy; indeed the maximal infinite vertex groups are precisely the maximal one-ended subgroups of $G$.

Dunwoody's accessibility theorem [D1] states that all finitely presented groups are accessible. It turns out that f.g. groups are in general not accessible but the particular case that we will need is covered by the following theorem of P. Linnell ([L]).

**Theorem 6.5.** *Let $G$ be a f.g. group. Suppose that there exists some constant $C$ such that any finite subgroup $H$ of $G$ is of order at most $C$. Then $G$ is accessible.*

Now let $\Gamma$ be a hyperbolic group and $G$ a $\Gamma$-limit group. As the order of finite subgroups of $G$ is bounded by $N(\Gamma)$ it follows that Theorem 6.5 applies to $G$, i.e. $G$ admits a Dunwoody decomposition $\mathbb{D}$. As modular automorphisms of the vertex groups of $\mathbb{D}$ restrict to the identiy on all finite subgroups it follows that they extend to automorphisms of $G$.

In the following we call the subgroup of $\mathrm{Aut}(G)$ consisting of those automorphisms that restrict (up to conjugation) to modular automorphism of the vertex groups of $\mathbb{D}$ the modular group of $G$ and denote it by $\mathrm{Mod}(G)$. In the case of a one-ended group this recovers our original definition.

Let now $G$ be an accessible group and $\Gamma$ be a group. Then we call a homomorphism $\psi : G \to \Gamma$ *locally injective* if $\psi$ is injective when restricted to the vertex groups of some (and therefore all) Dunwoody decomposition of $G$. In particular, if $\psi$ is locally injective then it is injective when restricted to 1-ended subgroups of $G$.

## 6.3 MR-diagrams for weakly equationally Noetherian hyperbolic groups

In this section we give a proof of the main theorem of this thesis, i.e. the description of $\mathrm{Hom}(G, \Gamma)$ for some finitely generated group $G$ and some hyperbolic group $\Gamma$ under the additional assumption that $\Gamma$ is weakly equationally Noetherian. It will then be the purpose of chapter 7 to establish that all hyperbolic groups have this property.

**Theorem 6.6.** *Let $\Gamma$ be a weakly equationally Noetherian hyperbolic group and $G$ be a finitely generated group. Then there exists a finite directed rooted tree $T$ with root $v_0$ satisfying*

1. *The vertex $v_0$ is labeled by $G$.*
2. *Any vertex $v \in VT$, $v \neq v_0$, is labeled by a group $G_v$ that is fully residually $\Gamma$.*
3. *Any edge $e \in ET$ is labeled by an epimorphism $\pi_e : G_{\alpha(e)} \to G_{\omega(e)}$*

*such that for any homomorphism $\phi : G \to \Gamma$ there exists a directed path $e_1, \ldots, e_k$ from $v_0$ to some vertex $\omega(e_k)$ such that*

$$\phi = \psi \circ \pi_{e_k} \circ \alpha_{k-1} \circ \ldots \circ \alpha_1 \circ \pi_{e_1}$$

*where $\alpha_i \in Mod(G_{\omega(e_i)})$ for $1 \leq i \leq k$ and $\psi$ is locally injective.*

*Remark* 6.7. In case $G$ is fully residually $\Gamma$, the factorization of homomorphisms from $G$ to $\Gamma$ as in Theorem 6.6 requires modular automorphisms of $G$ before the first proper quotient map. Thus in this case the diagram has precisely one edge $e$ satisfying $\alpha(e) = v_0$, and $\pi_e : G \to G_{\omega(e)}$ is an isomorphism.

*Proof of Theorem 6.6.* In view of Lemma 6.4 and Corollary 6.2 it clearly suffices to show that any group $G$ that is fully residually $\Gamma$ admits a set $\{q_i : G \to \Gamma_i\}$ of proper quotient maps such that any homomorphism $\varphi : G \to \Gamma$ which is not locally injective factors through some $q_i$ after precomposition with an element of $\mathrm{Mod}(G)$.

If $G$ is one-ended then injective is the same as locally injective and the assertion is just Theorem 4.2. If $G$ is not one-ended then we choose a Dunwoody decomposition $\mathbb{D}$ of $G$. For each (one-ended) vertex group $D_v$ there is a factor set

$$S_v = \{q_i^v : D_v \to D_v/N_v^i\}$$

by Theorem 4.2. For each $q_i^v$ we denote by $Q_i^v : G \to G/N_v^i$ the quotient of $G$ by the kernel $N_v^i$ and define the factor set for $G$ to be

$$\{Q_i^v \,|\, v \in VA, q_i^v \in S_v\}.$$

To see that this is a factor set let $\varphi : G \to \Gamma$ be a non-locally injective homomorphism. Choose $v \in VA$ such that $\varphi|_{A_v}$ is non-injective. Thus there exists $\alpha \in \mathrm{Mod}(A_v)$ such that $\varphi \circ \alpha : A_v \to \Gamma$ factors through some $q_i^v$. As $\alpha$ extends to an automorphism $\alpha' \in \mathrm{Aut}(G)$ it follows that $\varphi \circ \alpha'$ factors through $Q_i^v$. $\square$

# Chapter 7

# Shortening quotients and applications

In the previous section we have constructed Makanin-Razborov diagrams for weakly equationally Noetherian hyperbolic groups. It is the purpose of this chapter to establish that all hyperbolic groups are weakly equationally Noetherian, i.e. that the construction of the Makanin-Razborov diagrams applies to all hyperbolic groups.

## 7.1 Shortening quotients

In chapter 4, see Remark 4.7, we have seen that if $(\varphi_i) \subset \mathrm{Hom}(G, \Gamma)$ is a stable sequence such that $\underrightarrow{\ker}(\varphi_i) = 1$, i.e. that $L = G/\underrightarrow{\ker}(\varphi_i) = G$, then we can construct a proper quotient $G/\underrightarrow{\ker}(\hat{\varphi}_i)$ of $G = L$ where the $\hat{\varphi}_i$ are the shortened $\varphi_i$. This quotient is clearly again a $\Gamma$-limit group and is called a shortening quotient.

This construction only works if $G$ is fully residually $\Gamma$. It is the main purpose of this section to construct shortening quotients for arbitrary $\Gamma$-limit groups. In the end, see Corollary 7.6, it will turn out that all $\Gamma$-limit groups are fully residually $\Gamma$. We will first treat one-ended Γ-limit groups and then deal with the general case.

Let $L = G/\underrightarrow{\ker}(\varphi_i)$ be a one-ended Γ-limit group and $\mathbb{A}$ be an almost abelian JSJ-decomposition of $L$. Lemma 7.1 below guarantees that we can approximate $L$ by a sequence of groups $(W_i)$ that are endowed with splittings that appoximate $\mathbb{A}$.

**Lemma 7.1.** *Let $G$ be a finitely presented group and $L = G/\underrightarrow{\ker}(\varphi_i)$ be a one-ended Γ-limit group with associated Γ-limit map $\varphi : G \to L$. Let $\mathbb{A}$ be an almost abelian JSJ-decomposition of $L$, in particular $L = \pi_1(\mathbb{A}, v_0)$.*

*Then there exists a sequence of graphs of groups $(\mathbb{A}^i)$ with underlying graph $A$ (the graph underlying $\mathbb{A}$) and finitely presented fundamental groups $W_i = \pi_1(\mathbb{A}^i, v_0)$, surjective morphisms $f^i : \mathbb{A}^i \to \mathbb{A}^{i+1}$ and $h^i : \mathbb{A}^i \to \mathbb{A}$ and an epimorphism $\gamma : G \to W_0$ such that the following hold.*

1. $\varphi = h^0_* \circ \gamma : G \to L$.
2. $h^i = h^{i+1} \circ f^i$ *for all* $i$.
3. $L$ *is the direct limit of the sequence* $W_i$, *i.e.*
$$\underrightarrow{\ker}(\varphi_i) = \bigcup_{k=1}^{\infty} \ker(f^k_* \circ f^{k-1}_* \circ \ldots \circ f^1_* \circ f^0_* \circ \gamma).$$
4. *If $A_v$ is an orbifold type vertex group then $\psi^{h^i}_v : A^i_v \to A_v$ is an isomorphism for all $i$.*
5. *If $A_v$ is of axial type then $\psi^{h^i}_v : A^i_v \to A_v$ is injective for all $i$.*
6. *The maps $\psi^{h^i}_e : A^i_e \to A_e$ are injective for all $i$ and $e \in EA$.*
7. *For any $v \in VA$ we have $\bigcup \psi^{h_i}_v(A^i_v) = A_v$.*
8. *For any $e \in EA$ we have $\bigcup \psi^{h_i}_e(A^i_e) = A_e$.*

*Proof.* This is a simple application of foldings as discussed in [BF3] and Dunwoody's vertex morphisms [D2]. Let $T = \tilde{\mathbb{A}}$ be the Bass-Serre tree corresponding to $\mathbb{A}$, thus $T$

is an $L$-tree. The Dunwoody Resolution Lemma guarantees that there is a $G$-tree $Y$ with finitely generated edge and vertex stabilizers and a surjective morphism $(id, p)$ from $Y$ to the $L$-tree $T$, see [D1, DF] or [BF3].

After applying finitely many folds to the $G$-tree $Y$ we obtain a $G$-tree $Y'$ such that the induced map on the graphs of groups $Y'/G \to T/L = \mathbb{A}$ is bijective on the level of graphs and surjective for the edge and vertex groups of $\mathbb{A}$ that are finitely generated, see [BF3].

We now apply vertex morphisms to quotient out the kernels of the homomorphisms of edge groups and of the vertex groups whose targets are QH-subgroups or almost abelian groups. This clearly adds only finitely many relations. Denote the resulting graph of groups by $\mathbb{A}_0$, the morphism from $\mathbb{A}_0$ to $\mathbb{A}$ clearly satisfies (4)-(6).

We can now continue to apply folds of type IIA and IIB, see [BF3], and get a sequence of graphs of groups satisfying (7) and (8). At each step we further add all relators to edge groups and vertex group mapped to almost abelian vertex groups which makes this sequence preserve properties (4)-(6). Finally we add at each step the shortest relator to the vertex groups that are mapped to rigid vertex groups that does not yet hold. This then implies that all relations of $L$ will hold eventually, i.e. that (3) holds. Part (1) and (2) hold by construction. $\square$

Let now $G$ be finitely presented with fixed finite generating set $S$, $(\varphi_i) \subset \text{Hom}(G, \Gamma)$ a stable sequence such that $L = G/\underrightarrow{\ker}(\varphi_i)$ is a one-ended $\Gamma$-limit group. Let $\mathbb{A}$ be an almost abelian JSJ-decomposition of $L$. Choose sequences $(\mathbb{A}_i)$, $(h^i)$ and $(f^i)$, $(W_i)$ and $\gamma$ as in Lemma 7.1.

Let

$$\xi_k := f_*^{k-1} \circ \ldots \circ f_*^2 \circ f_*^1 \circ \gamma : G \to W_k$$

be the epimorphism induced by the $f^i$ and $\gamma$. Then $S_i := \xi_i(S)$ is a generating set of $W_i$. After dropping finitely many $\varphi_i$ we can assume that $\varphi_i$ factors through $\xi_i$ for all

$i$, i.e. that

$$\varphi_i = \lambda_i \circ \xi_i$$

for some $\lambda_i : W_i \to \Gamma$. This is clearly possible as the finitely many defining relations of $W_i$ lie in $\underrightarrow{\ker}(\varphi_i)$ and therefore in the kernel of $\varphi_j$ for sufficiently large $j$ .

Let now $\hat{\lambda}_i : W_i \to \Gamma$ be the homomorphism obtained from shortening $\lambda_i$ by precomposition with elements of $\text{Mod}_{\mathbb{A}^i}(W_i)$ and postcomposition with an inner automorphism. Here shortness is measured with respect to the generating set $S_i$. We then put $\eta_i := \hat{\lambda}_i \circ \xi_i$. After passing to a subsequence we can assume that $(\eta_i)$ is stable. We then put

$$Q := G/\underrightarrow{\ker}(\eta_i)$$

and call $Q$ a shortening quotient of $L$. It is clear from the construction that $Q$ is a quotient of $L$. Indeed if $g \in \underrightarrow{\ker}(\varphi_i)$ then $g \in \ker \xi_i$ for large $i$ and therefore also $g \in \ker \eta_i = \ker \hat{\lambda}_i \circ \xi_i$ for large $i$, thus $g \in \underrightarrow{\ker}(\eta_i)$. We denote the projection from $L$ to $Q$ by $\pi$, thus we have $\eta = \pi \circ \varphi$ if $\eta$ and $\varphi$ are the $\Gamma$-limit maps associated to the sequences $(\eta_i)$ and $(\varphi_i)$.

**Proposition 7.2.** *Let $L = G/\underrightarrow{\ker}(\varphi_i)$ and $Q = G/\underrightarrow{\ker}(\eta_i)$ be as above and $\mathbb{A}$ be an almost abelian JSJ-decomposition of $L$. Then the following hold.*

1. *The epimorphism $\pi : L \to Q$ is injective on rigid vertex groups of $\mathbb{A}$.*

2. *If $(\eta_i)$ is not contained in finitely many conjugacy classes then $Q$ is a proper quotient of $L$.*

3. *If all almost abelian subgroups of $Q$ are finitely generated then the following hold:*

   (a) *If a subsequence of $(\eta_i)$ factors through $\eta : G \to Q$ then a subsequence of $(\varphi_i)$ factors through $\varphi : G \to L$.*

   (b) *Almost abelian subgroups of $L$ are finitely generated.*

*Proof.* (1) Let $g \in L$ be an element that is conjugate to an element $h$ of a rigid vertex group $A_v$ of $\mathbb{A}$ such that $\pi(g) = 1$. We need to show that $g = 1$.

For some $i_0$ there exists $g_{i_0} \in W_{i_0}$ that is conjugate to some $k_{i_0} \in A_v^{i_0}$ such that $h_*^{i_0}(g_{i_0}) = g$. Choose $\tilde{g} \in G$ such that $\xi_{i_0}(\tilde{g}) = g_{i_0}$ and put $g_i = \xi_i(\tilde{g})$ for all $i$. Note that for $i \geq i_0$ the element $g_i$ is conjugate to some element $k_i \in A_v^i$.

Now $\pi(g) = 1$ means that $\eta_i(\tilde{g}) = \hat{\lambda}_i \circ \xi_i(\tilde{g}) = \hat{\lambda}_i(g_i) = 1$ for large $i$, in particular any element conjugate to $g_i$ lies in the kernel of $\hat{\lambda}_i$. Recall that $\hat{\lambda}_i = \lambda_i \circ \alpha_i$ for some modular automorphism $\alpha_i$ of $W_i$. As $g_i$ is conjugate to the element $k_i \in A_v^i$ and as modular automorphisms act on rigid groups by conjugation it follows that $\alpha_i(g_i)$ is also conjugate to $k_i$ and therefore conjugate to $g_i$. Thus $\hat{\lambda}_i \circ \alpha_i(g_i) = \lambda_i(g_i) = 1$. This implies that $\varphi_i(\tilde{g}) = \lambda_i \circ \xi_i(\tilde{g}) = \lambda_i(g_i) = 1$ for large $i$, it follows that $\tilde{g} \in \underrightarrow{\ker}(\varphi_i)$. Thus $g = 1$.

(2) Assume to the contrary that $(\eta_i)$ contains infinitely many conjugacy classes and that $L = Q$, i.e. that $L = Q = G/\underrightarrow{\ker}(\eta_i)$. After passing to a subsequence we can assume that $(\eta_i)$ converges to an action on an $\mathbb{R}$-tree $T$ satisfying the assumptions of Theorem 2.4. Let now $\mathcal{G}$ be the graph of actions decomposition corresponding to this action. As in chapter 4 we distinguish 3 different cases.

If the graph of actions has an orbifold type vertex then there is an automorphism of its vertex group that extends to a modular automorphism $\alpha$ of $L$ which shortens the action of $L$ on $T$, in particular $\alpha \circ \hat{\lambda}_i$ is shorter than $\hat{\lambda}_i$ for large $i$. Now this orbifold type vertex group corresponds to a suborbifold of one of the orbifold type vertices of the JSJ-decomposition of $L$. Thus $\alpha$ can be lifted to any $W_i$ as the morphisms $h^i$ are isomorphisms when restricted to QH-subgroups. Thus there exists $\alpha_i \in \mathrm{Mod}(W_i)$ such that $\alpha \circ \hat{\lambda}_i = \hat{\lambda}_i \circ \alpha_i$. In particular $\hat{\lambda}_i \circ \alpha_i$ is shorter than $\hat{\lambda}_i$, contradicting the shortness of $\hat{\lambda}_i$.

If the action has an axial type vertex then we can choose $\alpha$ as in the case of an orbifold type vertex but the lifting is slightly more subtle. Note first that the vertex group corresponding to this axial vertex space is also a vertex group of the JSJ-decomposition of $L$. This is true as the group must be elliptic in the JSJ as it is an almost abelian subgroup that is not 2-ended. We can assume that it is a vertex group as we could otherwise refine the JSJ contradicting its maximality.

Now the morphisms $\psi_v^{h_i} : A_v^i \to A_v$ are not necessarily surjective on almost abelian vertex groups. However for large $i$ the group $\psi_v^{h_i}(A_v^i) \leq A_v$ contains all generators, and therefore all elements, of $A_v$ that act non-trivially on the axial tree. This means we can define an automorphism of $A_v^i$ that extends to an automorphism of $W_i$ such that $\eta_i \circ \alpha_i$ is shorter than $\eta_i$ by analyzing the action of $A_v^i$ on the axial tree via $\psi_v^{h_i}$. Note that this is easier than in chapter 4 as the group $A_v^i$ is finitely generated.

In the simplicial case the edge group along which the Dehn twist is performed either corresponds to an edge group of the JSJ or to a simple closed curve in a QH-subgroup of the JSJ. In both cases we can simply lift the Dehn twists to $W_i$ and thereby shorten the homomorphism $\eta_i$. In the case where the edge group corresponds to a simple closed curve of a QH-subgroup this is obvious, in the other case it follows as an element that is central in the edge group of the graph of actions is also central in the corresponding edge group of $\mathbb{A}^i$.

(3) Note first that any edge group of $\mathbb{A}$, the JSJ-decomposition of $L$, is contained in either a rigid or an orbifold type vertex group. It follows that all edge groups of $\mathbb{A}$ are finitely generated. Indeed if the edge group is contained in a rigid vertex group then it follows from (1) that the edge group embeds into $Q$ and is therefore finitely generated. Otherwise the edge group is virtually cyclic and the modular automorphisms act on the group by conjugation, it therefore follows as in the proof of (1) that it is embedded into $Q$ and is therefore finitely generated.

As $L = \pi_1(\mathbb{A})$ is finitely generated and all edge groups of $\mathbb{A}$ are finitely generated it follows that also all vertex groups of $\mathbb{A}$ are finitely generated. Thus there exist $i_0$ such that for $i \geq i_0$ the morphism $h^i$ is bijective on all edge groups and non-rigid vertex groups. On the rigid vertex groups $h^i$ is surjective, i.e. the morphism consists just of vertex morphisms in the sense of Dunwoody [D2]. As almost all $\xi_i$ factor through $W_{i_0}$ we can pass to a subsequence and assume that $i_0 = 0$.

Suppose now that $\eta_i$ factors through $Q$. For any $v \in VA$ denote the kernel of the map $\psi_v^{h_0} : A_v^0 \to A_v$ by $K_v$. As $\eta_i$ factors through $Q$ and therefore through $L$ it follows that $K_v \subset \ker \eta_i$ for any vertex group $A_v$. As the $\eta_i$ and the $\varphi_i$ only differ by precomposition with an automorphism that acts by conjugation on rigid vertex groups this implies that $K_v \subset \ker \varphi_i$ for all rigid vertex groups $A_v$. Thus $\varphi_i$ factors through $L$ as all other relations of $L$ already hold in $W_0$. The second assertion follows immediately from the proof. □

The above construction only works for one-ended Γ-limit groups as we need the existence of an almost abelian JSJ-decomposition. In the remainder of this section we will show that the concept of a shortening quotient generalizes naturally to all Γ-limit groups.

Let now $G$ be a finitely presented group and $(\varphi_i) \subset \mathrm{Hom}(G, \Gamma)$ be a stable sequence. Put $L := G/\underrightarrow{\ker}(\varphi_i)$ and denote the associated Γ-limit map by $\varphi$. Let $\mathbb{D}$ be the Dunwoody decomposition of $L$, i.e. $L = \pi_1(\mathbb{D}, v_0)$, all edge groups of $\mathbb{D}$ are finite and no vertex group splits over finite groups, i.e. every vertex group is either finite or one-ended.

As in the proof of Lemma 7.1 we see that there is a graph of groups $\mathbb{D}'$ whose underlying graph $D$ is the same graph that is underlying $\mathbb{D}$, a morphism $f : \mathbb{D}' \to \mathbb{D}$ and an epimorphism $\gamma : G \to \pi_1(\mathbb{D}', v_0)$ such that the following hold.

1. $\varphi = f_* \circ \gamma$.
2. $\pi_1(\mathbb{D}', v_0)$ is finitely presented.
3. The morphism $f$ is injective on edge groups, thus $f$ only consists of a collection of vertex morphisms on some vertices.

Now as $\pi_1(\mathbb{D}', v_0)$ is finitely presented almost all $\varphi_i$ factor through $\gamma$. Thus after omitting finitely many elements from $(\varphi_i)$ we can assume that for all $i$ there exists $\bar{\varphi}_i : \pi_1(\mathbb{D}', v_0) \to L$ such that $\varphi_i = \bar{\varphi}_i \circ \gamma$.

For each vertex $v \in VD$ we get a stable sequence $(\bar{\varphi}_i^v)$ where $\bar{\varphi}_i^v : D'_v \to \Gamma$ is the restriction of $\bar{\varphi}_i$ to $D'_v$. Note that this restriction is only unique up to inner automorphisms of $\Gamma$ unless we choose a preferred conjugate of $D'_v$ in $\pi_1(\mathbb{D}', v_0)$. Independently of these conjugacy factors the obtained sequence is stable for all $v \in VD$ and we have $D_v = D'_v / \underrightarrow{\ker}(\bar{\varphi}_i^v)$.

Now for every one-ended $D_v$ we can apply the construction of the shortening quotient to the sequence $(\bar{\varphi}_i^v)$ and obtain (after passing to a subsequence) a new stable sequence $(\bar{\eta}_i^v) \subset \mathrm{Hom}(D'_v, \Gamma)$ such that $\underrightarrow{\ker}(\bar{\varphi}_i^v) \leq \underrightarrow{\ker}(\bar{\eta}_i^v)$ and that all conclusions of Proposition 7.2 hold for the quotient map

$$\pi_v : D_v = D'_v / \underrightarrow{\ker}(\bar{\varphi}_i^v) \to Q_v := D'_v / \underrightarrow{\ker}(\bar{\eta}_i^v).$$

If $D_v$ is finite we put $\bar{\eta}_i^v = \bar{\varphi}_i^v$ for all $i$.

Now as the shortening automorphisms act on finite subgroups by conjugation it follows that for each $i$ there exists a (not unique) homomorphism $\bar{\eta}_i : \pi_1(\mathbb{D}', v_0) \to \Gamma$ such that the restriction of $\bar{\eta}_i$ to $D'_v$ is conjugate to $\bar{\eta}_i^v$ for all $v \in V$.

We put $\eta_i = \bar{\eta}_i \circ \gamma : G \to \Gamma$. After passing to a subsequence we can assume that $(\eta_i)$ is stable and we put $Q := G / \underrightarrow{\ker}(\eta_i)$. As $\underrightarrow{\ker}(\varphi_i) \leq \underrightarrow{\ker}(\eta_i)$ by construction we have a natural epimorphism $\pi : L \to Q$. As in the one-ended case it follows that $\eta = \pi \circ \varphi$ if $\eta$ and $\varphi$ are the $\Gamma$-limit maps associated to the sequences $(\eta_i)$ and $(\varphi_i)$, respectively.

It is clear that the epimorphism $\pi$ maps the vertex groups $D_v$ of $\mathbb{D}$ to subgroups of $Q$ that are isomorphic to their shortening quotients $Q_v$, but we do not claim that the $Q_v$ are vertex groups of the Dunwoody decomposition of $Q$. It is however clear that $\pi$ factors naturally through $\pi_1(\bar{\mathbb{D}}, v_0)$ where $\bar{\mathbb{D}}$ is obtained from $\mathbb{D}$ by replacing the $D_v$ by the $Q_v$, i.e. by quotienting out the kernels of the quotient maps $\pi_v : D_v \to Q_v$.

**Theorem 7.3.** *Let $G$ be a finitely presented group and $L = G/\underrightarrow{\ker}(\varphi_i)$ a $\Gamma$-limit group. Let $(\eta_i)$ be as above and $Q = G/\underrightarrow{\ker}(\eta_i)$. Let $\pi : L \to Q$ be the natural quotient map. Then one of the following holds.*

1. $\ker \pi \neq 1$.
2. *A subsequence of $(\eta_i)$ factors through $\eta$ and all almost abelian subgroups of $Q$ are finitely generated.*

*If moreover all almost abelian subgroups of $Q$ are finitely generated then the following hold.*

(a) *If a subsequence of $(\eta_i)$ factors through $\eta : G \to Q$ then a subsequence of $(\varphi_i)$ factors through $\varphi : G \to L$.*

(b) *Almost abelian subgroups of $L$ are finitely generated.*

*Proof.* We first prove (1). Assume that $\ker \pi = 1$, i.e. that $L = G/\underrightarrow{\ker}(\varphi_i) = G/\underrightarrow{\ker}(\eta_i) = Q$. Thus for each $v \in VD$, the epimorphism $\pi_v : D_v \to Q_v$ is an isomorphism, hence by Proposition 7.2, $(\bar{\eta}_i^v)$ contains only finitely many conjugacy classes. After passing to a subsequence we can assume that for each $v$, all $\bar{\eta}_i^v$ are conjugate, i.e. that $\underrightarrow{\ker}(\bar{\eta}_i^v) = \ker \bar{\eta}_i^v$ for all $i$ and all $v \in VD$, in particular $D_v \cong Q_v = G/\underrightarrow{\ker}(\bar{\eta}_i^v) \cong \bar{\eta}_i^v(G) \leq \Gamma$. As almost abelian subgroups of hyperbolic groups are 2-ended it follows that all almost abelian subgroups of vertex groups of $\mathbb{D}$ and therefore of $\pi_1(\mathbb{D}, v_0) = L = Q$ are finitely generated.

To see that a subsequence of $(\eta_i)$ factors through $\eta$ note first that all $\eta_i = \bar{\eta}_i \circ \gamma$ and therefore also the associated $\Gamma$-limit map $\eta$ factor through $\gamma$, choose $\bar{\eta}$ such that $\eta = \bar{\eta} \circ \gamma$. Now the kernel of $\bar{\eta}$ is normally generated by the stable kernels $\underrightarrow{\ker}(\bar{\eta}_i^v)$. By the above remark there is a subsequence of $(\bar{\eta}_i)$ for which $\underrightarrow{\ker}(\bar{\eta}_i^v) = \ker \bar{\eta}_i^v$ for all $v \in VD$ and $i$, it follows that this subsequence factor through $\eta$.

Now suppose that all almost abelian subgroups are finitely generated. The shortening quotient $Q_v$ of $D_v$ embeds into $Q$ for all $v$, thus all almost abelian subgroups of $Q_v$ are finitely generated. It thus follows from Proposition 7.2 that all almost abelian subgroups of $D_v$ and therefore also $L = \pi_1(D_v)$ are finitely generated, this proves (b). The proof of (a) is similar to the proof of the first part; it suffices to show for a subseqence of $(\varphi_i)$ we have $\underrightarrow{\ker}(\bar{\varphi}_i^v) \subset \ker \bar{\varphi}_i^v$. By assumption a subsequence of $(\eta_{j_i})$ of $(\eta_i)$ factors through $\eta$ which implies that the sequences $(\bar{\eta}_{j_i}^v)$ factor through $D'_v \to Q_v = D'_v/\underrightarrow{\ker}(\bar{\eta}_i^v)$ for all $v \in DV$. As almost abelian subgroups of $Q_v$ are finitely generated it follows from Proposition 7.2 that a subsequence of $(\bar{\varphi}_i^v)$ factors through $D'_v \to D_v = D'_v/\underrightarrow{\ker}(\bar{\varphi}_i^v)$ for all $v \in VD$, thus a subsequence of $(\bar{\varphi}_i)$ factors $\pi_1(\mathbb{D}', v_0) \to L = \pi_1(\mathbb{D}, v_0)$, i.e. a subsequence of $(\varphi_i) = (\bar{\varphi}_i \circ \gamma)$ factors through $\varphi : G \to L$. □

If $L = G/\underrightarrow{\ker}(\varphi_i)$ and $Q = G/\underrightarrow{\ker}(\eta_i)$ are as in Theorem 7.3 then we call $Q$ a *shortening quotient* of $L$ and we say that $(\eta_i)$ is obtained from $(\varphi_i)$ by shortening or by the shortening procedure. It will be important in the next section that $\eta$ factors through $\varphi$ if $\eta$ and $\varphi$ are the $\Gamma$-limit maps corresponding to $(\eta_i)$ and $(\varphi_i)$.

## 7.2 Hyperbolic groups are weakly equationally Noetherian

In this chapter we show that hyperbolic groups are weakly equationally Noetherian. We fix a hyperbolic group $\Gamma$. Crucial to the argument is a partial order on $\Gamma$-limit maps as follows.

**Definition 7.4.** Let $G$ be f.g. and $\varphi : G \to L_\varphi$, $\eta : G \to L_\eta$ be $\Gamma$-limit maps. We say that $\eta \leq \varphi$ if $\eta = \pi \circ \varphi$ for some epimorphism $\pi : L_\varphi \to L_\eta$. We further say $\eta < \varphi$ if $\eta \leq \varphi$ and $\varphi \not\leq \eta$.

Fix a f.g. group $G$. By definition, every homomorphism $\eta : G \to \Gamma$ is a $\Gamma$-limit map, arising from the constant sequence $(\eta)$. Note further that the relation $\leq$ on the set of all $\Gamma$-limit maps from $G$ is transitive. We will show that there are only finitely many maximal $\Gamma$-limit maps with respect to $\leq$. The main technical step is the proof of the following theorem.

**Theorem 7.5.** *Let $(\varphi_i) \subset Hom(G, \Gamma)$ be a stable sequence and $\varphi : G \to G/\underrightarrow{\ker}(\varphi_i)$ the corresponding $\Gamma$-limit map. Then a subsequence of $(\varphi_i)$ factors through $\varphi$.*

As an immediate consequence of Theorem 7.5 we get the following.

**Corollary 7.6.** *$\Gamma$-limit groups are fully residually $\Gamma$.*

*Proof.* Let $G$ be f.g., $(\eta_i) \subset \mathrm{Hom}(G, \Gamma)$ a stable sequence with $\Gamma$-limit map $\eta : G \to L := G/\underrightarrow{\ker}(\eta_i)$. Let $E = \{g_1, \ldots, g_k\} \subset L$, we need to show that there is a homomorphism from $L$ to $\Gamma$ that maps $E$ injectively.

Choose $\tilde{E} = \{\tilde{g}_1, \ldots, \tilde{g}_k\} \subset G$ such that $\eta(\tilde{g}_j) = g_j$ for $j = 1, \ldots, k$. As $\eta|_{\tilde{E}}$ is injective, there exists an $i_0 \in \mathbb{N}$ such that for $i \geq i_0$, $\eta_i|_{\tilde{E}}$ is injective. Moreover, by Theorem 7.5, there is an $i \geq i_0$ such that $\eta_i = \bar{\eta}_i \circ \eta$ for some $\bar{\eta}_i \in \mathrm{Hom}(L, \Gamma)$. Clearly, $\bar{\eta}_i|_E$ is injective. □

Theorem 7.5 is an immediate consequence of Lemma 7.7 and Lemma 7.8.

**Lemma 7.7.** *Let $(\varphi_i) \subset Hom(F_k, \Gamma)$ be a stable sequence and $\varphi$ its associated $\Gamma$-limit map. Then one of the following holds.*

1. *There exists an infinite descending sequence of $\Gamma$-limit maps*

$$\varphi > \eta^1 > \eta^2 > \eta^3 > \ldots .$$

2. *For infinitely many $i$, $\varphi_i$ factors through $\varphi$.*

*Proof.* Assume that (1) does not hold. Let $(\eta_i^1)$ be a stable sequence obtained from $(\varphi_i)$ by shortening (and passing to a subsequence) and let $\eta^1$ be the corresponding Γ-limit map. If $\eta_1 < \varphi$ then we choose a sequence $(\eta_i^2)$ with Γ-limit map $\eta^2$ by shortening $(\eta_i^1)$ and so on. By assumption this process terminates, i.e. for some $s$ we have $\ker\, \eta^s = \ker\, \eta^{s+1}$.

By Theorem 7.3 (2) a subsequence of $(\bar{\eta}_i^{s+1})$ factors through $\eta^{s+1}$ and all almost abelian subgroups of $L_{\eta^{s+1}}$ are finitely generated. Applying Theorem 7.3 (a) and (b) $s+1$ times implies that a subsequence of $(\varphi_i)$ factors through $\varphi$, i.e. that situation 2 occurs. □

**Lemma 7.8.** *There exists no infinite descending sequence of* Γ*-limit maps.*

*Proof.* Assume that an infinite descending sequence of Γ-limit maps exists. For each $k \in \mathbb{N}$, choose a stable sequence $(\eta_i^k) \subset \mathrm{Hom}(G, \Gamma)$ with associated Γ-limit map $\eta^k : G \to G/\underrightarrow{\ker}(\eta_i^k)$ such that

1. $\eta^1 > \eta^2 > \ldots$ is an infinite descending sequence of Γ-limit maps,
2. for each $n > 1$, if $\bar{\eta}^n$ is a Γ-limit map such that $\bar{\eta}^n < \eta^{n-1}$ and there is an infinite descending sequence $\eta^{n-1} > \bar{\eta}^n > \ldots$ of Γ-limit maps, then
$$|\ker\, \bar{\eta}^n \cap B_n| \leq |\ker\, \eta^n \cap B_n|,$$
where $B_n$ is the Ball of radius $n$ in $G$ around the identity with respect to some fixed finite generating set.

It is clear that such a sequence exists, as the $\eta^i$ can be chosen inductively to satisfy property 2. For each $n$ choose an index $i_n$ such that

3. $\ker\, \eta_{i_n}^n \cap B_n = \ker\, \eta^n \cap B_n$,
4. $\ker\, \eta^{n+1} \not\leq \ker\, \eta_{i_n}^n$.

As $\eta^{n+1} < \eta^n$, it is clear that these conditions are satisfied if $i_n$ is chosen sufficiently large.

By construction, the diagonal sequence $(\eta^n_{i_n})_{n\in\mathbb{N}} \subset \mathrm{Hom}(G,\Gamma)$ is stable. Denote its $\Gamma$-limit map by $\eta^\infty$, clearly $\eta^\infty < \eta^n$ for all $n$.

It suffices to show that $\eta^\infty$ does not allow an infinite descending sequence of $\Gamma$-limit maps

$$\eta^\infty > \varphi^1 > \varphi^2 > \dots$$

as it follows then from Lemma 7.7 that infinitely many $\eta^n_{i_n}$ factor through $\eta^\infty$, which clearly contradicts condition 4 of the construction, as $\ker\,\eta^{n+1} \not\leq \ker\,\eta^n_{i_n}$ implies that $\eta^n_{i_n}$ does not factor through $\eta^{n+1}$ and therefore not through $\eta^\infty$.

So assume that an infinite descending sequence $\eta^\infty > \varphi^1 > \varphi^2 > \dots$ exists. Choose an element $g \in G$ with $\eta^\infty(g) \neq 1$, but $\varphi^1(g) = 1$. Assume that $|g| = n$. Then

$$\eta^1 > \eta^2 > \dots > \eta^{n-1} > \varphi^1 > \varphi^2 \dots$$

is an infinite descending sequence of $\Gamma$-limit maps and

$$|\ker\,\varphi^1 \cap B_n| > |\ker\,\eta^n \cap B_n|,$$

in contradiction to condition 2. □

**Corollary 7.9.** *Hyperbolic groups are Hopfian.*

*Proof.* Let $\Gamma$ be a hyperbolic group. We need to show that any epimorphism $\eta : \Gamma \to \Gamma$ is an isomorphism, i.e. has trivial kernel.

Note that $\eta^n : \Gamma \to \Gamma$ (the $n$th power of $\eta$) is also an epimorphism. If $\eta$ has non-trivial kernel then $\ker\,\eta^{n+1} < \ker\,\eta^n$ for all $n$. Thus we have an infinite sequence

$$id > \eta > \eta^2 > \eta^3 > \dots$$

of $\Gamma$ limit maps (recall that all homomorphisms to $\Gamma$ are $\Gamma$-limit maps coming from the constant sequence), a contradiction to Lemma 7.8. □

We can now establish the existence of maximal Γ-limit quotients.

**Theorem 7.10.** *Let*

$$\eta^1 < \eta^2 < \eta^3 < \dots$$

*be an infinite ascending sequence of* Γ*-limit maps. There exists a* Γ*-limit map* $\eta$ *such that for every* $n \in \mathbb{N}$*,* $\eta^n < \eta$*.*

*Proof.* Assume that for each $n$, $\eta^n$ is the Γ-limit map of $(\eta_i^n)_{i \in \mathbb{N}}$. Choose an index $i_n$ such that $\ker\, \eta_{i_n}^n \cap B_n = \ker\, \eta^n \cap B_n$.

By construction, the sequence $(\eta_{i_n}^n)_{n \in \mathbb{N}}$ is stable. Denote its associated Γ-limit map by $\eta$. We claim that for every $n$, $\eta^n < \eta$. Assume that for some $n_0$, $\eta^{n_0} \not< \eta$. Then there is an element $g \in \ker\, \eta$ such that $g \notin \ker\, \eta^{n_0}$. It follows that $g \notin \ker\, \eta^n$ for all $n \geq n_0$, and so for each $n \geq \max\{n_0, |g|\}$, $g \notin \ker\, \eta_{i_n}^n$. This implies that $g \notin \ker\, \eta$, a contradiction. □

**Theorem 7.11.** *Let* $G$ *be f.g. There are only finitely many maximal* Γ*-limit maps from* $G$*.*

*Proof.* The proof is by contradiction. If there are infinitely many maximal Γ-limit maps then it is easily verified that there is a sequence $(\eta^i)$ of pairwise distinct maximal Γ-limit maps such that for each $j, k \geq i$ we have

$$\ker\, \eta^j \cap B_i = \ker\, \eta^k \cap B_i,$$

where $B_i$ is the Ball of radius $i$ in $G$ around the identity with respect to some fixed finite generating set.

For each $i$ choose $\eta_i : G \to \Gamma$ such that $\ker\, \eta^i \cap B_i = \ker\, \eta_i \cap B_i$. The sequence $(\eta^i)$ is clearly stable. Let $\eta : G \to G/\underrightarrow{\ker}(\eta_i)$ be the corresponding Γ-limit map.

After possibly removing a single $\eta^i$ from the sequence we can assume that $\eta \not\geq \eta^i$ for all $i$ as we would otherwise get a contradiction to the maximality of the $\eta^i$. Thus for each $i$ there exists $g_i \in G$ such that $\eta^i(g_i) \neq 1$ and $\eta(g_i) = 1$.

It follows that there exists a stable sequence $(\varphi_i) \subset \mathrm{Hom}(G, \Gamma)$ such that for each $i$ we have $\ker \varphi_i \cap B_i = \ker \eta^i \cap B_i$ and $\varphi_i(g_i) \neq 1$, in particular no $\varphi_i$ factors through $\eta$ as $\eta(g_i) = 1$.

Let $\varphi : G \to G/\underrightarrow{\ker}(\varphi_i)$ be the associated $\Gamma$-limit map, we clearly get $\varphi = \eta$. By Theorem 7.5 a subsequence of $(\varphi_i)$ factors through $\varphi = \eta$, a contradiction. □

**Lemma 7.12.** *Let $\Gamma$ be a hyperbolic group and $\varphi : F_k \to H$ be an epimorphism.*

*Assume that $S \subset F_k$ is such that for every finite $S_0 \subset S$,*

$$\mathrm{rad}(\ker\, \varphi \cup S) \subsetneq \mathrm{rad}(\ker\, \varphi \cup S_0).$$

*Then there is a $\Gamma$-limit map $\eta : F_k \to F_k/\underrightarrow{\ker}(\eta_i)$ such that* $\ker \varphi \leq \ker \eta$ *and that*

$$\mathrm{rad}(\ker\, \eta \cup S) \subsetneq \mathrm{rad}(\ker\, \eta \cup S_0)$$

*for every finite $S_0 \subset S'$.*

*Proof.* By Theorem 7.11, there are only finitely many maximal $\Gamma$-limit maps

$$\varphi_1, \ldots, \varphi_k : H \to H_i.$$

Put $\eta_i = \varphi_i \circ \varphi$ for $1 \leq i \leq k$. Note that

$$\bigcup \mathrm{rad}(\ker\, \eta_i) = \mathrm{rad}(\ker\, \varphi)$$

as any homomorphism from $F_k$ to $\Gamma$ that factors through $\varphi$ must factor through some maximal $\Gamma$-limit map and therefore through some $\eta_i$.

Assume that for each $i$ there is a finite set $S_0^i \subset S$ such that $\mathrm{rad}(\ker\, \eta_i \cup S_0^i) =$

$\mathrm{rad}(\ker \eta_i \cup S)$. Putting $S_0 := \bigcup S_0^i$, we get

$$\begin{aligned}
\mathrm{rad}(\ker \varphi \cup S_0) &= \bigcup \mathrm{rad}(\ker \eta_i \cup S_0) \\
&= \bigcup \mathrm{rad}(\ker \eta_i \cup S) \\
&= \mathrm{rad}(S) \cap \bigcup \mathrm{rad}(\ker \eta_i) \\
&= \mathrm{rad}(S) \cap \mathrm{rad}(\ker \varphi) \\
&= \mathrm{rad}(S \cup \ker \varphi),
\end{aligned}$$

which is a contradiction. Thus for some $i_0$ such a set $S_0^{i_0}$ does not exist and the conclusion follows by putting $\eta = \eta_{i_0}$. □

**Corollary 7.13.** *Hyperbolic groups are weakly equationally Noetherian.*

*Proof.* Assume that $\Gamma$ is hyperbolic, $k \in \mathbb{N}$ and $S = \{w_1, w_2, \ldots\} \subset F_k$ such that $\mathrm{rad}(S) \subsetneq \mathrm{rad}(S_0)$ for every finite $S_0 \subset S$. We show that this implies the existence of an infinite descending sequence of $\Gamma$-limit maps, contradicting Lemma 7.8. Let $\varphi_1 : F_k \to F_k/\langle\langle w_1 \rangle\rangle$ and $\eta_1$ a $\Gamma$-limit map with $\ker \varphi_1 \leq \ker \eta_1$ as in Lemma 7.12. Then inductively for each $i$, pick $w_{j_i} \in S \setminus \ker \eta_{i-1}$ and put

$$\varphi_i : F_k/\langle\langle \ker \eta_i \cup w_{j_i} \rangle\rangle$$

and apply Lemma 7.12 to obtain a $\Gamma$-limit map $\eta_i$ with $\ker \varphi_i \leq \ker \eta_i$. Then all $\eta_i$ are $\Gamma$-limit maps and

$$\eta_1 > \eta_2 > \eta_3 > \ldots .$$

□

# Chapter 8

# Systems of equations in hyperbolic groups

In the previous section we have shown that hyperbolic groups are weakly equationally noetherian, i.e. every system of equations in a hyperbolic group $\Gamma$ without constants is equivalent to a finite subsystem. In this section we will briefly illustrate how constants affect the situation and that now new techniques are needed to deal with them.

An equation in $\Gamma$ in the variables $x_1, \dots, x_m$ (with constants) is of the form

$$\gamma_1 x_{i_1}^{\pm 1} \gamma_2 x_{i_2}^{\pm 1} \dots \gamma_k x_{i_k}^{\pm 1} \gamma_{k+1} = 1,$$

where $\gamma_i \in \Gamma$ and $j_i \in \{1, \dots, m\}$. Thus an equation is associated to an element $\eta \in \Gamma[x_1, \dots, x_n]$. In other words, a system of equations in $\Gamma$ is a subset $S \subset \Gamma[x_1, \dots, x_n]$, and a solution of the system $S$ corresponds to a homomorphism

$$\varphi : \Gamma[x_1, \dots, x_n]/\langle\langle S \rangle\rangle \to \Gamma,$$

satisfying $\varphi(\pi(\gamma)) = \gamma$ for each $\gamma \in \Gamma$, where

$$\pi : \Gamma[x_1, \dots, x_n] \to \Gamma[x_1, \dots, x_n]/\langle\langle S \rangle\rangle$$

is the canonical quotient map. Clearly, if $S$ has any solutions then $\pi$ is injective when restricted to $\Gamma$. The target is to give a description of the set of all such homomorphisms. We approach this setup more formally in the following section.

## 8.1 Restricted $\Gamma$-limit groups

Let $G$ be a f.g. group and $\zeta : \Gamma \hookrightarrow G$ an embedding of $\Gamma$ into $G$. We define

$$\mathrm{Hom}_\zeta(G, \Gamma) := \{\varphi \in \mathrm{Hom}(G, \Gamma) \mid \varphi \circ \zeta = \mathrm{id}_\Gamma\}.$$

Our goal is to construct a restricted Makanin-Razborov diagram which encodes precisely these restricted homomorphisms. For this purpose we define restricted $\Gamma$-limit groups.

**Definition 8.1.** Let $G$ be a group, $\zeta : \Gamma \hookrightarrow G$ an embedding and $N \trianglelefteq G$. The projection $\varphi : G \to G/N$ is called a *$\zeta$-restricted $\Gamma$-limit map* if there exists a stable sequence $(\varphi_i) \subset \mathrm{Hom}_\zeta(G, \Gamma)$ such that $N = \underrightarrow{\ker}(\varphi_i)$. If $\varphi$ is a $\zeta$-restricted $\Gamma$-limit map then its image, denoted $L_\varphi$, is called a *$\zeta$-restricted $\Gamma$-limit group.*

Clearly, a $\zeta$-restricted $\Gamma$-limit map $\varphi$ is injective on $\zeta(\Gamma)$. Therefore a $\zeta$-restricted $\Gamma$-limit group $L = L_\varphi$ comes with a natural embedding $\varphi \circ \zeta$ of $\Gamma$, which we denote by $\zeta_\varphi$ for brevity.

Now assume that $(\varphi_i) \subset \mathrm{Hom}_\zeta(G, \Gamma)$ is a stable sequence of pairwise distinct homomorphisms with $\zeta$-restricted $\Gamma$-limit map $\varphi : G \to L_\varphi$. Again, we obtain a limit action of $G$ on a real tree $T$ by Lemma 1.4. Note that in the general case, this limit action may not be non-trivial unless the homomorphisms $\varphi_i$ are short with respect to postcomposition by inner automorphisms of $\Gamma$ (cf. Theorem 1.9). However, in the restricted case we are not free to shorten the homomorphisms by postcomposing with arbitrary inner automorphisms of $\Gamma$, as they do not preserve $\mathrm{Hom}_\zeta(G, \Gamma)$. But the following lemma shows that if $L_\varphi$ is $\zeta$-restricted, the limit action will be non-trivial if the automorphisms are short with respect to postcomposition of conjugation by elements of $Z(\Gamma)$.

**Lemma 8.2.** *Let $\Gamma$ be hyperbolic, $G$ a f.g. group and $\zeta : \Gamma \hookrightarrow G$ be an embedding. If $(\varphi_i) \subset Hom_\zeta(G, \Gamma)$ is a sequence of pairwise distinct homomorphisms, then there exists a based real $G$-tree $T$ such that the induced sequence $\left(\frac{1}{|\varphi_i|} d_{\varphi_i}\right)$ of (scaled) pseudo-metrics on $G$ has a subsequence converging to $d_{G,T}$. If moreover for each $i \in \mathbb{N}$ and $g \in Z(\Gamma)$*

$$|i_g \circ \varphi_i| \geq |\varphi_i|, \tag{8.1}$$

*then the $G$-action on $T$ is non-trivial.*

*Proof.* The existence of $T$ is provided by Lemma 1.4, so it remains to show that the action on $T$ is non-trivial if (8.1) holds.

Assume first that $\Gamma$ is finite-by-abelian. As $\Gamma$ is f.g., the center $Z(\Gamma)$ is of finite index in $\Gamma$, and the argument is almost identical to the one that proves Theorem 1.9.

So we assume that $\Gamma$ is not finite-by-abelian. Assume further (after passing to a subsequence) that $(\varphi_i)$ is stable with associated $\zeta$-restricted $\Gamma$-limit map $\varphi$. By construction, for each $g \in \zeta(\Gamma)$ we get that

$$\lim_{i\to\infty} \frac{1}{|\varphi_i|} d_{\varphi_i}(1, g) = 0,$$

which implies that $\zeta(\Gamma)$ fixes the basepoint $x_0$ of $T$ in the limit action. But $x_0$ is not a globally fixed point by construction. Assume that $x_1 \in T$ is a globally fixed point. Then $\zeta(\Gamma)$ fixes the non-degenerate segment $[x_0, x_1] \subset T$. It follows from Theorem 1.11 that the image $\zeta_\varphi(\Gamma) \leq L_\varphi$ is finite-by-abelian. But $\zeta_\varphi$ is injective, thus $\Gamma$ is finite-by-abelian, a contradiction. □

As in the non-restricted case (cf. Remark 1.5), the above $G$-action induces an action of the $\zeta$-restricted $\Gamma$-limit group $L = L_\varphi$ on $T$. Note that a $\zeta$-restricted $\Gamma$-limit group is a $\Gamma$-limit group and the stability assertions of Theorem 1.11 hold. Thus we can apply Theorem 2.4 to $T$.

Assume in the following that $L$ does not admit any splitting along a finite subgroup in which $\zeta(\Gamma)$ is elliptic. This case is the natural analogue of the one-ended $\Gamma$-limit groups in the non-restricted case, we will later deal with arbitrary $\zeta$-restricted $\Gamma$-limit groups.

Under this assumption, in particular $L$ does not split over the stabilizer of an unstable arc of $T$, as $\zeta_\varphi(\Gamma)$ acts with a fixed point on $T$ and would therefore be elliptic in such a splitting. It follows from Theorem 2.4 that $L$ splits as a metric graph of actions, and we obtain a corresponding graph of groups splitting $\mathbb{A}$ of $L$ in which $\zeta_\varphi(\Gamma)$ is elliptic.

Let $L = \pi_1(\mathbb{A}, v_0)$, where $v_0$ is chosen to be the vertex such that $\zeta_\varphi(\Gamma)$ is conjugate into $A_{v_0}$. If $\Gamma$ is not 2-ended and therefore not almost abelian, $v_0$ cannot be axial. Subgroups of orbifold type vertex groups act without fixed points unless they are contained in edge groups of adjacent edges, which are again almost abelian. It follows that $v_0$ is a rigid vertex of $\mathbb{A}$. Moreover, we assume without loss of generality that $\zeta_\varphi(\Gamma) \leq [A_{v_0}]$.

Now all the shortening automorphisms constructed in section 4.2 are either Dehn twists or natural extensions of vertex group automorphisms with respect to the base vertex $v_0$, and therefore the identity when restricted to $[A_{v_0}]$. In particular, they fix $\zeta_\varphi(\Gamma)$ pointwise and thus preserve $\mathrm{Hom}_\zeta(G, \Gamma)$ under precomposition. Denoting the set of those automorphisms by $\mathrm{Aut}_\zeta(G)$, the shortening argument now yields a slightly modified result about the existence of $\Gamma$-factor sets as in Lemma 8.3 below. In the following we say that a splitting $\mathbb{A}$ of a $\zeta$-restricted $\Gamma$-limit group is *$\zeta$-restricted* if $\zeta(\Gamma)$ is elliptic in $\mathbb{A}$.

**Lemma 8.3.** *Let $G$ be f.g., $\Gamma$ hyperbolic and $\zeta : \Gamma \hookrightarrow G$ an embedding. Assume that $G$ does not admit any $\zeta$-restricted splittings over finite groups. Then there is a finite set of proper quotient maps $\{q_i : G \to \Gamma_i\}$ such that each $q_i$ is injective when restricted to $\zeta(\Gamma)$, and for each non-injective homomorphism $q \in Hom_\zeta(G, \Gamma)$, there exists $\alpha \in Aut_\zeta(G)$ such that $q \circ \alpha$ factors through some $q_i$. We call this set a $\zeta$-restricted $\Gamma$-factor set.*

By the same construction as in chapter 3, we obtain a *$\zeta$-restricted almost abelian*

*JSJ-decompsition* of $L$ as follows.

**Definition 8.4.** Let $L$ be a $\zeta$-restricted $\Gamma$-limit group which does not admit any $\zeta$-restricted splitting over finite subgroups. Let $\mathbb{A}$ be an almost abelian compatible $\zeta$-restricted splitting of $L$. Then $\mathbb{A}$ is called a $\zeta$-restricted almost abelian JSJ-decomposition of $L$ if the following hold.

1. Every $\zeta$-restricted splitting over a 2-ended group that is hyperbolic-hyperbolic with respect to another $\zeta$-restricted splitting over a 2-ended group is geometric with respect to a QH-subgroup of $\mathbb{A}$.
2. Any edge group of $\mathbb{A}$ that can be unfolded to be finite-by-abelian is finite-by-abelian.
3. For any almost abelian vertex group $A_v$, the rank of $A_v^+/\bar{P}_v$ cannot be increased by unfoldings.
4. $\mathbb{A}$ is in normal form and of maximal complexity amongst all $\zeta$-restricted splittings satisfying (1)-(3).

As in the non-restricted case (cf. Theorem 3.12), the $\zeta$-restricted JSJ-decomposition is unique up to edge slides and boundary slides, and any $\zeta$-restricted splitting $\mathbb{B}$ whose QH-subgroups are elliptic in the JSJ, is visible in it after unfoldings, foldings and edge slides.

To a $\zeta$-restricted almost abelian graph of groups decomposition $\mathbb{A}$ of $L$ we define a *$\zeta$-restricted modular group.*

**Definition 8.5.** Let $L = L_\varphi$ be a $\zeta$-restricted $\Gamma$-limit group which does not admit any $\zeta$-restricted splitting over finite subgroups. Assume that $L = \pi_1(\mathbb{A}, v_0)$, where $\zeta_\varphi(\Gamma) \leq [A_{v_0}]$. Then $\mathrm{Mod}_{\mathbb{A},\zeta}(L)$ is the subgroup of $\mathrm{Aut}(L)$ generated by

1. conjugation by elements of $Z(\zeta_\varphi(\Gamma))$,

2. Dehn twists along any edge $e \in EA$ by an element of $Z(MA(A_e))$ if $A_e$ is finite-by-abelian,

3. natural extensions of geometric automorphisms of vertex groups of QH-vertices $v \neq v_0$,

4. natural extensions of automorphisms of vertex groups of almost abelian subgroups $v \neq v_0$.

As the restricted modular group does not contain arbitrary inner automorphisms and the natural extensions of vertex automorphisms depend on the base vertex, the choice of $v_0$ as base vertex is crucial, as it ensures that $\zeta_\varphi(\Gamma) \leq [A_{v_0}]$ is fixed by $\mathrm{Mod}_{\mathbb{A},\zeta}(L)$, and therefore $\mathrm{Mod}_{\mathbb{A},\zeta}(L) \leq \mathrm{Aut}_\zeta(L)$.

## 8.2 Hyperbolic groups are equationally Noetherian

Let $L = L_\varphi$ be a $\zeta$-restricted $\Gamma$-limit group. If $L$ does not admit any $\zeta$-restricted splittings over finite groups, we may construct a *$\zeta$-restricted shortening quotient* similarly to the construction of shortening quotients of one-ended $\Gamma$-limit groups in chapter 7.1, using the $\zeta$-restricted modular group instead of the non-restricted modular group of $L$.

If $L$ does admit $\zeta$-restricted splittings over finite groups, we generalize the construction of the $\zeta$-restricted shortening quotient in the following way. We denote by $\mathbb{D}_\zeta$ the *$\zeta$-restricted Dunwoody-decomposition* of $L$, satisfying the following.

1. $\mathbb{D}_\zeta$ is $\zeta$-restricted.

2. All edge groups are finite.

3. If $v_0$ is the vertex such that $\zeta(\Gamma)$ is conjugate into $D_{v_0}$, then $D_{v_0}$ does not admit any $\zeta$-restricted splitting over finite groups.
4. For each vertex $v \neq v_o$, $D_v$ is one-ended.

Now choose a vertex $v \in VD_\zeta$. If $v = v_0$, construct a $\zeta$-restricted shortening quotient of $A_v = A_{v_0}$ as introduced above, otherwise construct a (non-restricted) shortening quotient of $A_v$. In either way, we extend this to a $\zeta$-restricted shortening quotient of $L$ in the same way as in the non-restricted case.

Note that a $\zeta$-restricted shortening quotient $Q$ of $L$ is again a $\zeta$-restricted $\Gamma$-limit group. Although $Q$ is not necessarily a shortening quotient in the non-restricted sense because homomorphisms which are shortened with respect to $\mathrm{Hom}_\zeta(G, \Gamma)$ may not be short with respect to $\mathrm{Hom}(G, \Gamma)$, we obtain entirely analogous results as in the non-restricted case. In particular, Theorem 7.3 holds in the case where $L$ is $\zeta$-restricted and $Q$ is a $\zeta$-restricted shortening quotient of $L$, the proof is identical.

For a given f.g. group $G$ with embedding $\zeta : \Gamma \hookrightarrow G$, we restrict the partial order defined in Definition 7.4 to the set of $\zeta$-restricted $\Gamma$-limit maps from $G$.

Theorem 7.5 and Lemma 7.7 trivially hold for $\zeta$-restricted $\Gamma$-limit maps as they are $\Gamma$-limit maps, and in the same fashion as in the non-restricted case (cf. Theorems 7.10 and 7.11) we prove that the set of all $\zeta$-restricted $\Gamma$-limit maps which are smaller than $\varphi$ has finitely many maximal elements. We obtain the analogous result of Lemma 7.12 and, similarly to Corollary 7.13, the consequence that hyperbolic groups are equationally Noetherian.

For the conclusion of this chapter, it follows the existence of a $\zeta$-restricted Makanin-Razborov diagram for a given f.g. group $G$ with an embedding $\zeta$ of a hyperbolic group $\Gamma$ in $G$, describing $\mathrm{Hom}_\zeta(G, \Gamma)$.

**Theorem 8.6.** *Let $\Gamma$ be a hyperbolic group, $G$ be a finitely generated group and $\zeta : \Gamma \hookrightarrow G$ an embedding. Then there exists a finite directed rooted tree $T$ with root $v_0$ satisfying*

*1. The vertex $v_0$ is labeled by $(G, \zeta)$,*

*2. Any vertex $v \in VT$, $v \neq v_0$ is labeled by a pair $(G_v, \zeta_v)$ of a fully residually $\Gamma$ group $G_v$ and an embedding $\zeta_v : \Gamma \hookrightarrow G_v$,*

*3. Any edge $e \in ET$ is labeled by an epimorphism $\pi_e : G_{\alpha(e)} \to G_{\omega(e)}$ satisfying $\pi_e \circ \zeta_{\alpha(e)} = \zeta_{\omega(e)}$,*

*such that for any homomorphism $\phi \in Hom_\zeta(G, \Gamma)$ there exists a directed path $e_1, \ldots, e_k$ from $v_0$ to some vertex $\omega(e_k)$ such that*

$$\phi = \psi \circ \pi_{e_k} \circ \alpha_{k-1} \circ \ldots \circ \alpha_1 \circ \pi_{e_1}$$

*where $\alpha_i \in Aut_{\zeta_{\omega(e_i)}}(G_{\omega(e_i)})$ for $1 \leq i \leq k$ and $\psi$ is locally injective.*

*Proof.* Having shown that hyperbolic groups are equationally Noetherian, we are left to construct a set $\{q_i : G \to \Gamma_i\}$ of proper quotient maps such that any non-locally injective $\zeta$-restricted homomorphism $\varphi : G \to \Gamma$ factors through some $q_i$.

Choose a $\zeta$-restricted Dunwoody decomposition $G = \pi_1(\mathbb{D}, v_0)$ where $\zeta(\Gamma) \leq [A_{v_0}]$. For each vertex group $v \in VA$ fix a set of quotients

$$S_v = \{q_i^v : A_v \to A_v/N_v^i\}$$

such that $S_v$ is a $\Gamma$-factor set of $A_v$ if $v \neq v_0$ and a $\zeta$-restricted $\Gamma$-factor set of $A_{v_0}$ for $v = v_0$. Put

$$S := \{Q_i^v : G \to G/N_v^i \,|\, v \in VA, q_i^v \in S_v, Q_i^v|_{\zeta(\Gamma)} \text{ is injective}\}.$$

To see that this is the desired set let $\varphi \in \mathrm{Hom}_\zeta(G, \Gamma)$ be non-locally injective. Choose $v \in VA$ such that $\varphi|_{A_v}$ in non-injective. Thus there exists a modular automorphism $\alpha$ of $A_v$ such that $\varphi|_{A_v} \circ \alpha$ factors through some $q_i^v$ (if $v = v_0$ then $\alpha$ is a $\zeta$-restricted modular automorphism). As the restriction of $\alpha$ to any finite group is conjugation, $\alpha$ can be naturally extended to an automorphism $\alpha_G$ of $G$. Then $\alpha_G \in \mathrm{Aut}_\zeta(G)$, and clearly $\varphi \circ \alpha_G$ factors through $q_i^v$. □

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