Two-Loop String Theory and the DVV Vertex

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Abstract

We compute the two-loop contributions to the free energy in the null compactification of perturbative string theory at finite temperature. The cases of bosonic, Type II and heterotic strings are all treated. The calculation exploits an explicit reductive parametrization of the moduli space of infinite-momentum frame string worldsheets in terms of branched cover instantons. Various arithmetic and physical properties of the instanton sums are described.

Applications to symmetric product orbifold conformal field theories and to the matrix string theory conjecture are investigated by analyzing the correspondence between the two-loop thermal partition function of DLCQ strings in flat space and the integrated two-point correlator of twist fields in a symmetric product orbifold conformal field theory at one-loop order. This is carried out by deriving combinatorial expressions for generic twist field correlation functions in permutation orbifolds using the covering surface method, by deriving the one-loop modification of the twist field interaction vertex, and by relating the two-loop finite temperature DLCQ string theory to the theory of Prym varieties for genus two covers of an elliptic curve. The case of bosonic $\mathbb{Z}_2$ orbifolds is worked out explicitly and precise agreement between both amplitudes is found. We use these techniques to derive explicit expressions for $\mathbb{Z}_2$ orbifold spin twist field correlation functions in the Type II and heterotic string theories.

To my Parents

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Contents

List of Publications

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Chapter 1

Introduction

Large N matrix field theories obtained as dimensional reductions of maximally supersymmetric $U(N)$ Yang-Mills theory in ten spacetime dimensions provide nonperturbative descriptions of M-theory and string theory in various backgrounds, and associated superconformal field theories (see [1] for a review). The best understood example is matrix string theory [2]–[3] which takes the form of maximally supersymmetric $U(N)$ Yang-Mills theory in two dimensions. In this case the gauge coupling is inversely proportional to the string coupling, so that the free string limit corresponds to the infrared limit and the first order interaction term to the least irrelevant operator in the gauge theory. In this strong coupling limit the supersymmetric Yang-Mills theory approaches a superconformal fixed point which is conjectured to be the supersymmetric sigma model on the symmetric product orbifold $(\mathbb{R}^8)^N/S_N$. The spectrum of this orbifold superconformal field theory can be canonically identified with that of the free second quantized Type IIA string [3, 4].

This equivalence may be given a geometric interpretation by introducing a finite temperature [5, 6]. This is done by further compactifying Euclidean time so that two target space directions are compactified on a torus $\mathbb{T}^2_\tau$ of a particular modulus τ. For the present discussion, the thermodynamic partition function is simply regarded as a generating function for the energy spectrum of free string theory and thermal instabilities such as the gravitational Jeans instability or the stringy Hagedorn transition will be ignored. On the Type IIA side, the one-loop free energy is given by a sum over unramified coverings of the torus $\mathbb{T}^2_\tau$ of degree N [5, 6]. On the superconformal field theory side, the partition function on $\mathbb{T}^2_\tau$ is given by a sum over twisted sectors imposing S_N-twisted boundary conditions on the string embedding fields. An extra summation over elements of S_N is required to define a projection onto the S_N-invariant subspace of the Hilbert space, resulting in a sum over commuting pairs of permutations assuring that the twists in time and space directions commute. The twisted sectors have a natural interpretation in terms of "long" strings formed from "short" fundamental string bits. The partition function from N fundamental single strings are combined together to give the partition function of one long string with a modified modular parameter, i.e. the worldsheet of the long strings is an N-fold cover of the torus. The pertinent combinatorics is summarized by the action of the Hecke operator [4] which maps a modular form into another one with the same weight. The action of the Hecke algebra admits an interpretation in terms of the creation of a long string background along with the addition of short string excitations to it [7]. In this comparison it is of course more natural to work with the grand canonical partition function by taking an ensemble of sigma-models on $\mathrm{Sym}^N(\mathbb{R}^8)$ for all $N \in \mathbb{N}$.

In DLCQ string theory at finite temperature, the g-loop free energy receives contributions from only those genus g string worldsheets which are branched covers of the spacetime torus $\mathbb{T}^2_\tau$ [6]. This gives a partial discretization of the moduli space $\mathcal{M}_g$ of genus g Riemann surfaces which reduces its complex dimension from $3g-3$ to $2g-3$ (from 1 to 0 for $g=1$). Thus perturbative string theory can be formulated entirely in terms of covering Riemann surfaces, a scenario familiar from the Gross-Taylor string expansion of the two-dimensional Yang-Mills theory [8, 9]. In this thesis we will work out explicitly the two-loop free energy which is computed from genus two worldsheets which are branched covers of $\mathbb{T}^2_\tau$. A surface of genus two can be realized as a double cover of the complex plane with three distinct branch cuts. Since any elliptic curve is a double cover of the plane with two branch cuts, a genus two surface can be built from two tori by identifying one of their branch cuts and gluing them together along the cut. This means that the two-loop partition function should coincide with the correlator of two twist fields (2.3.11) in the symmetric orbifold conformal field theory on $\mathbb{T}^2_\tau$. Such a coincidence is not entirely surprising, given that correlation functions of twist fields can be computed by means of free string partition functions on the appropriate covering space [10, 11]. Indeed, many aspects of string theory (at zero temperature) can be recovered from the sigma-model with target space $S^N\mathbb{R}^8$ [12, 13]. However, while the twist field correlator appears to be expressed in terms of branch point loci, the DLCQ string free energy is naturally parametrized in terms of pinching parameters corresponding to the sewing construction of the genus two cover from an unramified covering of the spacetime torus and an auxiliary torus (to be related to the Prym Variety). This suggests an interpretation of the correlation function $\langle V_{\rm int}\, V_{\rm int}\rangle$ as the overlap between a long string state and a fundamental string state, a result which is consistent with the physical interpretation of the Hecke algebra mentioned above.

It is conjectured [3] that the equivalence holds generally in the interacting string theory as well. Strings interact by means of splitting and joining, and the interaction points correspond to insertions of twist field operators in the orbifold superconformal field theory. It has been recently argued [14]–[15] that the structure of the contact interactions in Green-Schwarz light-cone superstring field theory simplifies within the twist field formulation of matrix string theory. Unlike the light-cone string field theory, however, the matrix model provides a full nonperturbative definition of the string dynamics in the large N limit. In this thesis we will investigate this conjectural perturbative correspondence further by examining the relationship between the thermodynamic free energy of Type II superstring theory in DLCQ and correlation functions of the leading irrelevant twist field operators in the symmetric product orbifold conformal field theory. The Polyakov path integral for the former quantity is known [6] to truncate the sum over contributing string worldsheets to those which are branched covers of the spacetime torus arising from the null compactification at finite temperature. The free energy at the leading non-vanishing order in the string coupling constant is the two-loop string amplitude which has been calculated in [**?**]. In order to check the conjecture one needs to compute the corresponding amplitude in the orbifold conformal field theory, which is given by the one-loop two-point function of appropriate twist fields. These operators create twisted sectors out of the vacuum state, in that the local fields of the sigma model acquire non-trivial monodromy about the twist field insertion points. Computing their correlation functions is thus not straightforward, and a good portion of our analysis will centre around the technicalities involved in these calculations.

In this Thesis we will examine this correspondence for the interacting string theory with $g_s > 0$ which arises by relaxing the free string infrared limit. This is obtained by perturbing the orbifold conformal field theory on $S^N\mathbb{R}^8$. To leading order, this perturbation is described by the Dijkgraaf-Verlinde-Verlinde (DVV) twist field [3] which perturbs the free Hamiltonian H via the density

$$V_{\text{int}} = g_s \sum_{1\le a<b\le N} \left(\tau^i\,\Sigma_i \otimes \overline{\tau}^j\,\overline{\Sigma}_j\right)_{a,b} + O\left(g_s^2\right) \ , \tag{1.0.1}$$

where τ^i, $i = 1, \ldots, 8$ are the excited bosonic twist fields and Σ_i are the fermionic spin fields. This defines a conformal field of weight $(\frac{3}{2}, \frac{3}{2})$ which is the unique least irrelevant perturbation that preserves $Spin(8)$ spacetime rotations and spacetime supersymmetry, and which creates a square-root branch cut in the sigma-model with coordinates $x_a^i - x_b^i$. It intertwines between different topological sectors of the worldsheet theory on $\mathbb{T}_\tau^2$ that are related by a basic splitting and joining interaction between pairs of strings. Thus if we use the Hamiltonian density (2.3.11) for computing scattering amplitudes via standard perturbation theory, then we should reproduce the conventional perturbative expansion of Type IIA superstring theory [12]. This expectation is supported by the fact [14] that the DVV twist field exactly reproduces the Lorentz-invariant Mandelstam cubic interaction vertex that describes the joining and splitting of Type II strings in light-cone gauge. Analysis of higher-order contact terms reveals that the structure of superstring field theory simplifies when expressed in terms of twist field correlators [14, 16, 17].

There are several strategies presented in the literature for computing twist field correlation functions. The stress tensor method was originally introduced in [18] and used to compute $\mathbb{Z}_2$ orbifold [18, 19, 20], and more generally $\mathbb{Z}_N$ orbifold [21, 22], correlation functions on worldsheets of arbitrary topology, and S_N orbifold correlation functions on the sphere [12, 23]. In this method one first determines the twisted Green's function (the n-point function of the stress energy tensor in the twisted sector) by demanding the correct short distance behaviour and monodromy about the twist field insertion points. A closely related but more general technique is the covering space method. It makes direct use of the fact that a monodromy is associated to a covering surface. If a field is multi-valued when transported around a closed curve, then it is well-defined as a single-valued function on the appropriate cover of the worldsheet without any special points. In this way the twist field correlation functions can be expressed as vacuum amplitudes of the free conformal field theory on the covering surfaces. This method was exploited in [10, 11, 24]. It is also the main principle behind computing essentially all quantities in permutation orbifolds as shown in [25] where, in particular, the partition function was given for arbitrary orbifold twist group.

In this thesis we use the covering space method for the definition and computation of twist field correlation functions in symmetric products defined on worldsheets of non-trivial topology. The vacuum amplitudes of these conformal field theories are known in complete generality, *i.e.*, for worldsheets of arbitrary genus and arbitrary finite twist group [26]. We generalize these results to the n-point correlation functions of twist field operators. When the worldsheet has non-trivial fundamental group and the twist group is nonabelian, the definition of the corresponding twisted Green's functions is problematic and the covering space technique is the only possible way to define the amplitudes. To make these formulae completely explicit, one needs to determine the dependence of the complex structure of the covering space on that of the worldsheet and the location of the twist field operator insertions. This is a very difficult problem in the general case when the covering surface does not admit any conformal automorphisms. We have not been able to solve this problem in full generality and are not aware of any solution to it for any specific cases of such a cover. All known computations of twist field correlation functions are done with respect to covers with automorphisms (this is the case, in particular, for the $\mathbb{Z}_N$ orbifolds), or to worldsheets of trivial topology when the covering space can be parametrized explicitly in terms of the complex coordinate z of the sphere. Nevertheless, using our technique we are able to determine the bosonic two-point twist field correlation function of the orbifold $\mathbb{R}^{24} \wr \mathbb{Z}_2 := (\mathbb{R}^{24} \times \mathbb{R}^{24})/\mathbb{Z}_2$ and compare it to the appropriate power of the $\mathbb{Z}_2$ orbifold twist field correlation function of the one-dimensional free boson computed in [27], yielding a highly non-trivial check of our methods. Although throughout we deal only with orbifolds of flat space $\mathbb{R}^d$, most of our considerations and results apply to more general symmetric products as well.

When writing down generating functions of amplitudes in symmetric products, one has to sum over all covers of the worldsheet in such a way that only the connected covering surfaces contribute. This fact lies behind the conjecture that these amplitudes naturally arise in physical string theories. We generalize the resummation procedure which was done originally for the torus partition function in [4] and for the Klein bottle amplitude in [28] for the case of closed strings, and then for the annulus and Möbius diagrams in [31, 32] for the case of open strings. The generalization to the twist field n-point function is possible due to a general combinatorial formula [28, 29, 30] which is the crux of all of these calculations.

The main technical achievement of the two-loop calculation of [**?**] was a modification of the Weierstrass-Poincaré theory of reduction. Reduction may be described entirely in terms of the Riemann matrix of periods of a curve, and it has the effect of expressing theta functions at a given genus in terms of lower dimensional theta functions. This happens exactly when the curve in question covers a surface of lower genus (but it may also occur without there being a covering map). The remarkable feature of this reduction is the simple universal form that the genus two DLCQ free energy takes in terms of Jacobi elliptic functions on the base torus. For the contributions from double covers of the torus to the two-loop free energy of the critical bosonic string, we find perfect agreement between the string free energy and the correlator of twist fields computed as the appropriate power of the $\mathbb{Z}_2$ orbifold twist field two-point function of [27]. We will find generally that the original genus zero interaction vertex proposed in [3] must be modified at one-loop order to ensure equivariance under the action of the non-trivial modular group in this instance. Since the structure of the result depends only on the orbifold twist group and not on the data of the specific string theory, we use this equivalence and the known formulae from [**?**] for the two-loop DLCQ free energy of the Type II and heterotic strings to derive the two-point functions of the appropriate spin twist fields in the corresponding $\mathbb{Z}_2$ orbifold superconformal field theories. To the best of our knowledge, these correlation functions have not been previously computed, and our explicit formulae should be useful for further clarifying the role of the twist field interaction vertex in light-cone string field theory.

The difficulty in establishing the correspondence is writing down the period matrix of the covering surface explicitly in terms of the modulus of the worldsheet torus and the branch point loci. This is achieved in part by elucidating the geometric meaning of the reduced genus two period matrix. In [?] it was shown that this period depends on two elliptic moduli, one of which lies in a modular orbit of an unramified (one-loop) cover of the base torus. Here we show that the second elliptic modulus determines the complex structure of a Prym variety. Prym varieties arise in special instances of covering surfaces. A theorem due to Mumford asserts that there are only three types of branched covers which give rise to Prym varieties, namely unramified double covers, ramified double covers with two branch points, and precisely our instance of genus two covers over an elliptic curve. When this is in addition a double cover of the torus, we use the canonical involution of the genus two surface to explicitly construct the dependence of the periods on the branch points (which are the images of the fixed points of the involution). This procedure unfortunately doesn't generalize to higher degree covering surfaces (although the identification with a Prym variety always holds).

Our detailed computations and results could also shed further light on aspects of more complicated symmetric orbifold conformal field theories. An important example is when the orbifold target space is taken to be $\mathrm{Sym}^N(\mathcal{M})$ with $\mathcal{M} = \mathrm{K3}$ or $\mathcal{M} = \mathbb{T}^4$ [33]. With $N = k\,n$, a particular deformation of the superconformal field theory is the sigma-model on the moduli space of k instantons in $U(n)$ gauge theory on $\mathcal{M}$ which is believed to be dual, via the AdS/CFT correspondence, to Type II string theory on the background geometry $\mathrm{AdS}_3 \times \mathbb{S}^3 \times \mathcal{M}$. The primary evidence for these particular correspondences comes from the matching of their BPS spectra. Finally, from a mathematical perspective our results are related to the computation of elliptic genera [4, 34] and topological Euler characteristics of Hilbert schemes [35], which in the case $\mathcal{M} = \mathrm{K3}$ is related to generalized Kac-Moody algebras [36, 37].

Plan of the Work

The organisation of chapter 1 is as follows. In section 1 we briefly describe the matrix string theory conjecture, the description of the symmetric orbifold Hilbert space. In section 2 we describe the light cone theory and make the connection between second quantised string theory and super conformal sigma models with symmetric orbifold target space. In section 3 we discuss interactions on the orbifold theory using the DVV interaction.

The organisation chapter 2 is as follows. In Section 1 we review the basic arguments establishing that DLCQ string theory at finite temperature is a theory of branched coverings of a torus [6]. We also outline some generic aspects of a certain reduction technique for the Hurwitz moduli space of branched covers which will be central to our analysis throughout this thesis. We conclude by reviewing the one-loop calculation [5, 6] in this light for later comparison with the two-loop results.

In Section 2 we begin the construction of the two-loop free energy. We present an explicit description of the moduli space of genus two branched covers using a particular reduction technique. As an example, we compute the bosonic free energy in terms of genus one theta-functions of the elliptic curve $\mathbb{T}^2_\tau$. While bosonic string theory cannot emerge from a gauge theory (since the necessary supersymmetric cancellations of fluctuation determinants do not occur), this calculation can be compared to the bosonic sigma-model with target space $\mathrm{Sym}^N(\mathbb{R}^{24})$ and the interaction density (2.3.11) modified by replacing $\tau^i\,\Sigma_i$ with the unexcited twist field [38]

$$\sigma = \prod_{i=1}^{24} \sigma^i \tag{1.0.2}$$

of dimension $\frac{3}{2}$ having the supersymmetry variation $G^{\dot a}_{-1/2}(\sigma\,\Sigma_{\dot a}) = \tau^i\,\Sigma_i$.

In Section 3 we compute the two-loop superstring free energy. Our calculation draws heavily on recent progress [39] in two-loop superstring perturbation theory in the NSR formalism which yields explicit unambiguous expressions for the chiral superstring measure in terms of genus two modular forms. With the appropriate modification of the genus two GSO projection at finite temperature [40], we find a formula for the superstring free energy in terms of theta-functions on $\mathbb{T}^2_\tau$.

In Section 4 we perform the analogous calculation for the heterotic string. In this case the pertinent conformal field theory is the supersymmetric heterotic sigma-model defined on the symmetric product orbifold [38, 41]

$$\mathrm{Sym}^N\big(\mathbb{R}^8 \times G\big) = \big(\mathbb{R}^8 \times G\big)^N / S_N \ltimes (\mathbb{Z}_2)^N \tag{1.0.3}$$

for the heterotic gauge group G. The interaction density (2.3.11) should be modified to contain the bosonic twist field $\overline{\sigma}$ given by (1.0.2) in the right-moving sector and the supersymmetric twist field $\tau^i\,\Sigma_i$ in the left-moving sector. The relevant gauge dynamics is conjectured to be governed by heterotic matrix string theory [38, 42, 43, 41, 44], i.e. two-dimensional supersymmetric Yang-Mills theory with chiral anomaly-free matter fields and gauge group $O(N)$.

Our formulas for the free energies, while in principle being explicit, are quite complicated. In Section 5 we consider various degeneration limits of the genus two covers in which these expressions drastically simplify, and hence elucidate various arithmetic and physical properties of our amplitudes. We find the appropriate modification of the action of the Hecke algebra for twist field correlators. In a certain collapsing limit, we also find effective one-loop string theories which resemble non-supersymmetric strings on particular $\mathbb{Z}_2$-orbifolds. In another collapsing limit, the partition function resembles the one-loop instanton sum over long string configurations. Finally, in Appendix A we present an alternative reductive description of the moduli space of genus two branched covers which may be of independent interest and use in other applications, while Appendix B contains some technical details of the calculations performed in the main text.

The organisation of chapter 3 is as follows. In Section 4.1 we give a general introduction to the theory of bosonic permutation orbifolds, and use the one-loop sigma model to illustrate the typical combinatorial structure of amplitudes therein. We apply the combinatorial resummation formula for symmetric products to compute a large class of correlation functions which are invariant under the action of the twist group. We show how to generalize these formulae to correlation functions of twist field operators, and briefly review the structure of the DLCQ string partition function. In Section 4.2 we present detailed and explicit calculations for $\mathbb{Z}_2$ orbifolds. In the course of this analysis, we make the generic connection between DLCQ string theory and the theory of Prym varieties, and also derive the explicit modification of the twist field interaction vertex for toroidal worldsheets in the symmetric product sigma model. In Section 4.3 we discuss the technical issues surrounding the generalizations of these results to S_N orbifolds with $N > 2$. We examine the uniformization construction, which is used to build vacuum amplitudes, in the context of a generic twist field n-point function, and the problem of determining the period of the genus two covering surface in terms of the branch point data. We also study the combinatorial expansion in more detail and indicate that, while computable in principle, the combinatorics become very non-trivial for $N > 2$. Finally, in Section 4.4 we describe the modifications of permutation orbifolds required in the presence of fermionic degrees of freedom, and of twist field correlation functions therein. We then apply these and previous considerations to derive explicit formulae for the one-loop spin twist field correlation functions in the $\mathbb{Z}_2$ orbifold supersymmetric and heterotic string theories.

Chapter 2

Matrix String Theory

2.1 Matrix String Theory

The[1] matrix model [45, 47] is conjectured to describe M theory in the infinite momentum frame. The theory is $U(N)$ supersymmetric quantum mechanics.

Compactification of the matrix model along a spacial direction produces $2D$ $\mathcal{N} = 8$ $U(N)$ supersymmetric Yang-Mills (SYM) theory [3, 48] with world sheet $\Sigma = (\mathbb{R} \times S^1)^{1,1}$.

$$S = \frac{1}{2\pi} \int_\Sigma \mathrm{Tr}\big\{ F_{\mu\nu}^2 + (D_\mu X^i)^2 - g_{YM}^2 \sum_{i<j} [X^i, X^j]^2 + \theta^T \not{D} \theta + g_{YM}^2 \theta^T \gamma_i [X^i, \theta] \big\}. \quad (2.1.1)$$

The 8 scalar fields X^i and 8 fermionic fields θ_L^α and $\theta_R^{\dot{\alpha}}$ are $N \times N$ hermitian matrices. The fields X^i, θ^α and $\theta^{\dot{\alpha}}$ transform in the $\mathbf{8_v}$,$\mathbf{8_s}$ and $\mathbf{8_c}$ representations of the $SO(8)$ R-symmetry group of transversal rotations. $2D$ SYM can also be derived from the dimensional reduction of $10D$ maximall SYM. Green Schwartz superstring in the light Cone can be used to costruct $2D$ SYM if the fields are promoted to $N \times N$ hermitian matrices with the appropriate interactions inserted. The fields are then minimally coupled to a gauge potential to take advantage of the $U(N)$ gauge symmetry. The $2D$ SYM's coupling constant g_{YM} has dimensions one over length, SYM is therefore not a conformal theory.

The Matrix String Theory conjecture states that $2D$ SYM is a non-perturbative definition of IIA string theory. In a similar vein to the Matrix Model the X^i matrix

[1]This chapter is largely based on [34].

fields are identified as N D-strings interacting in a non abelian fashion. To make the correspondence, the YM coupling constant is identified with the inverse of the string coupling g_s.

$$g_s = \frac{1}{g_{\mathrm{YM}}\,\ell_s}\ , \tag{2.1.2}$$

where $\ell_s = \sqrt{\alpha'}$ is the string length. In the infrared limit the SYM theory looses its length scale and is expected to reach a superconformal fixed point. In this limit the commutators of the fields tend to zero to ensure the theory remains finite. The fields take values in the Cartan subalgebra. In other words the matrices are simultaneously diagonalizable and the theory reduces to N copies of the Green Schwartz Superstring in the light cone where the free fields are the eigenvalues.

$$S[X,\theta] = \sum_{i=1}^{8}\sum_{I=1}^{N}\int_{\Sigma} d^2z\partial X_I^i\partial X_I^i + \theta_I \not{D}\theta_I \tag{2.1.3}$$

The gauge fields in the IR limit are ignored because they decouple from the rest of the theory and any field in a twisted sector can be made periodic by gauge transformations[5]. The over all contribution to the partition function is normalized to 1. The remaining S_N symmetry of the theory comes from the Weyl group of $U(N)$. The theory is invariant under permutations of the eigenvalues. In other words the theory in this IR Limit is a $\mathcal{N} = 8$ superconformal sigma model with a symmetric orbifold of $\mathbb{R}^8$ ($S^N(\mathbb{R}^8) = \mathbb{R}^{8N}/S_N$)as its target space. The eigenvalues lie in twisted sectors determined by their boundary conditions,

$$X^I(\tau,\sigma+2\pi) = g^{IJ}X^J(\tau,\sigma). \tag{2.1.4}$$

If the field X^I lies in the twisted sector $\mathcal{H}_g$ then the field $(h\cdot X)^I$ lies in the twisted sector $\mathcal{H}_{hgh^{-1}}$.

$$(hgh^{-1})hX(\tau,\sigma) = hX(\tau,\sigma+2\pi) \tag{2.1.5}$$

We see that the boundary conditions are defined up to cojugacy classes. Since the theory is invariant under permutations in the eigenvalues, the twisted sector $\mathcal{H}_{hgh^{-1}}$ should describe the same physics as $\mathcal{H}_g$. We therefore decompose the Hilbert space for the symmetric orbifold $S^N\mathbb{R}^8$ into a sum over twisted sectors labeled by conjugacy classes $[g]$. The center C_g with respect to a given element within the conjugacy class

$[g]$ acts within a given twisted sector. We must project out states not invariant under C_g to obtain the twisted sector $\mathcal{H}_{[g]}^{C_g}$.

$$\mathcal{H}(S^N\mathbb{R}^8) = \bigoplus_{[g]} \mathcal{H}_{[g]}^{C_g} \tag{2.1.6}$$

Each conjugacy class of S_N can be further decomposed into products of cycles,

$$[g] = (1)^{N_1}(2)^{N_2}\cdots(k)_k^N. \tag{2.1.7}$$

Each cycle is now labeled by a partition $\{N_n|\sum_n nN_n = N\}$ of N. The center is given by

$$C_g = S_{N_1} \times (S_{N_2} \ltimes \mathbb{Z}_2^{N_2}) \times \cdots (S_k \ltimes \mathbb{Z}_k^{N_k}). \tag{2.1.8}$$

The factor S_{N_n} within the center C_g acts on the decomposition (2.1.7) by permuting the N_n cycles of length n while the $\mathbb{Z}_n$ is the usual cyclic action on the cycle of length n. The total Hilbert space is,

$$\mathcal{H}(S^N\mathbb{R}^8) = \bigoplus_{\substack{N_n \\ \sum nN_n=N}} \bigotimes_{n>0} S^{N_n}\mathcal{H}_{(n)}^{\mathbb{Z}_n}. \tag{2.1.9}$$

The symmetric product is defined as,

$$S^N\mathcal{H} = \Big(\underbrace{\mathcal{H}\otimes\cdots\otimes\mathcal{H}}_{\text{N times}}\Big)^{S_N}. \tag{2.1.10}$$

The space $\mathcal{H}_{(n)}^{\mathbb{Z}_n}$ is the $\mathbb{Z}_n$ invariant Hilbert space of a string on the space $\mathbb{R}^8\times\mathbb{S}^1$. The fields, x^i in the twisted sectors $\mathcal{H}_{(n)}^{\mathbb{Z}_n}$ have cyclic boundary conditions,

$$x_I(\sigma+2\pi) = x_{I+1}(\sigma) \qquad I\in\{1,\ldots,n\}. \tag{2.1.11}$$

$\mathbb{Z}_n$ has a natural cyclic action on the fields x_I, it is possible to glue the fields together by considering a field $X(\sigma)$ defined on the extended interval $[0,2\pi n]$. In this way we have constructed a long string of length n. It is natural to introduce fractional modeing, so that translations along the σ direction are well behaved in a given twisted sector,

$$\mathrm{e}^{P\theta} : X(\sigma) = X(\sigma+\theta/n) \tag{2.1.12}$$

The $L_0^{(n)}$ Virasoro operator in a given twisted sector are also renormalised relative to a single string that wraps the $\mathbb{S}^1$ ounce,

$$L_0^{(n)} = \frac{L_0}{n}\ ,\quad \bar{L}_0^{(n)} = \frac{\bar{L}_0}{n} \tag{2.1.13}$$

The Hamiltonian of the total theory takes the form,

$$H = \sum_n (L_0^{(n)} + \bar{L}_0^{(n)}) \tag{2.1.14}$$

and can be seen to have fractional values, the momentum however takes integer values,

$$P = \sum_n (L_0^{(n)} - \bar{L}_0^{(n)}). \tag{2.1.15}$$

This is a consequence of the $\mathbb{Z}_n$ part of the C_g projecting out states with non integer modeing. We conclude that $L_0 - \bar{L}_0 = 0 \mod n$. So $\mathcal{H}_{(n)}^{\mathbb{Z}_n}$ is the Hilbert space of a single string which winds the $\mathbb{S}^1$ n times and is isomorphic to the hilbert space of a single string which satisfies

$$L_0 - \bar{L}_0 = 0 \mod n. \tag{2.1.16}$$

It should be pointed out that the description of the symmetric orbifold Hilbert space for $\mathbb{R}^8$ can be extended to Kähler manifolds. In [4] they compute partition functions for the symmetric products of Kähler manifolds which coincides with the elliptic genus, this is defined as the trace over the Ramond-Ramond sector of the sigma model of the operator $(-1)^F y^{F_L} q^{L_0 - \frac{c}{24}}$. They prove the identity

$$\sum_{N=0}^{\infty} p^N Z(S^N M; q, y) = \prod_{n>0, m\geq 0, \ell} \frac{1}{(1 - p^n q^m y^\ell)^{c(mn,\ell)}} \tag{2.1.17}$$

where the coefficients $c(n, m, \ell)$ are defined by the expansion

$$Z(M; q, y) = \sum_{m,\ell} c(m, \ell) q^m y^\ell = \mathrm{Tr}_{\mathcal{H}(M)} (-1)^F y^{F_L} q^{L_0 - \frac{c}{24}}, \tag{2.1.18}$$

and $p = \mathrm{e}^{2\pi i\rho}$ $q = \mathrm{e}^{2\pi i\tau}$, $y = \mathrm{e}^{2\pi i z}$. τ will be the complex structure of a $\mathbb{T}_\tau$, ρ is taken to be the Kälher parameter and z is a point on the Jacobian of $\mathbb{T}_\tau$. If we compute the logarithm of the above partition function, we obtain [4],

$$\mathcal{F}(p, q, y) = \sum_{N>0} p^N \sum_{kn=N} \frac{1}{k} \sum_{m\geq 0,\ell} c(mn, \ell) q^{km} y^{k\ell}. \tag{2.1.19}$$

Now we will introduce the Hecke operator [57] defined as,

$$\mathbf{H}_N \phi(\tau, z) := \frac{1}{N} \sum_{\substack{ad=N \\ b \mod d}} \phi(\frac{a\tau + b}{d}, az). \tag{2.1.20}$$

This is a map from the space of weak Jacobi forms of weight zero and index r into the space of weak Jacobi forms of weight zero and index Nr. Weak Jacobi forms of weight k and index $r \in \frac{1}{2}\mathbb{Z}$ denoted $\phi_{k,r}$are holomorphic functions on $\mathcal{H} \times \mathbb{C}$. Their transformation under the action of $Sl(2,\mathbb{Z})$ is

$$\phi_{k,r}(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}) = (c\tau+d)^k \exp{(\pi i \frac{rcz^2}{c\tau+d})}\phi_{k,r}(\tau,z),$$
$$\phi_{k,r}(\tau, z+m\tau+n) = \exp{(-\pi i r(m^2\tau+2nz))}\phi_{k,r}(\tau,z). \tag{2.1.21}$$

An important example are the genus one theta functions.

The free energy (2.1.19) is seen to be,

$$\mathcal{F}(p,q,y) = \sum_{N>0} p^N \mathbf{H}_N Z(M,q,y). \tag{2.1.22}$$

This expression plays a significant role in the rest of the thesis for the special case of $S^N\mathbb{R}^8$. The Hecke operator is a map from first to second quantised string theory and has the interpretation as a sum over degree N un-branched covers, $f : \mathbb{T}_{\frac{k\tau+b}{n}} \rightarrow \mathbb{T}_\tau$.

2.2 Strings in the Light Cone

In the light cone coordinates the space $\mathbb{R}^{1,9}$ splits into $\mathbb{R}^{1,1} \times \mathbb{R}^8$. The light cone, $\mathbb{R}^{1,1}$ is the longitudinal space and the $\mathbb{R}^8$ is the physical transverse space. The Green Schwartz superstring in the light cone is given by the following action,

$$S = \int_\Sigma \partial x^i \bar{\partial} x^i + \theta^a \bar{\partial}\theta^a + \bar{\theta}^a \partial \bar{\theta}^a. \tag{2.2.1}$$

The sections that make up the left and right moving sectors of the string theory come from the following bundles,

$$\partial x \in \Gamma(K_\Sigma \otimes x^*T\mathbb{R}^8) \qquad \bar{\partial} x \in \Gamma(K_\Sigma \otimes x^*T\mathbb{R}^8), \tag{2.2.2}$$

and

$$\theta \in \Gamma(K^2_\Sigma \otimes x^*S^+) \qquad \bar{\theta} \in \Gamma(\bar{K}^2_\Sigma \otimes x^*S^\pm). \tag{2.2.3}$$

K is the canonical line bundle on the Riemann surface Σ and $S^\pm$ are the spin bundles over Σ. A choice of chirality in $\bar{\theta}$ will determine which sting theory we study. The

$SO(8)$ representation S^- will lead to IIA while S^+ will lead to IIB. The virasoro constraints in the light cone are,

$$\partial x^- = \frac{1}{p_+}(\partial x^i)^2 \qquad \bar{\partial} x^- = \frac{1}{p^+}(\bar{\partial} x^i)^2. \tag{2.2.4}$$

The level matching condition imposes the following on worldsheet the momentum,

$$P = L_0 - L_0 = 0. \tag{2.2.5}$$

The mass shell relation in the light cone takes the form,

$$p^- = \frac{1}{p^+}(L_0 + \bar{L}_0) = \frac{1}{p^+}H. \tag{2.2.6}$$

The Hilbert space for the IIA GS superstring is,

$$\mathcal{H} = L^2(\mathbb{R}^8) \otimes \mathcal{V} \otimes \mathcal{F} \otimes \bar{\mathcal{F}}. \tag{2.2.7}$$

$L^2(\mathbb{R}^8)$ describes the quantum mechanics of the bossonic zero mode $\oint x_I$ which is the center of mass of the string. The fermionic zero modes $\oint \theta^a$ and $\oint \bar{\theta}^{\dot{a}}$ form a 16×16 dimensional vector space $\mathcal{V} = (V \oplus S^-) \otimes (V \oplus S^+)$ of ground states. The Fock space $\mathcal{F}$ generated by non-zero modes is given by the formal generating series in q,

$$\mathcal{F}_q = \bigotimes_{n>0} \Big(\bigwedge\nolimits_{q^n} S^- \otimes S_{q^n} V \Big). \tag{2.2.8}$$

Where the generating series is defined as

$$S_q V = \bigoplus_{n>0} q^n S^n V. \tag{2.2.9}$$

Now in the case of DLCQ, the longitudinal null coordinate x^- is periodically identified with radius R. The string now is able to wind around the compactified direction.

$$w^- = \int_{\mathbb{S}^1} dx^- = 2\pi m R \qquad m \in \mathbb{Z}. \tag{2.2.10}$$

To make sense of translations in the x^- direction p^+ becomes quantized,

$$p^+ = \frac{n}{R} \qquad n \in \mathbb{Z}_{>0}. \tag{2.2.11}$$

We see that the constraints ensure that

$$w^- = \frac{2\pi}{p^+}(L_0 - \bar{L}_0) = \frac{2\pi R}{n}(L_0 - \bar{L}_0). \tag{2.2.12}$$

If the winding number w^- is an integer then the worldsheet momentum must satisfy the level matching condition,

$$P = L_0 - \bar{L}_0 = 0 \mod n. \tag{2.2.13}$$

The Hilbert space of states for this compactified string is $\mathcal{H}_{(n)}^{\mathbb{Z}_n}$. This establishes the equivalence of a second quantised IIA string theory and that of the symmetric product SCFT. We can then equate partition functions,

$$Z^{\text{string}}(\mathbb{R}^8; p, q, \bar{q}) = Z^{\text{SCFT}}(S_p\mathbb{R}^8; q, \bar{q}). \tag{2.2.14}$$

2.3 Interactions

If MST is a non-perturbative definition of IIA superstring theory then it must be possible to derive the perturbation theory of the string theory. Rather then carry out a perturbative expansion of MST directly we deform the symmetric product orbifold conformal field theory by introducing an interaction density V_{int}.

$$S = S_{SCFT} + \lambda \int d^2z \; V_{int}. \tag{2.3.1}$$

The deformation is supposed to be an irrelevant operator that respects the space time supersymmetry and the $SO(8)$ R-symmetry. The constant $\lambda \sim g_s\sqrt{\alpha'}$ to ensure that an . In string perturbation theory the strings split and join to form new strings. In order to model this behavior we use twist filds σ_P , $P \in S^N$ which map one twisted sector of the hilbert space to another. In effect their insertion on the cylinder gives the free fields non trivial monodromy about the insertion point and in our case permute the eigenvalues. This monodromy defines a topologicaly distinct cover of the world sheet. Since the base has a complex structure this is pulled back to the cover giving it the structure of a Riemann surface, this complex structure is unique up to biholomorphic map. The twist field $\sigma_{(n)}(w, \bar{w})$ (with (n) a cycle) is inserted on the world sheet, its defining property is the OPE,

$$\partial X^I(z)\sigma_{(n)}(w, \bar{w}) \sim (z - w)^{-(1-\frac{1}{n})} \, \mathrm{e}^{\frac{2\pi i I}{n}} \tau^I_{(n)}(w). \tag{2.3.2}$$

$\tau^i_{(n)} = \alpha^i_{-1/2}\sigma_{(n)}$ is the excited twist field. The bossonic field $\partial X^I(z)$ has the series expansion near zero,

$$\partial X^i(z) = -\frac{i}{n}\sum_m \alpha^i_m \,\mathrm{e}^{\,2\pi i\frac{Im}{n}}\, z^{\frac{m}{n}-1}, \tag{2.3.3}$$

with mode algebra

$$[\alpha^i_m, \alpha^j_n] = m\delta^{ij}\delta_{m,-n}. \tag{2.3.4}$$

The twist field gives rise to a twisted vacuum, $\sigma_{(n)}\big|0>$ which is anhilated by the $\alpha_m\ ,\ \ m \geq 0$ modes. The states are generated by the $\alpha_{-m}\ ,\ \ m > 0$ modes acting on the twisted sector. One can compute the conformal dimension $\Delta_{(n)}$ of $\sigma_{(n)}(0)$ by computing the stress energy of the twisted vacuum with stress energy tensor given by

$$T(z) = -\frac{1}{2}\lim_{z\to w}\Big(\partial X^i_I(z)\partial X^i_I(w) - \frac{1}{(z-w)^2}\Big), \tag{2.3.5}$$

and the stress enegy of the twisted vacuum given by

$$< n|T(z)|n >= \frac{\Delta_{(n)}}{z^2} < n|n > . \tag{2.3.6}$$

The conformal dimension is seen to be

$$\Delta_{(n)} = \frac{D}{24}\Big(n - \frac{1}{n}\Big). \tag{2.3.7}$$

The fermionic fields θ^a and $\bar{\theta}^{\dot{a}}$ rely on the spin fields $\Sigma^i\ ,\ \Sigma^a\ ,\ \Sigma^{\dot{a}}$ to provide non trivial monodromy. Their OPEs are,

$$\theta^a\Sigma^i(0) \sim \frac{\eta^*}{\sqrt{2}z}\gamma^i_{a\dot{a}}\Sigma^{\dot{a}} \qquad \theta^a\Sigma^{\dot{a}}(0) \sim \frac{\eta^*}{\sqrt{2}z}\gamma^i_{a\dot{a}}\Sigma^i \tag{2.3.8}$$

The $\gamma^i_{a\dot{a}}$ are the $SO(8)$ gamma matrices which act as the CG-coefficients between the three eight dimensional representations of $SO(8)$, the $\mathbf{8_v}, \mathbf{8_s}$ and the $\mathbf{8_c}$ under triality. The charges are,

$$Q^a = \sqrt{n}\oint d\sigma \sum_{I=1}^{N}\theta^a_I \qquad Q^{\dot{a}} = \frac{1}{\sqrt{N}}\oint d\sigma G^{\dot{a}} \tag{2.3.9}$$

Where $\sum_{I=1}^{N}\theta^a_I$ is a sum over fermionic zero modes and $G^{\dot{a}}$ is the supersymmetry current with mode expansion about the point zero,

$$G^{\dot{a}} = \sum_{n+\frac{1}{2}} G^{\dot{a}}_{-n} z^{n-3/2}. \tag{2.3.10}$$

We will take the following density as our proposed interaction,

$$V_{\text{int}} = g_s \sum_{1\leq a<b\leq N} \left(\tau^i\, \Sigma_i \otimes \overline{\tau}^j\, \overline{\Sigma}_j\right)_{a,b} + O\left(g_s^2\right) \ . \tag{2.3.11}$$

The supersymmetric variation of the operator V_{int} is seen to be a total derivative by the following relations,

$$\left[G^{\dot{a}}_{-1/2}, \sigma\Sigma^{\dot{a}}\right] = \frac{1}{2}\tau^I\Sigma^I, \quad \left[G^{\dot{a}}_{-1/2}, \tau^I\Sigma^I\right] = \partial(\sigma\Sigma^{\dot{a}}). \tag{2.3.12}$$

The total deformation is shown to be the least irrelevant operator which respects the $SO(8)$ R-symmetry and space time supersymmetry, this will be the DVV vertex.

2.4 Symmetric Product Twist Field Correlators on Spheres

Twist field correlators on spheres are computed in [10] and [11] for any symmetric product orbifold $S^N(M)$. They employ the covering space technique which defines the correlation function of twist fields as the partition function Z of a free theory with modified boundary conditions for the fields around the insertion points of the twist fields. These boundary conditions are determined by the defining property of the twist fields, their OPE with the free field. The worldsheet is a punctured disc where the positions of the twist fields are the punctures. The radii of the punctures are regularisation parameters which are sent to zero at the end of the calculation. The two point function is defined as the following ratio

$$\langle\sigma^\epsilon(z_1)\sigma^\epsilon(z_2)\rangle_\delta \equiv \frac{Z_{\delta,\epsilon}[\sigma(z_1), \sigma(z_2)]}{(Z_\delta)^N}. \tag{2.4.1}$$

$Z(\delta)$ is the partition function of a free theory on the disc of radius $\frac{1}{\delta}$. The identity operator is on the boundary. The insertion points z_i are on the disc in other words $|z_i| \ll \frac{1}{\delta}$ and ϵ is the radius of a puncture at the location of the twist field insertion. $Z_{\delta,\epsilon}[\sigma(z_1), \sigma(z_2)]$ is the partition function with modified boundary conditions on the disc. It is argued that the twisted partition function in (2.4.1) is given by the partition function of a free theory with target space M and with no other operator insertions after the regularisation parameters have been sent to zero. The argument is as follows, the fields on the disc are multi-valued due to the twist fields. One constructs the cover by gluing patches of the disc together taking into account the monodromy of the fields. The metric on the cover is uniquely determined by the gluing. This cover will have punctures with the identity operator on their boundaries which can now be sealed. The twisted partition function is seen to be equal to the partition function on the cover by noting that the action of the twisted theory on individual coordinate patches is equivalent to the action of the partition function on the pre-image of these patches using the pullback of the metric. In general there is a conformal anomaly which manifests itself in terms of a Liouville factor in front of the partition function on the cover after a change in metric.

2.5 Tree Level Amplitudes

The MST conjecture is checked at tree level by deriving the Virasoro amplitude [12] and the four graviton [23] scattering amplitude. The method employed to compute the correlation functions is the stress energy method allthough the covering space is explicitly called upon to further the calculation. It is not allways possible to construct the covering map or the complex structure on the cover using this method. In [21] they determine the $\mathbb{Z}_N$ twist field correlators in terms of "cut" differential forms, we will identify the $\mathbb{Z}_2$ case with the Prym form.

We will briefly sumerise the Virasoro calculation in [12]. First of all take the bossonic conformal field theory and perturb it by the bossonic analog of the DVV vertex V_{int}. The action is,

$$S_{CFT} + V_{int}, \tag{2.5.1}$$

and

$$V_{int} = -\frac{\lambda N}{2\pi} \int d^2 z |z| \sigma_{IJ}(z, \bar{z}), \tag{2.5.2}$$

where

$$\sigma_{IJ} = \prod_{i=1}^{24} \sigma_{IJ}^i \otimes \bar{\sigma}_{IJ}^i. \tag{2.5.3}$$

The calculation in [12] is the second order term in the perturbative expansion of the

S-matrix.

$$\langle f|S|i\rangle = -\frac{1}{2}(\frac{\lambda N}{2\pi})^2\langle f|\int d^2z_1 d^2z_2|z_1||z_2|\mathcal{T}(\mathcal{L}(z_1,\bar{z}_1)\mathcal{L}(z_2,\bar{z}_2))|i\rangle, \qquad (2.5.4)$$

where $\mathcal{T}$ is time ordering operator and

$$\mathcal{L}(z_i,\bar{z}_i) = \sum_{I<J}\sigma_{IJ}(z_i). \qquad (2.5.5)$$

The in and out states are obtained by the following operator,

$$\sigma_{[g]}[\{\boldsymbol{k}_\alpha\}](z,\bar{z}) = \frac{1}{N!}\sum_{h\in S_N} : \exp(i\frac{k_\alpha^i}{\sqrt{n_\alpha}}Y_\alpha^i[h](z,\bar{z})) : \sigma_{h^{-1}g_c h}(z,\bar{z}). \qquad (2.5.6)$$

The field $Y_\alpha^i[h](z,\bar{z}))$ is a suitable linear combination of the N bosonic fields X_I^i which have trivial monodomy under the canonical element g_c within the conjugacy class [g]. The initial state describes tachyons with momenta k_1 and k_2 and is given by

$$|i\rangle = C_0\sigma_{g_0}[k_1,k_2](0,0)|0\rangle. \qquad (2.5.7)$$

The final state describes two tachyons with momenta k_3 and k_4 and is given by

$$|f\rangle = C_\infty \lim_{z_\infty\to\infty}|z_\infty|^{4\Delta_\infty}\langle 0|\sigma_{g_\infty}[k_3,k_4](z_\infty,\bar{z}_\infty)|0\rangle. \qquad (2.5.8)$$

C_0 and C_∞ are normalisation constants while g_0 is the product of cycles $(n_0)(N-n_0)$ and $g_\infty = (n_\infty)(N-n_\infty)$. The S-matrix element can be expressed as

$$\langle f|S|i\rangle = -\frac{1}{2}(\frac{\lambda N}{2\pi})^2\int \partial^2 z_1|z_1|^{2\Delta_\infty-2\Delta_0-2}\int d^2u\langle f||u|\mathcal{T}(\mathcal{L}(1,1)\mathcal{L}(,\bar{u}))|i\rangle, \quad (2.5.9)$$

after the following conformal transformation $z\longrightarrow \frac{z}{z_1}$ and change of variables $u=\frac{z_2}{z_1}$. In the light cone the conformal dimension of the twist fields $\sigma_{g_\infty}[k_3,k_4]$ and $\sigma_{g_0}[k_1,k_2]$ take the form,

$$\Delta_0 = N - \frac{k_1^- + k_2^-}{8N},$$
$$\Delta_\infty = N + \frac{k_3^- + k_4^-}{8N}. \qquad (2.5.10)$$

Performing a wick rotation and integrating over the worldsheet produces

$$\langle f|S|i\rangle = -i2\lambda^2N^3\delta(k_1^- + k_2^- + k_3^- + k_4^-)\int d^2u\langle f||u|\mathcal{T}(\mathcal{L}(1,1)\mathcal{L}(,\bar{u}))|i\rangle, \quad (2.5.11)$$

where the δ function imposes light cone momenta conservation. Using the twist field OPE,

$$\sigma_{g_1}(z,\bar{z})\sigma_{g_2}(0,0) = \frac{1}{|z|^{2\Delta_{g_1}+2\Delta_{g_2}-2\Delta_{g_1g_2}}}\left(C^{g_1g_2}_{g_1,g_2}\sigma_{g_1g_2}(0) + C^{g_2g_1}_{g_1,g_2}\sigma_{g_2g_1}(0)\right) + \cdots \quad (2.5.12)$$

and the homotopy condition on spheres one finds the S-matrix element simplifies further.

$$\begin{aligned}
&\langle f|\mathcal{T}(\mathcal{L}(1,1)\mathcal{L}(,\bar{u}))|i\rangle \\
&= C_0C_\infty \sum_{I<J,K<L} \langle \sigma_{[g_\infty]}[k_3,k_4](\infty)\sigma_{IJ}(1)\sigma_{K,L}(u,\bar{u})\sigma_{[g_0]}[k_1,k_2](0)\rangle \\
&= C_0C_\infty \sum_{\substack{h_\infty \in S_N \\ h_\infty^{-1}g_\infty h_\infty g_{IJ}g_{kL}g_0=1}} \sum_{I<J,K<L} \langle \sigma_{[h_\infty^{-1}g_\infty h_\infty]}[k_3,k_4](\infty)\sigma_{IJ}(1)\sigma_{K,L}(u,\bar{u})\sigma_{[g_0]}[k_1,k_2](0)\rangle.
\end{aligned} \quad (2.5.13)$$

[12] now set about computing the correlation functions,

$$G(u,\bar{u}) = \langle \sigma_{[g_\infty]}[k_3,k_4](\infty)\sigma_{IJ}(1)\sigma_{K,L}(u,\bar{u})\sigma_{[g_0]}[k_1,k_2](0)\rangle, \quad (2.5.14)$$

using the stress energy tensor method. Primary fields $\phi(w)$ of a given CFT have the following OPE with the stress energy tensor,

$$T(z)\phi(w) = \frac{\Delta}{(z-w)^2} + \frac{\partial\phi(w)}{z-w} + \cdots . \quad (2.5.15)$$

The following expression,

$$\frac{\langle T(z)\sigma_{[g_\infty]}[k_3,k_4](\infty)\sigma_{IJ}(1)\sigma_{K,L}(u,\bar{u})\sigma_{[g_0]}[k_1,k_2](0)\rangle}{\langle \sigma_{[g_\infty]}[k_3,k_4](\infty)\sigma_{IJ}(1)\sigma_{K,L}(u,\bar{u})\sigma_{[g_0]}[k_1,k_2](0)\rangle}, \quad (2.5.16)$$

can be deduced from the expression for $T(z)$ (2.3.5) and the OPE of free fields and twist fields (2.3.2). Using (2.5.15) one obtains the following differential equation for $G(u,\bar{u})$,

$$\partial_u G(u,\bar{u}) = H(u,\bar{u}), \quad (2.5.17)$$

where $H(u,\bar{u})$ is determined from (2.5.16),

$$\frac{\langle T(z)\sigma_{[g_\infty]}[k_3,k_4](\infty)\sigma_{IJ}(1)\sigma_{K,L}(u,\bar{u})\sigma_{[g_0]}[k_1,k_2](0)\rangle}{\langle \sigma_{[g_\infty]}[k_3,k_4](\infty)\sigma_{IJ}(1)\sigma_{K,L}(u,\bar{u})\sigma_{[g_0]}[k_1,k_2](0)\rangle} = \frac{\Delta_{KL}}{(z-w)^2} + \frac{H(u,\bar{u})}{z-w} + \cdots . \quad (2.5.18)$$

To progress further the N-fold covering map is needed,

$$u(t) = \frac{t^{n_0}(t-t_0)^{N-n_0}}{(t-t_\infty)^{N-n_\infty}} \frac{(t_1-t_\infty)^{N-n_\infty}}{t_1^{n_0}(t_1-t_0)^{N-n_0}}. \tag{2.5.19}$$

The triple $(u(t_0), u(t_1), u(t_\infty)) = (0, 1, \infty)$. Using projective transformations we define a new variable x in terms of (t_0, t_1, t_∞),

$$(t_0, t_1, t_\infty) = \left(x-1, x - \frac{(N-n_\infty)x}{(N-n_0)x+n_0}, \frac{N-n_0-n_\infty}{n_\infty} + \frac{n_0 x}{n_\infty} - \frac{N(N-n_\infty)x}{n_\infty(N-n_0)x+n_0} \right). \tag{2.5.20}$$

The differential equation for $G(u(x), \bar{u}(\bar{x}))$ is

$$\begin{aligned} \partial_x G(u, \bar{u}(\bar{x})) = \quad & -\frac{D}{16}\frac{d}{dx}\log u + \frac{d_0}{x} + \frac{d_1}{x-1} + \frac{d_2}{x+\frac{n_0}{N-n_0}} \\ & + \frac{d_3}{x - \frac{N-n_\infty-n_0}{N-n_0}} + \frac{d_4}{x-\frac{n_0}{n_0-n_\infty}} - \frac{D}{24}\left(\frac{1}{x-\alpha_1} + \frac{1}{x-\alpha_2}\right), \end{aligned} \tag{2.5.21}$$

where the parameters d_i and α_i depend on the momenta k_i and the integers n_0, n_∞ and N. The solution of (2.5.21) is

$$\begin{aligned} G(u,\bar{u}) \quad = \quad & C(g_0, g_\infty)\delta^D(\boldsymbol{k_1}+\boldsymbol{k_2}+\boldsymbol{k_3}+\boldsymbol{k_4})|u|^{-\frac{D}{8}}|x-\alpha_1|^{-\frac{D}{12}}|x-\alpha_2|^{-\frac{D}{12}} \\ & \times |x|^{2d_0}|x-1|^{2d_1}|x+\frac{n_0}{N-n_0}|^{2d_2}|x-\frac{N-n_\infty-n_0}{N-n_0}|^{2d_3}|x-\frac{n_0}{n_0-n_\infty}|^{2d_4}. \end{aligned} \tag{2.5.22}$$

The constant $C(g_0, g_\infty)$ is found to be,

$$\frac{2^{-11}}{n_0 n_\infty (N-n_0)(N-n_\infty)(n_0-n_\infty)^2}\left(\frac{N-n_0}{n_\infty - n_0}\right)^{2+\frac{1}{2}(k_1+k_3)k_4}. \tag{2.5.23}$$

When the appropriate expression for the correlator is substituted into (2.5.11) one finds that in the large N limit the S-matrix element is,

$$-i\frac{\delta^{D+2}(\sum_i k_i^\mu)}{\sqrt{k_1^+ k_2^+ k_3^+ k_4^+}} A(1,2,3,4), \tag{2.5.24}$$

where $A(1,2,3,4)$ is the Virasoro amplitude,

$$A(1,2,3,4) = \lambda^2 2^{-9} \int d^2 z |z|^{\frac{1}{2}k_1 k_4}|1-z|^{\frac{1}{2}k_3 k_4}. \tag{2.5.25}$$

Chapter 3

Two-Loop String Theory on Null Compactifications

In this chapter we compute partition functions for discrete light cone quantized strings as part of our investigation of the Matrix String Theory conjecture. These partition functions will be weighted sums over branched covers and will necessitate a parametrization of Hurwitz space, (the moduli space of branched covers).

3.1 Discrete Light-Cone Quantization of String Theory at Finite Temperature

Consider the discrete light-cone quantization (DLCQ) of Type II superstring theory at finite temperature using the Polyakov path integral [6]. We work throughout in the Neveu-Schwarz-Ramond formalism. In string perturbation theory, the gauge-fixed action in the conformal gauge and in Euclidean spacetime at genus g is $S[X]+\overline{S[X]}+S[B,C]+\overline{S[B,C]}$, where

$$S[X]+S[B,C]=\frac{1}{4\pi\,\alpha'}\int\limits_{\Sigma_g}\mathrm{d}^2z\,\left(\tfrac{1}{2}\,|\partial x^\mu|^2+\psi_\mu\,\overline{\partial}\psi^\mu+b\,\overline{\partial}c+\beta\,\overline{\partial}\gamma\right)\tag{3.1.1}$$

and $\sqrt{\alpha'}$ is the string scale. Here $X=(x^\mu,\psi^\mu)_{\mu=0}^9$ denotes the spacetime matter fields, while B and C denote the b,β and c,γ ghost fields, respectively, with (b,c) the spin $(2,1)$ conformal ghost fields and (β,γ) the spin $(\frac{3}{2},\frac{1}{2})$ superconformal ghost fields.

The worldsheet is an oriented compact Riemann surface Σ_g of genus g whose first homology group is generated by a set of canonical one-cycles $\boldsymbol{a} = (a_i)_{i=1}^g$, $\boldsymbol{b} = (b_i)_{i=1}^g$ with intersection numbers

$$a_i \cap a_j = b_i \cap b_j = 0 \qquad , \qquad a_i \cap b_j = -b_j \cap a_i = \delta_{ij} \ . \tag{3.1.2}$$

This intersection form is summarized by the matrix

$$\mathsf{J}_g = \begin{pmatrix} \mathbf{0}_g & \mathbb{1}_g \\ -\mathbb{1}_g & \mathbf{0}_g \end{pmatrix} \tag{3.1.3}$$

with $\mathsf{J}_g^2 = -\mathbb{1}_g$ which makes $H_1(\Sigma_g, \mathbb{R})$ into a symplectic vector space. The first cohomology group $H^{1,0}(\Sigma_g, \mathbb{C})$ is spanned by a set of holomorphic one-differentials $\boldsymbol{\omega} = (\omega_i)_{i=1}^g$ which have the period normalizations

$$\oint\limits_{a_i} \omega_j = \delta_{ij} \qquad , \qquad \oint\limits_{b_i} \omega_j = \Omega_{ij} \ , \tag{3.1.4}$$

where Ω is the period matrix of Σ_g which lives in the Siegal upper half-plane $\mathcal{H}_g$ of $g \times g$ complex-valued, symmetric matrices of positive definite imaginary part. We shall throughout write $\Omega = \Omega_1 + \mathrm{i}\,\Omega_2$, where Ω_1 and Ω_2 are real-valued symmetric matrices with $\Omega_2 > 0$.

The DLCQ and finite temperature conditions are imposed by two spacetime compactifications which may be described by the respective identifications

$$\begin{aligned} \big(x^0\,,\,\boldsymbol{x}\,,\,x^9\big) \ &\sim \ \big(x^0 + \sqrt{2}\,\pi\,\mathrm{i}\,R\,,\,\boldsymbol{x}\,,\,x^9 - \sqrt{2}\,\pi\,R\big)\ , \\ \big(x^0\,,\,\boldsymbol{x}\,,\,x^9\big) \ &\sim \ \big(x^0 + \beta\,,\,\boldsymbol{x}\,,\,x^9\big) \end{aligned} \tag{3.1.5}$$

where R is the radius of the light-cone in Minkowski space, and $\beta = 1/k_{\rm B}\,T$ with T the temperature and $k_{\rm B}$ the Boltzmann constant. The corresponding path integral, with the appropriate modification of the GSO projection to make spacetime fermions anti-periodic under $x^0 \to x^0 + \beta$, then computes the thermodynamic free energy of the superstring. The compactification conditions induce quantized zero modes in the mode expansions of the bosonic string embedding fields x^μ corresponding to the wrappings of the various homology cycles of Σ_g around the compact spacetime dimensions. The windings of $(\boldsymbol{a}, \boldsymbol{b})$ around the light-cone are labelled by integers

$(\boldsymbol{p}, \boldsymbol{q})$ and by $(\boldsymbol{n}, \boldsymbol{m})$ around the time direction. Apart from the modification of the GSO projection by the temperature winding numbers $(\boldsymbol{n}, \boldsymbol{m})$, the only place that these integers appear are as zero mode soliton contributions to the bosonic matter part of the action (3.1.1). To compute this contribution to the action expand the differentials dX^μ in terms of holomorphic, anti-holomorphic and exact differentials.

$$dX^0 = \sum_{i=1}^{g}(\lambda_i\omega_i + \bar{\lambda}_i\bar{\omega}_i) + \text{exact} \quad , dX^9 = \sum_{i=1}^{g}(\gamma_i\omega_i + \bar{\gamma}_i\bar{\omega}_i) + \text{exact} \tag{3.1.6}$$

Thewinding numbers come from the periods of the dX^0 and dX^9 differentials,

$$\oint_{a_i} dX^0 = \beta n_i + \sqrt{2}\pi Rip_i \quad , \quad \oint_{b_i} dX^0 = \beta m_i + \sqrt{2}\pi Riq_i \tag{3.1.7}$$

$$\oint_{a_i} dX^9 = \sqrt{2}\pi Rip_i \quad , \quad \oint_{a_i} dX^9 = \sqrt{2}\pi Riq_i. \tag{3.1.8}$$

This data is substituted into the bosonic part of the action,

$$-\frac{1}{4\pi\alpha'}\int dX^\mu \wedge *dX_\mu,$$

and using the Riemann Bilinear idenities

$$\int_{\Sigma_g} \omega_i \wedge \bar{\omega}_j = \sum_{k=1}^{g}(\oint_{a_k}\omega_i \oint_{b_k}\bar{\omega}_j - \oint_{b_k}\omega_i \oint_{a_k}\bar{\omega}_j) \tag{3.1.9}$$

we can compute the soliton piece,

$$S[X,\Psi] = \frac{\beta^2}{4\pi\alpha'}(\boldsymbol{n}\Omega^\dagger - \boldsymbol{m})\Omega_2^{-1}(\Omega\boldsymbol{n} - \boldsymbol{m})$$

$$+ 2\pi i\frac{\sqrt{2}\beta R}{4\pi\alpha'}\frac{1}{2}[(\boldsymbol{p}\Omega^\dagger - \boldsymbol{q})\Omega_2^{-1}(\Omega\boldsymbol{n} - \boldsymbol{m}) + (\boldsymbol{n}\Omega^\dagger - \boldsymbol{m})\Omega_2^{-1}(\Omega\boldsymbol{p} - \boldsymbol{q})] + \cdots \tag{3.1.10}$$

In the path integral one should sum over all possible topological winding sectors. The crucial point is that the action (3.1.1) depends linearly in a purely imaginary form on the set of integers $(\boldsymbol{p}, \boldsymbol{q})$, which when summed thereby produce periodic Dirac delta-functions.

In this way, the finite-temperature, DLCQ superstring free energy (per unit space-time volume) at genus g is found to be given by [6]

$$F_g = -g_s^{2g-2}\,\nu^{2g} \sum_{\mathbf{m},\mathbf{n}\in\mathbb{Z}^g}\ \sum_{\mathbf{r},\mathbf{s}\in\mathbb{Z}^g}\ \int_{\mathcal{F}_g} \mathrm{d}\mu_g \begin{bmatrix}\boldsymbol{n}\\ \boldsymbol{m}\end{bmatrix}(\Omega\,,\,\overline{\Omega})\ |\det\Omega_2|\ \mathrm{e}^{-\frac{\beta^2}{4\pi\alpha'}(\Omega\boldsymbol{n}-\boldsymbol{m})^\dagger(\Omega_2)^{-1}(\Omega\boldsymbol{n}-\boldsymbol{m})}$$

$$\times \prod_{j=1}^{g} \delta\left(\sum_{i=1}^{g} (n_i + \mathrm{i}\,\nu\, r_i)\, \Omega_{ij} - (m_j + \mathrm{i}\,\nu\, s_j) \right) \ , \tag{3.1.11}$$

where g_s is the string coupling constant and

$$\nu = \frac{4\pi\,\alpha'}{\sqrt{2}\,\beta\,R} \ . \tag{3.1.12}$$

The sums in (3.1.11) go over all four g-vectors of integers $\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{r}, \boldsymbol{s}$ such that the period matrix Ω is in a fundamental modular domain $\mathcal{F}_g$. The modular invariant, genus g superstring measure on moduli space $\mathcal{M}_g$ is denoted $\mathrm{d}\mu_g[{}^{\boldsymbol{n}}_{\boldsymbol{m}}](\Omega, \overline{\Omega}\,)$, and its dependence on the temperature winding integers arises from the modification of the sum over worldsheet spin structures that breaks supersymmetry in the finite temperature theory [40]. The expression (3.1.11) contains a constraint on the Riemann surfaces Σ_g which contribute to the partition function. As we now explain, it is equivalent to summing over all genus g branched covers Σ_g of the torus $\mathbb{T}^2_{\mathrm{i}\,\nu}$ whose Teichmüller parameter is $\mathrm{i}\,\nu$ [6]. It is worth mentioning that the DLCQ finite temperature PP-wave partition function also has a similar structure with the same sum over branched covers, it is not clear if this constraint holds at higher genus though.

Let $f : \Sigma_g \to \mathbb{T}^2_{\mathrm{i}\,\nu}$ be a holomorphic map, i.e. a branched covering. The covering map induces a homomorphism between the first homology groups via the push-forward

$$f_* \ : \ H_1(\Sigma_g, \mathbb{Z}) \ \longrightarrow \ H_1(\mathbb{T}^2_{\mathrm{i}\,\nu}, \mathbb{Z}) \ . \tag{3.1.13}$$

Choosing canonical homology bases $(\boldsymbol{a}, \boldsymbol{b})$ and (α, β) of the covering space Σ_g and the base space $\mathbb{T}^2_{\mathrm{i}\,\nu}$, respectively, this homomorphism can be written explicitly in terms of an integral $2 \times 2g$ matrix

$$\mathsf{M} = \begin{pmatrix} \boldsymbol{n} & \boldsymbol{m} \\ \boldsymbol{r} & \boldsymbol{s} \end{pmatrix} \tag{3.1.14}$$

of maximal rank acting on the homology generators of the base torus as

$$f_* \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} = \mathsf{M}^\top \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \ . \tag{3.1.15}$$

Similarly, the covering map induces through the pull-back a homomorphism $f^* : H^{1,0}(\mathbb{T}^2_{\mathrm{i}\,\nu}, \mathbb{C}) \to H^{1,0}(\Sigma_g, \mathbb{C})$ on the first cohomology groups, and there exists a complex $g \times 1$ matrix H of maximal rank which relates the normalized holomorphic differentials $\boldsymbol{\omega}$ and ω on Σ_g and $\mathbb{T}^2_{\mathrm{i}\,\nu}$ by

$$\omega = \mathsf{H}^\top \boldsymbol{\omega} \ . \tag{3.1.16}$$

The matrix H can be used to give an explicit formula for the covering map as $f(z) = (\Psi \circ \mathfrak{A})(z) := \mathsf{H}^\top \mathfrak{A}(z) \mod \mathbb{Z} \oplus \mathrm{i}\,\nu\,\mathbb{Z}$, where $\mathfrak{A}$ is the Abel map embedding Σ_g into its Jacobian variety $\mathrm{Jac}(\Sigma_g) := \mathbb{C}^g/\mathbb{Z}^g \oplus \Omega\,\mathbb{Z}^g$. This characterization exploits the fact that the Jacobian variety of the curve Σ_g represents a fibration over the elliptic curve $\mathbb{T}^2_{\mathrm{i}\,\nu}$ with the commutative diagram

$$\begin{array}{ccc} \Sigma_g & \xrightarrow{\ \mathfrak{A}\ } & \mathrm{Jac}(\Sigma_g) \ . \\ & \underset{f}{\searrow} & \downarrow{\scriptstyle \Psi} \\ & & \mathbb{T}^2_{\mathrm{i}\,\nu} \end{array} \tag{3.1.17}$$

Furthermore, by computing the α and β periods of both sides of (3.1.16) we arrive at the matrix equality

$$\mathsf{H}^\top \, (\mathbb{1}_g, \Omega) = (1, \mathrm{i}\,\nu)\, \mathsf{M} \ . \tag{3.1.18}$$

By using the explicit form (3.1.14) one finds $\mathsf{H} = \boldsymbol{n} + \mathrm{i}\,\nu\,\boldsymbol{r}$ and the equation (3.1.18) is equivalent to the period matrix constraint in (3.1.11). This argument leads to the following theorem proven in [5].

Theorem 3.1.1 *Σ_g is a branched cover of $\mathbb{T}^2_{i\nu}$ if and only if the period matrix obeys the constraint 3.1.18.*

The degree $\deg(f)$ of the covering map can be computed from the Hopf condition [49]

$$\mathsf{M}\, \mathsf{J}_g\, \mathsf{M}^\top = \deg(f)\ \mathsf{J}_1 \tag{3.1.19}$$

giving $\deg(f) = \boldsymbol{n} \cdot \boldsymbol{s} - \boldsymbol{m} \cdot \boldsymbol{r}$. The computation of the periods in (3.1.16) leads to a homogeneous linear equation in the variables $\boldsymbol{n}, \boldsymbol{m}, \boldsymbol{r}, \boldsymbol{s}$ and $\mathrm{i}\,\nu$ which has the compatibility condition

$$\det\begin{pmatrix} \mathbb{1}_g & & \Omega \\ \dots & \dots & \dots \\ & \mathsf{M} & \end{pmatrix} = 0 \ . \tag{3.1.20}$$

The formula (3.1.20) restricts the allowed Riemann period matrices of the curve Σ_g to lie in a Humbert variety inside $\mathcal{H}_g$.

3.1.1 Weierstrass-Poincaré Reduction

The superstring integration measure $\mathrm{d}\mu_g[{\boldsymbol{n} \atop \boldsymbol{m}}](\Omega,\overline{\Omega}\,)$ is invariant under the mapping class group of the covering Riemann surface. This invariance can be exploited in a manner which simplifies explicit calculations. The Siegal modular group of Σ_g is $Sp(2g,\mathbb{Z})$ which preserves the intersection form (3.1.3). With respect to the canonical basis of $\mathbb{R}^{g,g}$, it consists of matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad , \quad D^\top B - B^\top D = C^\top A - A^\top C = 0$$

$$, \quad A^\top D - C^\top B = \mathbb{1}_g \ . \tag{3.1.21}$$

This group acts on a canonical homology basis of Σ_g as

$$\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} \longmapsto \begin{pmatrix} \boldsymbol{a}' \\ \boldsymbol{b}' \end{pmatrix} = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} \ . \tag{3.1.22}$$

The temperature winding integers $(\boldsymbol{n},\boldsymbol{m})$ transform in the same way as $(\boldsymbol{a},\boldsymbol{b})$, and so do the integers $(\boldsymbol{r},\boldsymbol{s})$ which come from compactification of the light-cone. Using (3.1.21) the inverse of the transformation (3.1.22) is easily found to be

$$\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} = \begin{pmatrix} A^\top & -C^\top \\ -B^\top & D^\top \end{pmatrix} \begin{pmatrix} \boldsymbol{a}' \\ \boldsymbol{b}' \end{pmatrix} \ . \tag{3.1.23}$$

The projective modular group $PSp(2g,\mathbb{Z})$ acts naturally on the Siegal upper half-plane $\mathcal{H}_g$ of $g\times g$ period matrices as

$$\Omega \longmapsto \Omega' = (A\,\Omega + B)\,(C\,\Omega + D)^{-1} \ . \tag{3.1.24}$$

For genera $g = 1, 2, 3$, the moduli space of Σ_g is "almost" given by [50]

$$\mathcal{M}_g = \mathcal{H}_g / PSp(2g, \mathbb{Z}) \ . \tag{3.1.25}$$

For $g \geq 4$ the period matrix can still be used to parametrize moduli space by imposing Schottky relations on Ω. Note that the delta-function constraint of (3.1.11) is modular covariant.

We can now use the technique of reduction to simplify the constraint equation on Ω in (3.1.11) before solving it. The technique of reduction is described in [49] (see also [51, 52]) for the general case of coverings of Riemann surfaces of arbitrary genus. Reduction comprises the use of modular transformations on the base and cover in order to make a change in homology basis so that the number of homology cycles on the cover which effectively wind around the base is reduced. It yields a convenient canonical form for the underlying algebraic curve Σ_g which can be thought of as a higher genus version of the canonical Weierstrass parametrization of an elliptic curve. In the present case the periods meet the conditions of the fundamental Weierstrass-Poincaré theory of the complete reducibility of abelian integrals to lower genera [49], which deals with general abelian tori and their associated theta-functions. The main idea is that the curve Σ_g, being a covering of a torus, has a rich group of automorphisms which leads to a decomposition of its Jacobian variety. By considering the curve as a spectral variety, one can thereby reduce the corresponding theta-functions to lower genera. Furthermore, the technique greatly simplifies the analysis of moduli space integrals such as (3.1.11) by extending the usual Rankin-Selberg method of "unwrapping" modular integrals [53].

We will use the Poincaré reducibility theorem applied to the special case of a covering $f : \Sigma_g \to \mathbb{T}^2_{\mathrm{i}\nu}$. It relies [49] on the existence of a Frobenius normal form for the $2 \times 2g$ integral matrix (3.1.14), satisfying the Hopf condition (3.1.19), given by

$$\mathsf{M} = \mathsf{S}\,\mathsf{P}\,\mathsf{T} \tag{3.1.26}$$

where S and T are, respectively, 2×2 and $2g \times 2g$ symplectic unimodular matrices. The Poincaré normal form is given by the $2 \times 2g$ matrix

$$\mathsf{P} = r \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & s & 0 & \dots & 0 & t & 0 & 0 & \dots & 0 \end{pmatrix} \ , \tag{3.1.27}$$

where r,s,t are integers such that $r^2\,t=\deg(f)$ and s either vanishes or is a divisor of t. The cases $s=0$ can be ruled out by the requirement that block diagonal period matrices are not allowed [54], being contributions from a particular boundary component of moduli space. The existence of the form (3.1.27) implies, among other things, that there exists a modular transformation such that the windings around the temperature direction of spacetime occur *only* around the single homology cycle a_1, with all other cycles being periodic. This means that the compactification conditions can be chosen to be

$$\begin{aligned}
\oint\limits_{a_1'} \mathrm{d}x^0 &= \beta\,r+\sqrt{2}\,\pi\,R\,\mathrm{i}\,p_1' \ , \\
\oint\limits_{a_j'} \mathrm{d}x^0 &= \sqrt{2}\,\pi\,R\,\mathrm{i}\,p_j' \ , \quad j=2,\dots,g \ , \\
\oint\limits_{b_i'} \mathrm{d}x^0 &= \sqrt{2}\,\pi\,R\,\mathrm{i}\,q_i' \ , \\
\oint\limits_{a_i'} \mathrm{d}x^9 &= \sqrt{2}\,\pi\,R\,p_i' \ , \\
\oint\limits_{b_i'} \mathrm{d}x^9 &= \sqrt{2}\,\pi\,R\,q_i'
\end{aligned} \tag{3.1.28}$$

for $i=1,\dots,g$, with the transverse components $\boldsymbol{x}$ periodic around the new basis of homology cycles $\boldsymbol{a}',\boldsymbol{b}'$ of Σ_g. In addition, after summation over $\boldsymbol{p}',\boldsymbol{q}'$ only the homology cycles a_2 and b_1 wrap around the light-cone.

Reduction depends on the number theoretic properties of the entries of the integral matrix M and is explicitly carried out by using the $2g\times 2g$ symplectic matrices

$$\begin{pmatrix} \mathbb{1}_g & S \\ \mathbf{0}_g & \mathbb{1}_g \end{pmatrix} \ , \quad \begin{pmatrix} \mathbb{1}_g & \mathbf{0}_g \\ S & \mathbb{1}_g \end{pmatrix} \ , \quad \begin{pmatrix} A & \mathbf{0}_g \\ \mathbf{0}_g & (A^{-1})^\top \end{pmatrix} \ , \tag{3.1.29}$$

where S is a symmetric $g\times g$ integral matrix and $A\in SL(g,\mathbb{Z})$. By regarding the matrix (3.1.14) as consisting of two $2\times g$ block matrices $\mathsf{M}=(\mathsf{M}_1,\mathsf{M}_2)$, the matrices (3.1.29) interpolate between these blocks via elementary row and column operations.

Using the normal form (3.1.26) one can transform (3.1.18) into the equation of the Weierstrass-Poincaré theorem

$$(\mathbb{1}_g, \Omega)\,\mathsf{T} = \mathsf{F}\begin{pmatrix} 1 & 0 & \dots & 0 & -\frac{\sigma_1}{t\,\sigma_2} & \boldsymbol{q} \\ \mathbf{0}_{(g-1)\times 1} & & \mathbb{1}_{g-1} & \boldsymbol{q}^\top & \mathsf{Z} & \end{pmatrix}\,, \tag{3.1.30}$$

where F is a non-singular $g \times g$ complex matrix, $\boldsymbol{q} = (-\frac{s}{t}, 0, \dots, 0)$ is a $(g-1)$-vector, the complex numbers σ_1, σ_2 are defined by $(\sigma_1, \sigma_2) = (1, \mathrm{i}\,\nu)\,\mathsf{S}$, and Z is a $(g-1) \times (g-1)$ complex matrix satisfying the Riemann bilinear relations which can be found after explicit construction of the symplectic transformation. Because the vector $\boldsymbol{q}$ is rational-valued, the corresponding genus g theta-functions factorize into products of theta-functions of lower genera based on the curves with periods $\frac{\sigma_1}{t\,\sigma_2}$ and Z.

The reduction to the Poincaré normal form (3.1.27) can be thought of as a gauge fixing of the large diffeomorphism symmetry (the mapping class group) of the Riemann surface Σ_g. There is still then a residual gauge symmetry left over, which we will fix by restricting to those modular transformations which preserve the corresponding reduced compactification conditions. This defines a proper subgroup $\mathcal{G} \subset Sp(2g, \mathbb{Z})$, and so it will *extend* the fundamental modular region for the action of $Sp(2g, \mathbb{Z})$ on $\mathcal{H}_g$ from $\mathcal{F}_g$ to some domain $\mathcal{F}'_g$. Modular invariance is then restored via the observation [53] that the new region $\mathcal{F}'_g$ is composed of an infinite number of images of the fundamental domain $\mathcal{F}_g$ under certain elements of the modular group. The sum over all copies of $\mathcal{F}_g$ in $\mathcal{F}'_g$ may be implemented by a sum over all elements of the coset $Sp(2g, \mathbb{Z})/\mathcal{G}$. The corresponding constraints on the period matrix Ω in (3.1.30) reduce the complex dimension $3g-3$ of moduli space to $2g-3$. In addition, there is discrete data contained in the compactification integers, such as those arising from the requirement that the real-valued symmetric matrix Ω_2 be positive. This gives a partial discretization of the Riemann moduli space $\mathcal{M}_g$ to the Hurwitz moduli space of holomorphic maps, with the $2g-3$ moduli given by the branching data required to build the cover Σ_g from its base $\mathbb{T}^2_{\mathrm{i}\,\nu}$. The Hurwitz space can be embedded as an analytic subvariety of $\mathcal{M}_g$ [55, 56].

3.1.2 One-Loop Computation

It is instructive to recall the genus one situation [5, 6]. Then all covers $\Sigma_1 \to \mathbb{T}^2_{\mathrm{i}\nu}$ are unbranched. In this case one can deduce the period constraint of (3.1.11) by elementary methods which exhibit the geometric construction of the covering torus from the base torus in terms of the compactification integers specified by (3.1.14). For this, let us regard the torus $\mathbb{T}^2_{\mathrm{i}\nu}$ as the quotient of the complex plane $\mathbb{C}$ by a lattice $\Lambda = \langle e_1, e_2\rangle := \mathbb{Z}\, e_1 \oplus \mathbb{Z}\, e_2$ of rank 2 generated by two-vectors e_1 and e_2. The isomorphism classes of unramified covers $\Sigma_1 \to \mathbb{T}^2_{\mathrm{i}\nu}$ of degree N then correspond to the inequivalent sublattices $\Lambda' \subset \Lambda$ of index $[\Lambda : \Lambda'\,] = N$. These may be found as follows. Let $f_1 = r'\, e_1 \in \Lambda'$ be the smallest multiple of e_1. Then there exists $f_2 = s'\, e_1 + m'\, e_2 \in \Lambda'$ with $s' < r'$ such that Λ' is generated by f_1 and f_2 over $\mathbb{Z}$. The index of this lattice is $r'\, m'$. As a consequence, for each integer r' dividing the index N there are r' inequivalent sub-lattices

$$\big\langle r'\, e_1, \tfrac{N}{r'}\, e_2\big\rangle \quad , \quad \big\langle r'\, e_1, e_1 + \tfrac{N}{r'}\, e_2\big\rangle \quad , \quad \dots \quad , \quad \big\langle r'\, e_1, (r'-1)\, e_1 + \tfrac{N}{r'}\, e_2\big\rangle \ . \tag{3.1.31}$$

It follows that the number of inequivalent sublattices $\Lambda' \subset \Lambda$ of index $[\Lambda : \Lambda'\,] = N$ is

$$\sigma_1(N) = \sum_{r'|N} r' \ , \tag{3.1.32}$$

and the moduli of the corresponding covers are given by

$$\tau = \frac{s' + \frac{\mathrm{i}}{\nu}\, m'}{r'} \ . \tag{3.1.33}$$

We will now use the Weierstrass-Poincaré reduction to show that solving the reduced constraint in this case gives the same moduli (3.1.33) of the covers constructed from the base modular parameter $\mathrm{i}\,\nu$. The integers $n' = 0, m' \in \mathbb{Z}$ are defined by the $SL(2,\mathbb{Z})$ transformation

$$n' = 0 = D\, n + C\, m \qquad , \qquad -\, m' = B\, n + A\, m \ . \tag{3.1.34}$$

The first equation is solved by the relatively prime integers $C = -n/\gcd(n,m)$ and $D = m/\gcd(n,m)$. Now we use the fact that the set of integers $\mathbb{Z}$ is a principal ideal domain, which implies that there exists integers A and B such that

$$A\, m + B\, n = \gcd(n,m) \ . \tag{3.1.35}$$

Reduction for the genus one case is thus very simple, as all the windings of the cover Σ_1 around the temperature direction are put into the b cycle by the $SL(2,\mathbb{Z})$ transformation generated by the unimodular matrix

$$\mathsf{T}_1 = \begin{pmatrix} \frac{m}{\gcd(n,m)} & B \\ -\frac{n}{\gcd(n,m)} & A \end{pmatrix} . \tag{3.1.36}$$

Furthermore, from (3.1.34) it follows that the sole temperature winding integer is given by the greatest common divisor of the original two winding numbers as

$$m' = -\gcd(n,m) . \tag{3.1.37}$$

The constraint equation for the modulus τ of the cover Σ_1 is given by

$$\mathsf{H}^\top(1,\tau) = (1,\,\mathrm{i}\,\nu)\begin{pmatrix} n & m \\ r & s \end{pmatrix} = (1,\,\mathrm{i}\,\nu)\begin{pmatrix} 0 & -m' \\ r' & s' \end{pmatrix}\begin{pmatrix} A & -B \\ \frac{n}{\gcd(n,m)} & \frac{m}{\gcd(n,m)} \end{pmatrix}, \tag{3.1.38}$$

which can be solved explicitly to determine τ as in (3.1.33) with

$$r' = \frac{m\,r - n\,s}{\gcd(n,m)} \qquad , \qquad s' = B\,r + A\,s . \tag{3.1.39}$$

The genus one fundamental domain is given by

$$\Delta := \mathcal{F}_1 = \Big\{\tau \in \mathbb{C} \;\Big|\; -\tfrac{1}{2} < \tau_1 \le \tfrac{1}{2} \;,\; |\tau|^2 \ge 1 \;,\; \tau_2 > 0\Big\} . \tag{3.1.40}$$

Requiring the reduced compactification constraints to be modular invariant sets $C=0$ and $A=D=1$ in (3.1.21). Thus only the translations $\tau \mapsto \tau + B$, $B \in \mathbb{Z}$ survive under the action of the restricted modular group $\mathcal{G}$ on Teichmüller space, and the fundamental modular region is extended to the strip

$$\Delta' := \mathcal{F}_1' = \Big\{\tau \in \mathbb{C} \;\Big|\; -\tfrac{1}{2} < \tau_1 \le \tfrac{1}{2} \;,\; \tau_2 > 0\Big\} . \tag{3.1.41}$$

Requiring that $\tau \in \Delta'$ is then equivalent to $s' \in \mathbb{Z}/r'\,\mathbb{Z}$, $N := m'\,r' > 0$.

The integration measure on moduli space is obtained by computing the standard zero temperature, chiral Laplacian determinants on the torus induced by integrating out the ten worldsheet bosonic fields x^μ, their superpartners ψ^μ, and the ghosts, in a given spin structure. The GSO projection then dictates to sum over the three even spin structures in each of the left and right moving sectors of the worldsheet field theory (The odd spin structure $(1,1)$ yields a vanishing contribution due to the zero modes of the free worldsheet fermion fields ψ^μ). The appropriate modification which makes the spacetime fermion fields antiperiodic inserts an extra phase factor $(-1)^{m'}$ in front of the GSO phase associated with the $(0,1)$ spin structure. The modular invariant, finite-temperature superstring measure is thereby given as [40]

$$\mathrm{d}\mu_1^{(m')}(\tau\,,\,\overline{\tau}) = \left(\frac{1}{4\pi^2\,\alpha'}\right)^5 \frac{\mathrm{d}^2\tau}{(\tau_2)^6}\,\frac{1}{4\big|\eta(\tau)\big|^8}\,\Big|\theta_2(0|\tau)^4-\theta_3(0|\tau)^4+\mathrm{e}^{\,\pi\,\mathrm{i}\,m'}\,\theta_4(0|\tau)^4\Big|^2 \ . \tag{3.1.42}$$

Here the Jacobi-Erderlyi functions $\theta_a(z|\tau)$, $a = 2,3,4$ (which are induced by the spacetime fermion fields and the superconformal ghost fields) are defined in terms of the three even characteristic, genus one theta-functions as $\theta_2=\theta\binom{1}{0}$, $\theta_3=\theta\binom{0}{0}$, and $\theta_4=\theta\binom{0}{1}$, where

$$\theta\begin{pmatrix}a\\ b\end{pmatrix}(z|\tau) = \sum_{n=-\infty}^{\infty}\ \mathrm{e}^{\,\pi\,\mathrm{i}\,(n+\frac12\,a)^2\,\tau}\ \ \mathrm{e}^{\,2\pi\,\mathrm{i}\,(n+\frac12\,a)\,(z+\frac12\,b)} \tag{3.1.43}$$

are holomorphic functions of $(z|\tau)\in\mathbb{C}\times\mathcal{H}_1$ for $a,b\in\mathbb{R}$, while

$$\eta(\tau) = \tfrac12\,\theta_2(0|\tau)\,\theta_3(0|\tau)\,\theta_4(0|\tau) \tag{3.1.44}$$

is the Dedekind function (which is induced by the spacetime boson fields and the conformal ghost fields). By using the Jacobi abstruse identity

$$\theta_3(0|\tau)^4-\theta_4(0|\tau)^4-\theta_2(0|\tau)^4 = 0\ , \tag{3.1.45}$$

we can simplify the expression (3.1.42) to

$$\mathrm{d}\mu_1^{(m')}(\tau\,,\,\overline{\tau}) = \left(\frac{1}{4\pi^2\,\alpha'}\right)^5 \frac{\mathrm{d}^2\tau}{(\tau_2)^6}\,\frac{\big(1-\mathrm{e}^{\,\pi\,\mathrm{i}\,m'}\big)\,\big|\theta_4(0|\tau)\big|^8}{2\big|\eta(\tau)\big|^8}\ . \tag{3.1.46}$$

By substituting all of these expressions back into the genus one free energy (3.1.11) and integrating the delta-function with the appropriate Jacobian factor, we arrive finally at

$$F_1 = -\frac{1}{\sqrt2\,\pi\,R\,\beta}\sum_{N=1}^{\infty}\ \mathrm{e}^{\,-\frac{\beta\,N}{\sqrt2\,R}}\ \mathbf{H}_N * \left[\left(\frac{1}{4\pi^2\,\alpha'\,\tau_2}\right)^4\,\frac{\big|\theta_4(0|\tau)\big|^8}{\big|\eta(\tau)\big|^8}\right]\,\bigg|_{\tau=\mathrm{i}\,/\nu}\ , \tag{3.1.47}$$

where $\mathbf{H}_N$ are the (restricted) Hecke operators [57] whose actions on a modular invariant function $f(\tau,\overline{\tau}\,)$ on Teichmüller space are defined by

$$\mathbf{H}_N * f(\tau,\overline{\tau}\,) = \frac{1}{N} \sum_{\substack{m'\,r'=N \\ m'\,\mathrm{odd}}} \ \sum_{s'\in\mathbb{Z}/r'\,\mathbb{Z}} f\left(\tfrac{s'+\tau\,m'}{r'}, \tfrac{s'+\overline{\tau}\,m'}{r'}\right) \ . \tag{3.1.48}$$

By applying the modular transformation $\tau \mapsto -1/\tau$ and using the transformation rules

$$\theta_4(\tfrac{z}{\tau}|-\tfrac{1}{\tau}) = \sqrt{-\,\mathrm{i}\,\tau}\ \ \mathrm{e}^{\pi\,\mathrm{i}\,z^2/\tau}\ \theta_2(z|\tau) \qquad , \qquad \eta(-\tfrac{1}{\tau}) = \sqrt{-\,\mathrm{i}\,\tau}\ \eta(\tau)\ , \tag{3.1.49}$$

we can transform the expression (3.1.47) into the equivalent form

$$F_1 = -\frac{1}{\sqrt{2}\,\pi\,R\,\beta} \sum_{N=1}^{\infty} \mathrm{e}^{-\frac{\beta\,N}{\sqrt{2}\,R}}\ \mathbf{H}_N * \left[\left(\frac{1}{4\pi^2\,\alpha'\,\tau_2}\right)^4 \frac{\left|\theta_2(0|\tau)\right|^8}{\left|\eta(\tau)\right|^8} \right] \Bigg|_{\tau=\mathrm{i}\,\nu} \ . \tag{3.1.50}$$

The operand of the Hecke operators in (3.1.50) is the partition function of a first quantized Green-Schwarz superstring, so that the expression (3.1.50) has a natural interpretation as a map from a first quantized to a second quantized superstring theory [4]. The discrete Teichmüller parameters (3.1.33) indicate how the homology cycles of the base $\mathbb{T}^2_{\mathrm{i}\,\nu}$ wind around the cycles of the unbranched cover Σ_1. The combinatorics of enumerating unbranched covers of the torus $\mathbb{T}^2_{\mathrm{i}\,\nu}$ are thereby elegantly accounted for by the Hecke operators acting on the partition function of a superconformal field theory, with toroidal worldsheet and target space $\mathbb{R}^8$, in (3.1.50). This result agrees with both the computation using operator quantization in light-cone gauge and in matrix string theory [5], as well as in the superconformal field theory on the symmetric product orbifold [31]– [32]. The calculation can also be applied to bosonic and heterotic strings, with the final result always being the insertion of the appropriate one-loop light-cone Green-Schwarz string partition function in the operand of the Hecke operator in (3.1.50). In what follows we shall extend these one-loop calculations to the case of genus two branched covers Σ_2 of the torus $\mathbb{T}^2_{\mathrm{i}\,\nu}$.

3.2 Bosonic Strings

We will now extend the calculation of Section 3.1.2 by computing the two-loop free energy F_2 in (3.1.11). As a warm up, in this section we will look at the simpler setting of bosonic string theory (whose action is obtained from (3.1.1) by dropping all Grassmann fields in 26 spacetime dimensions) for which the moduli space integration measure is more manageable. This will make the various reduction techniques that we present more transparent. They will also carry through to the superstring and heterotic string cases which will be studied in the next two sections. There is a fairly complete picture of Teichmüller space and moduli space at genus two. Every genus two surface admits a hyperelliptic representation as a double cover of the complex plane with three quadratic branch cuts supported by six branch points. While this description is useful for describing interacting matrix strings [59, 60], it is not the natural parametrization for DLCQ strings.

3.2.1 Two-Loop Worldsheet Contributions

The two-loop free energy is given by a sum over (equivalence classes of) non-constant holomorphic maps $f : \Sigma_2 \to \mathbb{T}^2_{i\nu}$. Let us begin by summarizing some useful facts about these contributing worldsheets [9]. By the Riemann-Hurwitz theorem, the total branching number B for the branched cover of a torus by a genus two surface Σ_2 is $B = 2$. This means that a covering $f : \Sigma_2 \to \mathbb{T}^2_{i\nu}$ has three possible types of singularities: (a) Two simple branch points; (b) one branch point with two ramification points each of ramification index 2; or (c) one branch point with one ramification point of ramification index 3. The singularity types (b) and (c) can each be thought of as degenerate limits of type (a), which in this sense represents the generic situation.

The lifting of curves from $\mathbb{T}^2_{\imath\nu}$ to the covering space Σ_2 induces a group homomorphism

$$f_\# : \pi_1\big(\mathbb{T}^2_{\imath\nu} \setminus \mathcal{B}_f\big) \longrightarrow S_N \tag{3.2.1}$$

where $\mathcal{B}_f \subset \mathbb{T}^2_{\imath\nu}$ is the branch locus of the covering map f, $N = \deg(f)$, and $\pi_1(\mathbb{T}^2_{\imath\nu} \setminus \mathcal{B}_f) \cong \langle \alpha, \beta, \gamma_1, \gamma_2 \mid \alpha\,\beta\,\alpha^{-1}\,\beta^{-1}\,\gamma_1\,\gamma_2 = \mathbb{1} \rangle$ (with $\gamma_2 = \mathbb{1}$ in the case that $\mathcal{B}_f$ consists of a single non-simple branch point). Let γ_t be a homotopy generator which surrounds a branch point $t \in \mathcal{B}_f \subset \mathbb{T}^2_{\imath\nu}$. If t is simple, then the permutation $f_\#(\gamma_t) \in S_N$ has a single non-trivial cycle of length 2. Otherwise, $f_\#(\gamma_t)$ either contains two non-trivial cycles of length 2 or it has a single non-trivial cycle of length 3.

Together with the canonical homology generators α, β, these permutations generate a transitive subgroup $\mathcal{T}_{N,\mathcal{B}_f}$ of S_N and the induced map (3.2.1) is an isomorphism onto this subgroup. There is a one-to-one correspondence between elements of $\mathcal{T}_{N,\mathcal{B}_f}$ and irreducible branched covers. The two-loop free energy that we obtain in this and the subsequent sections are thus generating functions for the orbits in $\mathcal{T}_{N,\mathcal{B}_f}$ under conjugation by permutations in S_N.

3.2.2 Modular Parameters

We will now find the most general form of the 2×2 period matrix Ω of the covering surface Σ_2. This will be achieved by using a modified version of the reduction technique described in Section 3.1.1 to solve the genus two constraint which gives the moduli of the genus two branched covers of the torus $\mathbb{T}^2_{\mathrm{i}\nu}$. The constraint equation (3.1.18) in this case reads

$$\mathsf{H}^\top (\mathbb{1}_2, \Omega) = (1, \mathrm{i}\,\nu) \begin{pmatrix} n_1 & n_2 & m_1 & m_2 \\ r_1 & r_2 & s_1 & s_2 \end{pmatrix} , \tag{3.2.2}$$

where $\sum_{i=1,2} (n_i\, s_i - m_i\, r_i) = \deg(f) =: N$ is the degree of the cover.

As in the one-loop calculation, it is possible to calculate part of the matrix T appearing in the Frobenius normal form (3.1.26) by choosing integers A_i, B_i such that

$$A_i\, m_i + B_i\, n_i = \gcd(n_i, m_i) =: n_i' \ , \quad i = 1, 2 \ . \tag{3.2.3}$$

Then the $Sp(4, \mathbb{Z})$ matrix

$$\Lambda_a = \begin{pmatrix} B_1 & 0 & -\frac{m_1}{n_1'} & 0 \\ 0 & B_2 & 0 & -\frac{m_2}{n_2'} \\ A_1 & 0 & \frac{n_1}{n_1'} & 0 \\ 0 & A_2 & 0 & \frac{n_2}{n_2'} \end{pmatrix} \tag{3.2.4}$$

transfers all windings from the b_i homology cycles to the a_i cycles, i.e. it defines a Rankin-Selberg modular transformation (3.1.22) for which the 2×4 integral matrix M becomes

$$\mathsf{M} \ \longmapsto \ \begin{pmatrix} n_1' & n_2' & 0 & 0 \\ r_1' & r_2' & s_1' & s_2' \end{pmatrix} \tag{3.2.5}$$

with $r_i' = B_i\, r_i + A_i\, s_i$ and $s_i' = \frac{1}{n_i'}\,(n_i\, s_i - m_i\, s_i)$ for $i = 1, 2$. The matrix Λ_a belongs to an $SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$ subgroup of the full modular group $PSp(4,\mathbb{Z}) \cong SO(3,2,\mathbb{Z})$.

The next step is to move the temperature windings from the a_2 cycle to the a_1 cycle. For this, we introduce two further integers U_1 and U_2 with

$$U_1\, n_1' + U_2\, n_2' = \gcd(n_1', n_2') =: r \ . \tag{3.2.6}$$

Then the $Sp(4,\mathbb{Z})$ matrix Λ_b given by

$$\Lambda_b = \begin{pmatrix} U_1 & -\frac{n_2'}{r} & 0 & 0 \\ U_2 & \frac{n_1'}{r} & 0 & 0 \\ 0 & 0 & \frac{n_1'}{r} & -U_2 \\ 0 & 0 & \frac{n_2'}{r} & U_1 \end{pmatrix} \tag{3.2.7}$$

will perform the necessary operation. It belongs to an $SL(2,\mathbb{Z})$ subgroup of the mapping class group. The desired transformation of M for which all temperature windings have been transfered to the a_1 homology cycle is therefore described by the matrix

$$\mathsf{M}' := \mathsf{M}\, \Lambda_a\, \Lambda_b = \begin{pmatrix} r & 0 & 0 & 0 \\ x' & y' & z' & w \end{pmatrix} \tag{3.2.8}$$

where $x' = U_1\, r_1' + U_2\, r_2'$, $y' = \frac{1}{r}\,(n_1'\, r_2' - n_2'\, r_1')$, $z' = \frac{1}{r}\,(n_1'\, s_1' + n_2'\, s_2')$ and $w = U_1\, s_2' - U_2\, s_1'$.

We now construct a third transformation matrix $\Lambda_c \in Sp(4,\mathbb{Z})$ by disregarding the first and third columns of the matrix (3.2.8) and writing

$$\begin{pmatrix} 0 & 0 \\ y' & w \end{pmatrix} \begin{pmatrix} Y & -\frac{w}{z} \\ W & \frac{y'}{z} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix} , \tag{3.2.9}$$

where the integers Y and W obey

$$Y\, y' + W\, w = \gcd(y', w) =: z \ . \tag{3.2.10}$$

This does not affect the zeroes in the first row of (3.2.8), and the symplectic matrix $\mathsf{T} = \Lambda_a\, \Lambda_b\, \Lambda_c$ finally reduces the matrix M to the form

$$\mathsf{M}' \longmapsto \begin{pmatrix} r & 0 & 0 & 0 \\ x & y & z & 0 \end{pmatrix} \tag{3.2.11}$$

with $x, y, z \in \mathbb{Z}$. Note that we do not apply the 2×2 matrix S here, which affects an $SL(2, \mathbb{Z})$ modular transformation of the base $\mathbb{T}^2_{\mathrm{i}\,\nu}$. The complete Poincaré normal form (3.1.27) is derived in Appendix A.

In this way the constraint equation (3.2.2) reduces to

$$\mathsf{H}^\top (1\!\!1_2, \Omega) = (1, \mathrm{i}\,\nu) \begin{pmatrix} r & 0 & 0 & 0 \\ x & y & z & 0 \end{pmatrix} \mathsf{T} \ . \tag{3.2.12}$$

Now we factor out a symplectic unit on the right-hand side of this equation in order that the eventual solution of the constraint equation gives a period matrix with rational-valued off-diagonal elements. This gives

$$\mathsf{H}^\top (1\!\!1_2, \Omega) = (1, \mathrm{i}\,\nu) \begin{pmatrix} 0 & 0 & -r & 0 \\ z & 0 & -x & -y \end{pmatrix} \mathsf{J}_2\, \mathsf{T} \ . \tag{3.2.13}$$

The matrix $\mathsf{J}_2\, \mathsf{T} \in Sp(4, \mathbb{Z})$ is non-singular, and its inverse $(\mathsf{J}_2\, \mathsf{T})^{-1}$ acts on the left-hand side of (3.2.13) as a modular transformation of the period matrix Ω and the pullback vector $\mathsf{H}^\top$. Parametrizing it by a block matrix of the form (3.1.21), one has

$$\mathsf{H}^\top (1\!\!1_2, \Omega)\, (\mathsf{J}_2\, \mathsf{T})^{-1} = \mathsf{H}^\top (C\,\Omega + D) \Big(1\!\!1_2\,,\, (C\,\Omega + D)^{-1}\, (A\,\Omega + B)\Big) =: \mathsf{H}'\, (1\!\!1_2, \Omega'\,) \tag{3.2.14}$$

giving

$$\mathsf{H}'\, (1\!\!1_2, \Omega'\,) = (1, \mathrm{i}\,\nu) \begin{pmatrix} 0 & 0 & -r & 0 \\ z & 0 & -x & -y \end{pmatrix} \ . \tag{3.2.15}$$

We can now solve the constraint (3.2.15) to get

$$\mathsf{H} = (1, \mathrm{i}\,\nu) \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix} = (\,\mathrm{i}\,\nu\, z, 0) \tag{3.2.16}$$

and

$$\mathsf{H}\,\Omega = (\,\mathrm{i}\,\nu\, z, 0) \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12} & \Omega_{22} \end{pmatrix} = (1, \mathrm{i}\,\nu) \begin{pmatrix} -r & 0 \\ -x & -y \end{pmatrix} \ , \tag{3.2.17}$$

where for notational ease we have dropped the primes indicating the modular transformations (The free energy is modular invariant). The period matrix is finally given in the form

$$\Omega = \begin{pmatrix} -\frac{x + \frac{r}{\mathrm{i}\,\nu}}{z} & -\frac{y}{z} \\ -\frac{y}{z} & \Omega_{22} \end{pmatrix} \tag{3.2.18}$$

with $r, x, y, z \in \mathbb{Z}$ and $\Omega_{22} \in \mathcal{H}_1$. This form of the period matrix has a natural geometrical interpretation. The diagonal elements are related to the moduli of two tori which have been sewn together along the branch cut of $\mathbb{T}^2_{\mathrm{i}\,\nu}$ to form the genus two cover. The element $-\frac{x+r/\mathrm{i}\,\nu}{z}$ is the modulus of a degree $N = r\,z$ unbranched cover Σ_1 of the torus $\mathbb{T}^2_{\mathrm{i}\,\nu}$ as obtained in Section 3.1.2. The off-diagonal element is a measure of the radius and length of the cylinder joining the two tori when they are glued together along the branch cut of $\mathbb{T}^2_{\mathrm{i}\,\nu}$. This picture will be elucidated later on when we study degeneration limits of the branched covers Σ_2 in Section 3.5. Using the projective modular symmetry $PSp(4, \mathbb{Z})$ defining the moduli space $\mathcal{M}_2$, we will identify $\Omega \sim -\Omega$ in (3.2.18).

This calculation demonstrates that the existence of the covering $f : \Sigma_2 \to \mathbb{T}^2_{\mathrm{i}\,\nu}$, reducing a holomorphic differential on Σ_2 to an elliptic one (3.1.16), necessarily implies [52] the existence of another (generally distinct) covering $f' : \Sigma_2 \to \mathbb{T}^2_\tau$ which leads to a reduction of some other independent holomorphic differential on Σ_2 to an elliptic one. In this case, the Jacobian of the curve Σ_2 represents a fibration whose base and fibre are the elliptic curves $\mathbb{T}^2_{\mathrm{i}\,\nu}$ and $\mathbb{T}^2_\tau$, with the commutative diagram

$$
\begin{array}{ccc}
 & & \mathbb{T}^2_\tau \\
 & \overset{f'}{\nearrow} & \uparrow \Psi' \\
\Sigma_2 & \xrightarrow{\mathfrak{A}} & \mathrm{Jac}(\Sigma_2) \ . \\
 & \underset{f}{\searrow} & \downarrow \Psi \\
 & & \mathbb{T}^2_{\mathrm{i}\,\nu}
\end{array}
\tag{3.2.19}
$$

The curve Σ_2 is embedded by the Abel map $\mathfrak{A}$ into its Jacobian variety as a divisor. The relationship (3.2.19) will then split the contribution to the two-loop effective string action from Σ_2 into individual contributions from the two tori $\mathbb{T}^2_\tau$ and $\mathbb{T}^2_{\mathrm{i}\,\nu}$, as we shall see explicitly below.

3.2.3 Moduli Space

The subgroup $\mathcal{G}$ of $Sp(4,\mathbb{Z})$ transformations which leave the structure of the integral matrices

$$\begin{pmatrix} 0 & 0 & -r & 0 \\ z & 0 & -x & -y \end{pmatrix} \tag{3.2.20}$$

in (3.2.15) invariant has four generators and consists of unimodular matrices of the generic form

$$\begin{pmatrix} 1 & A_{12} & B_{11} & B_{12} \\ 0 & A_{22} & B_{12} & B_{22} \\ 0 & 0 & 1 & 0 \\ 0 & C_{22} & D_{21} & D_{22} \end{pmatrix} \tag{3.2.21}$$

which obey the non-linear constraints

$$\begin{aligned} A_{22}\,D_{22} - B_{22}\,C_{22} &= 1 \ , \\ A_{22}\,D_{21} - B_{21}\,C_{22} &= A_{12} \ , \\ B_{21}\,D_{22} - B_{22}\,D_{21} &= B_{12} \ . \end{aligned} \tag{3.2.22}$$

We choose B_{11} and B_{12} as arbitrary integers. From the Hopf condition (3.1.19) it follows that the subgroup $\mathcal{G}$ preserves the two integers r and z. Under a modular transformation by an element (3.2.21) of the group $\mathcal{G}$ the period matrix transforms according to (3.1.24). By using (3.2.22) one finds that the matrix elements of Ω have the transformation properties

$$\begin{aligned} \Omega_{11} &\longmapsto \Omega_{11} + B_{11} - \frac{C_{22}\,(\Omega_{12})^2 + 2\,D_{21}\,\Omega_{12} + B_{12}\,D_{21}}{C_{22}\,\Omega_{22} + D_{22}} \ , \\ \Omega_{12} &\longmapsto A_{22}\,\Omega_{12} + B_{21} - \frac{A_{22}\,\Omega_{22} + B_{22}}{C_{22}\,\Omega_{22} + D_{22}}\,(C_{22}\,\Omega_{12} + D_{21}) \ , \\ \Omega_{22} &\longmapsto \frac{A_{22}\,\Omega_{22} + B_{22}}{C_{22}\,\Omega_{22} + D_{22}} \ . \end{aligned} \tag{3.2.23}$$

Note that Ω_{22} transforms under a genus one $SL(2,\mathbb{Z})$ modular transformation. In addition the positivity of Ω_2 yields the quadratic constraints

$$\mathrm{Im}(\Omega_{11}) > 0 \ , \quad \mathrm{Im}(\Omega_{22}) > 0 \ , \quad (\mathrm{Im}\,\Omega_{12})^2 < \mathrm{Im}(\Omega_{11})\,\mathrm{Im}(\Omega_{22}) \ . \tag{3.2.24}$$

From (3.2.23) and (3.2.24) it follows that the period matrix take values in the extended fundamental domain

$$\mathcal{F}_2' = \Big\{ \Omega \in \mathcal{H}_2 \ \Big| \ \Omega_{11} \in \Delta' \ , \ \Omega_{22} \in \Delta \ , \ \Omega_{12} \in \mathcal{P}_{\Omega_{22}} \Big\} \tag{3.2.25}$$

written in terms of the elliptic fundamental domains (3.1.40) and (3.1.41) along with the parallelogram

$$\mathcal{P}_\tau = \Big\{ \sigma_1 + \tau\, \sigma_2 \ \Big| \ \sigma_1, \sigma_2 \in [0,1] \Big\} \tag{3.2.26}$$

in the upper complex half-plane. The domain (3.2.25) is the same as the modular region obtained using the ordinary Rankin-Selberg reduction [54]. This provides a complete picture of the moduli space of branched covers of a torus at genus two. The map which sends a Riemann surface Σ_2 to the equivalence class of the period matrix $\Omega \in \mathcal{M}_2$ is an isomorphism onto the subspace $\mathcal{M}_2 \setminus [\mathcal{H}_1 \times \mathcal{H}_1]$, where $[\mathcal{H}_1 \times \mathcal{H}_1]$ is the modular orbit of the space of diagonal period matrices in $\mathcal{H}_2$ corresponding to the boundary component of moduli space where the surface Σ_2 degenerates into two tori. The general task of finding an explicit set of inequalities on the matrix elements of Ω which characterizes the corresponding fundamental modular domain $\mathcal{F}_2$ is a difficult highly non-linear mathematical problem. Here an explicit representation of moduli space has been obtained by using reduction and unfolding techniques. This is the main motivation behind our modification of the Poincaré normal form, as it leads to a much simpler and tractable fundamental modular region. For completeness, the complete moduli space corresponding to the fully reduced Poincaré normal form (3.1.27) at genus two is worked out explicitly in Appendix A.

The sums over the integers in (3.1.11) which characterize the branched covers are restricted by the requirement that they count only the moduli (3.2.18) lying in the extended fundamental domain (3.2.25). The positivity constraint (3.2.24) and the Hopf condition (3.1.19) for the degree N of the covering map require $r, z \in \mathbb{N}$ such that $r\,z = N$. The two equivalence relations $\Omega_{12} \sim \Omega_{12} + 1$ and $\Omega_{11} \sim \Omega_{11} + 1$ imply that $x, y \in \mathbb{Z}/z\,\mathbb{Z}$, with $y \neq 0$ in order for the period matrix in $\mathcal{M}_2 \setminus [\mathcal{H}_1 \times \mathcal{H}_1]$ to correspond to a genus two curve Σ_2. The arbitrary complex number $\tau := -\Omega_{22} \in \mathcal{H}_1$ is integrated over the genus one fundamental domain Δ. These ranges are all defined so that the modular orbit of diagonal period matrices is removed from $\mathcal{H}_2$.

3.2.4 Counting Branched Covers

In section 3.1.2 we counted the number of inequivalent subcovers of a lattice $\mathbb{C}/\mathbb{Z}\oplus\tau\mathbb{Z}$ arriving at the expression (3.1.32). Identifying these sublattices with un-branched covers of elliptic curves. The equivalent problem in the genus two case is much harder. A generating function for the numbers of branched covers of genus two curves over tori is computed using algebraic geometric methods in [61].

$$F_D := \sum_N C_{N,D} q^N = \frac{1}{25920\mu_D}(5E_2^3 - 3E_2E_4 - 2E_6) + \frac{1-\mu_D}{5760}(2E_4 + 5E_2^2 + 10E_2 - 17), \tag{3.2.27}$$

where E_{2k} are the Eisenstein series ($E_k := 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(N)q^N$, $q = \mathrm{e}^{2\pi i\tau}$). It is shown that $F_D \in \mathbb{Q}[E_2, E_4, E_6]$ is a quasi modular form. If the map from the moduli space of curves into the space of abelian varieties is injective then one would expect the number of connected components of Hurwitz space, (the number of topologicaly distinct branched covers) to correspond to the number of connected components of the subspace of abelian varieties related to the branched covers. Our reduced moduli space has $\bar{F} = (\sigma_2(N) - \sigma_1(N))q^N$ connected components suggesting that information about the covers is packaged in a non trivial way within the discrete and continuous moduli. It is interesting that (3.2.27) has been partially computed in [8, 62, 63] by computting partition functions in $2D$ Yang Mills. The term proportional to $2E_4 + 5E_2^2 + 10E_2 - 17$ seems to be difficult to pin down using world sheet techniques.

3.2.5 Theta-Constants

The genus two theta-function with characteristics $\Theta : \mathrm{Jac}(\Sigma_2) \times \mathcal{H}_2 \to \mathbb{C}$ is defined as the Fourier series [64]

$$\Theta\begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{pmatrix}(\boldsymbol{z}|\Omega) = \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \exp\left[\pi\,\mathrm{i}\,\left(\boldsymbol{n} + \tfrac{1}{2}\boldsymbol{a}\right)\cdot\Omega\left(\boldsymbol{n} + \tfrac{1}{2}\boldsymbol{a}\right) + 2\pi\,\mathrm{i}\,\left(\boldsymbol{n} + \tfrac{1}{2}\boldsymbol{a}\right)\cdot\left(\boldsymbol{z} + \tfrac{1}{2}\boldsymbol{b}\right)\right] . \tag{3.2.28}$$

It is a holomorphic function of $(\boldsymbol{z}|\Omega) \in \mathbb{C}^2 \times \mathcal{H}_2$ for the characteristics $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^2$. For $\boldsymbol{a}, \boldsymbol{b} \in \{0,1\}\times\{0,1\}$ the theta-function is even if $\boldsymbol{a}\cdot\boldsymbol{b} \equiv 0 \bmod 2$, odd if $\boldsymbol{a}\cdot\boldsymbol{b} \equiv 1 \bmod 2$. There are ten even genus two theta-functions and six odd ones. We can write (3.2.28) in a form without characteristics by factorizing a phase to get

$$\Theta\begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{pmatrix}(\boldsymbol{z}|\Omega) = \mathrm{e}^{\frac{\pi\,\mathrm{i}}{4}\,\boldsymbol{a}\cdot\Omega\boldsymbol{a}+\pi\,\mathrm{i}\,\boldsymbol{a}\cdot(\boldsymbol{z}+\frac{1}{2}\,\boldsymbol{b})}\ \Theta\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{0}\end{pmatrix}\ \big(\boldsymbol{z}+\tfrac{1}{2}\,\Omega\boldsymbol{a}+\tfrac{1}{2}\,\boldsymbol{b}\big|\,\Omega\big) \tag{3.2.29}$$

We can now use the reduction (3.2.18) to decompose the genus two theta-function (3.2.28) in terms of genus one theta-functions [49]. The exponent of the theta-function with zero characteristic in (3.2.29) is given by the quantity

$$\boldsymbol{k}\cdot\Omega\boldsymbol{k}+2\,\boldsymbol{k}\cdot\big(\boldsymbol{z}+\tfrac{1}{2}\,\Omega\boldsymbol{a}+\tfrac{1}{2}\,\boldsymbol{b}\big) \tag{3.2.30}$$

$$= (k_1)^2\,\Omega_{11}+2\,k_1\ \big(z_1+k_2\,\Omega_{12}+\tfrac{1}{2}\,\Omega_{11}\,a_1+\tfrac{1}{2}\,\Omega_{12}\,a_2+\tfrac{1}{2}\,b_1\big)$$

$$+\ (k_2)^2\,\Omega_{22}+2\,k_2\ \big(z_2+\tfrac{1}{2}\,\Omega_{12}\,a_1+\tfrac{1}{2}\,\Omega_{22}\,a_2+\tfrac{1}{2}\,b_2\big) \tag{3.2.31}$$

for $\boldsymbol{k}=(k_1,k_2)\in\mathbb{Z}^2$. In the present case the period matrix (3.2.18) (after projective $\mathbb{Z}_2$ reflection) has rational-valued off-diagonal entries $\Omega_{12}=\frac{y}{z}=\frac{r\,y}{N}$. Let $k_2=n+N\,\tilde{k}_2$ where $\tilde{k}_2\in\mathbb{Z}$ and $0\le n\le N-1$. We may then rewrite (3.2.31) in the form

$$\boldsymbol{k}\cdot\Omega\boldsymbol{k}+2\,\boldsymbol{k}\cdot\big(\boldsymbol{z}+\tfrac{1}{2}\,\Omega\boldsymbol{a}+\tfrac{1}{2}\,\boldsymbol{b}\big)$$

$$= (k_1)^2\,\Omega_{11}+2\,k_1\ \big[z_1+\big(n+\tfrac{a_2}{2}\big)\ \Omega_{12}+\Omega_{11}\,\tfrac{a_1}{2}+\tfrac{b_1}{2}\big]+\ 2\,N\,\tilde{k}_2\,\Omega_{12}$$

$$+\left(\tfrac{n}{N}+\tilde{k}_2\right)^2\ N^2\,\Omega_{22}+\ 2\,N\ \left(\tfrac{n}{N}+\tilde{k}_2\right)\ \big(z_2+\Omega_{12}\,\tfrac{a_1}{2}+\Omega_{22}\,\tfrac{a_2}{2}+\tfrac{b_2}{2}\big)\ . \tag{3.2.32}$$

Once this expression is multiplied by $\pi\,\mathrm{i}$ and exponentiated, the term $2\pi\,\mathrm{i}\,N\,\tilde{k}_2\,\Omega_{12}$ can be dropped since it is an integer multiple of $2\pi\,\mathrm{i}$. In this way the genus two theta-function factorizes into elliptic theta-functions (3.1.43) as

$$\Theta\begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{pmatrix}(\boldsymbol{z}|\Omega)\ =\ \mathrm{e}^{-\pi\,\mathrm{i}\,a_1\,a_2\,r\,y/2\,N}\sum_{n=0}^{N-1}\ \mathrm{e}^{-\pi\,\mathrm{i}\,a_1\,n\,r\,y/N}\ \theta\begin{pmatrix}a_1\\ b_1\end{pmatrix}\left(z_1+\big(n+\tfrac{a_2}{2}\big)\,\tfrac{r\,y}{N}\,\middle|\,\tfrac{r\,x+\frac{r^2}{\mathrm{i}\,\nu}}{N}\right)$$

$$\times\,\theta\begin{pmatrix}\frac{2\,n+a_2}{N}\\ 0\end{pmatrix}\ \big(N\,\big(z_2+\tfrac{a_1\,r\,y}{2\,N}+\tfrac{b_2}{2}\big)\big|\,N^2\,\tau\big)\ . \tag{3.2.33}$$

Each term in the sum over n in (3.2.33) contains a pair of theta-functions, one for each of the tori in (3.2.19).

Let us now restrict to theta-constants by setting $\boldsymbol{z}=\boldsymbol{0}$ and to integer characteristics $\boldsymbol{a},\boldsymbol{b}\in\{0,1\}\times\{0,1\}$. The decomposition (3.2.33) into genus one theta-functions can then be simplified somewhat by applying a Poisson resummation to the second set of theta-functions with fractional characteristics. For those characteristics, this results in the modular transformation property [66]

$$\theta\binom{a}{b}(z|\tau) = \frac{\mathrm{e}^{-\pi\,\mathrm{i}\left(\frac{z^2}{\tau}+\frac{a\,b}{2}\right)}}{\sqrt{-\mathrm{i}\,\tau}}\,\theta\binom{b}{0}\left(-\tfrac{a}{2}-\tfrac{z}{\tau}\Big|-\tfrac{1}{\tau}\right)\ . \tag{3.2.34}$$

Then the elliptic theta-functions are all given by standard integer characteristic Jacobi-Erderlyi functions θ_a for $a = 1, 2, 3, 4$ and (3.2.33) becomes

$$\Theta\binom{\boldsymbol{a}}{\boldsymbol{b}}(\mathbf{0}|\Omega) = \frac{\mathrm{e}^{\pi\,\mathrm{i}\,a_2\,b_2/2}}{N\,\sqrt{-\mathrm{i}\,\tau}}\sum_{n=0}^{N-1}(-1)^{b_2\,n}\,\theta\binom{a_1}{b_1}\left(\left(n+\tfrac{a_2}{2}\right)\tfrac{r\,y}{N}\,\Bigg|\,\frac{r\,x+\frac{r^2}{\mathrm{i}\,\nu}}{N}\right)$$

$$\times\,\theta_\gamma\left(\frac{n+\frac{a_2}{2}}{N}\Bigg|-\frac{1}{N^2\,\tau}\right)\ , \tag{3.2.35}$$

where $\gamma = 2$ (resp. $\gamma = 3$) when the degree N and the connecting integer y are even (resp. odd).

3.2.6 Genus Two Siegel Modular Forms

Siegel modular forms[58] of weight k are holomorphic functions on the Siegel upper half plane satisfying

$$f(\gamma[\Omega]) = \det(C\Omega - D)^k f(\Omega), \tag{3.2.36}$$

where $\gamma \in PS(4, \mathbb{Z})$ and the action of γ was defined in (3.1.24). A cusp form is a Siegel modular form that vanishes under degeneration of the corresponding Riemann surface. The degenerations will be discussed within section (3.5). The Siegel modular forms admit a fourier decomposition since they are left invariant under translations in the Siegel upper half plane. The variables used in this decomposition are

$$q = \mathrm{e}^{2i\pi\Omega_{11}}, r = \mathrm{e}^{2i\pi\Omega_{11}}, s = \mathrm{e}^{2i\pi\Omega_{11}}. \tag{3.2.37}$$

The space of genus two Siegel Modular forms are a ring and are generated by forms of weight $4, 6, 10$ and 12 [65]. We note that the genus two theta constants transform as modular forms up to a phase under the $PS(4, \mathbb{Z})$ action [64].

3.2.7 Free Energy

We are finally ready to write down the genus two contribution to the bosonic free energy. In the critical bosonic string theory, the corresponding integration measure $\mathrm{d}\mu_2^{\mathrm{bos}}(\Omega,\overline{\Omega}\,)$ on moduli space is completely characterized by the fact that it is expressed in terms of the square of a holomorphic volume form on $\mathcal{M}_2$ [67, 50, 54], and by the requirement that it be free of global gravitational anomalies. It is independent of temperature and has no zeroes or singularities in the interior of moduli space, while on the boundary of moduli space (the $Sp(4,\mathbb{Z})$ orbit of $\mathcal{H}_1 \times \mathcal{H}_1$ and infinity in $\mathcal{H}_2$) it has a second order pole. This uniquely determines the bosonic integration measure as an expansion in terms of modular forms. Using holomorphic factorization, it can thereby be shown [67, 50, 54, 39] that the modular invariant, bosonic genus two moduli space measure is given by

$$\mathrm{d}\mu_2^{\mathrm{bos}}(\Omega\,,\,\overline{\Omega}\,) = \left(\frac{1}{4\pi^2\,\alpha'}\right)^{12}\ \mathrm{d}^2\Omega_{11}\wedge\mathrm{d}^2\Omega_{22}\wedge\mathrm{d}^2\Omega_{12}\ \Big(\det\Omega_2\Big)^{-13}\ \Big|\Psi_{10}(\Omega)\Big|^{-2}\ , \tag{3.2.38}$$

where Ψ_{10} is the unique, parabolic modular form of weight ten which vanishes on the diagonal period matrices of $\mathcal{H}_2$ (This generalizes the Dedekind function (3.1.44) which comprises the one-loop moduli space density for bosonic strings [54]). It can be expressed in terms of the ten even integer characteristic, genus two theta-constants as the holomorphic Siegel cusp form

$$\Psi_{10}(\Omega) = 2^{-12}\prod_{\boldsymbol{a}\cdot\boldsymbol{b}\equiv 0\,\mathrm{mod}\,2}\left[\Theta\begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{pmatrix}(\mathbf{0}\,|\Omega)\right]^2\ . \tag{3.2.39}$$

After integrating the delta-function in (3.1.11) with the necessary Jacobian from (3.2.17) and (3.2.18), the bosonic free energy is thereby found to be

$$\begin{aligned} F_2^{\mathrm{bos}} \ &= \ -g_s^2\left(\frac{1}{2\sqrt{2}\,\pi\,\beta\,R}\right)^{12}\sum_{N=1}^{\infty}\frac{\mathrm{e}^{-\frac{\beta\,N}{\sqrt{2}\,R}}}{N^2}\sum_{r\,z=N}\left(\frac{z}{r}\right)^{10} \\ &\quad\times\sum_{\substack{x,y\in\mathbb{Z}/z\,\mathbb{Z}\\ y\neq 0}}\int_{\Delta}\frac{\mathrm{d}^2\tau}{(\tau_2)^{12}}\prod_{\boldsymbol{a}\cdot\boldsymbol{b}\equiv 0\,\mathrm{mod}\,2}\left|\Theta\begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{pmatrix}(\mathbf{0}|\Omega)\right|^{-4}\ . \end{aligned} \tag{3.2.40}$$

Using (3.2.35) the product over genus two theta-functions in (3.2.40) can be expressed in terms of a long string of integer characteristic Jacobi-Erderlyi functions θ_a, $a =$

$1,2,3,4$. Generically these are *not* theta-constants of the elliptic curves $\mathbb{T}^2_{\mathrm{i}\,\nu}$ and $\mathbb{T}^2_\tau$, as the connecting integers $y \in \mathbb{Z}/z\,\mathbb{Z}$ gluing the two tori together appear in their arguments. The sums over the remaining integers N, r, z, x give the summation over worldsheet instanton sectors $\Sigma_1 \to \mathbb{T}^2_{\mathrm{i}\,\nu}$ characterizing the Hecke algebra. The τ-integral in (3.2.40) gives the integration over the location of the branch cut on the base $\mathbb{T}^2_{\mathrm{i}\,\nu}$ which is used to construct the covering surface Σ_2 by gluing. This identification can be established by using Thomae formulas to express the branch points of the genus two curve transcendentally in terms of the theta-constants (3.2.35) [52]. We will return to these features in Section 3.5.

As an aside, it is interesting to note that the cusp form (3.2.39) also arises in the computation [37] of the elliptic genus of the Kummer surface K3 as the one-loop free energy of a single string given by

$$\chi_{\mathrm{K3}}(\zeta|\tau) = 8 \sum_{a=2}^{4} \frac{\theta_a(\zeta|\tau)^2}{\theta_a(0|\tau)^2} , \tag{3.2.41}$$

where $\Omega_{12} = \zeta$ is the complexified Kähler form of the elliptic curve $\mathbb{T}^2_\tau$. The completion of the corresponding string partition function on the symmetric product orbifold of K3 to an automorphic form for the group $SO(3,2,\mathbb{Z})$ is simply (3.2.39). This form can also be interpreted as the denominator of a generalized Kac-Moody algebra [37, 36]. Our reduction formulas here and in what follows bear a remarkable similarity to this construction, with the K3 surface regarded as the resolution of the orbifold $(\mathbb{T}^2_{\mathrm{i}\,\nu} \times \mathbb{T}^2_\tau)/\mathbb{Z}_2$. It would be interesting to further pursue whether or not our two-loop partition functions admit deeper interpretations along these lines.

3.3 Superstrings

We now turn to our main object of interest, the two-loop superstring free energy. There are two new ingredients in this case that one must add to the calculation of the previous section. In the genus one case, the simplicity of the measure (3.1.42) is a result of the local cancellation between the longitudinal X and ghost B, C determinants on moduli space. This ceases to happen for genera $g > 1$, and in this case the calculations are notoriously subtle. We shall take the standard prescription for obtaining the measure $\mathrm{d}\mu_2[{\boldsymbol{n} \atop \boldsymbol{m}}](\Omega,\overline{\Omega})$ by integrating over the fermionic moduli [68]. The non-splitness of super-moduli space does not generically allow for a global, unambiguous reduction to ordinary moduli, since the Grassmann integrations lead to spurious gauge dependences in the form of total derivative terms on $\mathcal{M}_2$ [69]. The problem can be overcome by descending from super-moduli space to moduli space by projecting super-geometries onto super-period matrices [39]. The integration over Grassmann odd supermoduli is then performed by integrating over the fibers of this projection. With this, one can find a good global holomorphic gauge slice in Teichmüller space without spurious gauge dependences that could otherwise lead to modular anomalies in the measure on moduli space. For each even spin structure at $g = 2$, slice-independence allows an arbitrary choice of worldsheet gravitino field insertion point [70] and the split gauge choice leads to an expression for the chiral superstring measure in terms of modular forms [39]. The contributions from odd spin structures again vanish as a result of the integration over fermionic zero modes. For fixed spin structure, the chiral measure allows for a unique modular covariant GSO projection [39], which must be appropriately modified [40] due to the finite-temperature supersymmetry breaking effects analogously to the one-loop case.

3.3.1 Spin Structures

Let us begin by setting some useful shorthand notations. A reduced genus two integer characteristic is a pair of vectors $\binom{\boldsymbol{a}}{\boldsymbol{b}} = \binom{a_1\ b_1}{a_2\ b_2}$ where each a_i, b_i, $i = 1, 2$ are either 0 or 1. A characteristic is even if $\boldsymbol{a}\cdot\boldsymbol{b} \equiv 0 \bmod 2$, odd if $\boldsymbol{a}\cdot\boldsymbol{b} \equiv 1 \bmod 2$. There are ten even characteristics and six odd characteristics associated to the distinct choices of spin structures on the genus two Riemann surface Σ_2, i.e. to the choices of a square root of the canonical line bundle over Σ_2. The ten even characteristics (spin structures) are denoted

$$\delta_1 = \binom{00}{00} \qquad \delta_2 = \binom{00}{01} \qquad \delta_3 = \binom{01}{00} \qquad \delta_4 = \binom{01}{01} \qquad \delta_5 = \binom{00}{10}$$

$$\delta_6 = \binom{01}{10} \qquad \delta_7 = \binom{10}{00} \qquad \delta_8 = \binom{10}{01} \qquad \delta_9 = \binom{10}{10} \qquad \delta_0 = \binom{11}{11} \ . \quad (3.3.1)$$

The odd spin structures are denoted

$$\nu_1 = \binom{00}{11} \qquad \nu_2 = \binom{01}{11} \qquad \nu_3 = \binom{11}{00} \qquad \nu_4 = \binom{11}{01} \qquad \nu_5 = \binom{10}{11} \qquad \nu_6 = \binom{11}{10} \ . \tag{3.3.2}$$

Integer characteristics may be summed modulo 2, componentwise as if they were 2×2 matrices. For example, $\nu_1 + \nu_4 + \nu_6 = \delta_0$ and $\nu_2 + \nu_3 + \nu_5 = \delta_0$. There is a two-to-one map from triples of odd characteristics which are pairwise distinct onto even characteristics. The relative signature between any two spin structures is defined by

$$\left\langle \binom{\boldsymbol{a}}{\boldsymbol{b}} \middle| \binom{\boldsymbol{a}'}{\boldsymbol{b}'} \right\rangle = \exp\Big[\pi\,\mathrm{i}\,\big(\boldsymbol{a}\cdot\boldsymbol{b}' - \boldsymbol{b}\cdot\boldsymbol{a}'\big)\Big] \ . \tag{3.3.3}$$

3.3.2 GSO Projection

In order for spacetime fermions and spacetime bosons to have the correct statistics at finite temperature, the fermions must have antiperiodic boundary conditions and the bosons must be periodic around the temperature direction of the target space. The standard GSO projection is thus modified by phases which take into account the winding numbers $\boldsymbol{n}$ and $\boldsymbol{m}$ [40]. A genus two Riemann surface has 16 spin structures given by the generators (3.3.1) and (3.3.2) of the cohomology group $H^1(\Sigma_2, \mathbb{Z}/2\,\mathbb{Z}) = (\mathbb{Z}/2\,\mathbb{Z})^4$ which are in one-to-one correspondence with flat real line bundles $L \to \Sigma_2$. Define $\phi(L) = +1$ if the spin structure L is even and $\phi(L) = -1$ if it is odd. This quantity coincides with the mod 2 index [64]

$$\phi(L) = \exp\Big[\pi\,\mathrm{i}\,\dim H^0\big(\Sigma_2\,,\,\mathit{Spin}(\Sigma_2)\otimes L\big)\Big] \tag{3.3.4}$$

which counts the number of holomorphic sections of the twisted spinor bundle $\mathit{Spin}(\Sigma_2) \otimes L$ modulo 2. The reduction modulo 2 of $(\boldsymbol{n}, \boldsymbol{m})$ defines the characteristic class in $H^1(\Sigma_2, \mathbb{Z}/2\,\mathbb{Z})$ of a flat connection of a real line bundle $\mathcal{L}_{(\boldsymbol{n},\boldsymbol{m})} \to \Sigma_2$ such that a holomorphic section of $\mathcal{L}_{(\boldsymbol{n},\boldsymbol{m})}$ changes by a phase $(-1)^{n_i}$ as one goes once around the a_i homology cycles and by $(-1)^{m_i}$ as one goes once around the b_i homology cycles. Given a spin structure L, the tensor product $L \otimes \mathcal{L}_{(\boldsymbol{n},\boldsymbol{m})}$ is another spin structure for any $\boldsymbol{n}, \boldsymbol{m}$ and we define

$$U_L(\boldsymbol{n}, \boldsymbol{m}) = \phi(L)\,\phi(L \otimes \mathcal{L}_{(\boldsymbol{n},\boldsymbol{m})}) \ . \tag{3.3.5}$$

The quantity (3.3.5) takes values ± 1, and it is the correct phase to insert into the sum over spin structures L and winding numbers $\boldsymbol{n}, \boldsymbol{m}$ [40]. It is compatible with both factorization of Σ_2 to lower genus and modular invariance.

As an example, let us calculate $U_{\delta_5}(\boldsymbol{n},\boldsymbol{m}) = U_{\left(\substack{00\\10}\right)}(\boldsymbol{n},\boldsymbol{m})$. The spin structure δ_5 is even so $U_{\delta_5}(\boldsymbol{n},\boldsymbol{m}) = \phi(L_{\delta_5} \otimes \mathfrak{L}_{(\boldsymbol{n},\boldsymbol{m})})$. We first calculate $U_{\delta_5}(n_1,0,0,0)$. For $n_1 \in \mathbb{Z}$ odd one has $L_{\delta_5} \otimes \mathfrak{L}_{(n_1,0,0,0)} = L_{\delta_9}$ and so $U_{\delta_5}(n_1,0,0,0) = \phi(L_{\delta_9}) = 1$. The spin structure $L_{\delta_5} \otimes \mathfrak{L}$ is likewise even if $\mathfrak{L}$ corresponds to wrapping the a_2 and b_1 cycles around the temperature direction odd numbers of times n_2 and m_1, and so we have

$$U_{\delta_5}(n_1,0,0,0) = U_{\delta_5}(0,0,m_1,0) = U_{\delta_5}(0,n_2,0,0) = 1 \ . \tag{3.3.6}$$

When the b_2 cycle wraps around the temperature direction an odd number of times m_2 one gets the spin structure $L_{\delta_5} \otimes \mathfrak{L}_{(0,0,0,m_2)} = L_{\nu_1}$. The phase is then

$$U_{\delta_5}(0,0,0,m_2) = (-1)^{m_2} \ . \tag{3.3.7}$$

Taking into account pairs of cycles produces the phase factors

$$\begin{aligned}
U_{\delta_5}(0,n_2,m_1,0) &= 1 \ , \\
U_{\delta_5}(0,0,m_1,m_2) &= U_{\delta_5}(n_1,0,0,m_2) \ = \ (-1)^{m_2} \ , \\
U_{\delta_5}(n_1,0,m_1,0) &= \tfrac{1}{2}\left(1+(-1)^{n_1}+(-1)^{m_1}-(-1)^{n_1+m_1}\right) \ , \\
U_{\delta_5}(0,n_2,0,m_2) &= \tfrac{1}{2}\left(1-(-1)^{n_2}+(-1)^{m_2}+(-1)^{n_2+m_2}\right) \ .
\end{aligned} \tag{3.3.8}$$

For triples of cycles the phases are given by

$$\begin{aligned}
&U_{\delta_5}(n_1,0,m_1,m_2) = \tfrac{1}{2}\left((-1)^{m_2}+(-1)^{n_1+m_2}+(-1)^{m_1+m_2}-(-1)^{n_1+m_1+m_2}\right) \ , \\
&U_{\delta_5}(0,n_2,m_1,m_2) = U_{\delta_5}(0,n_2,0,m_2) \ , \\
&U_{\delta_5}(n_1,n_2,m_1,0) = U_{\delta_5}(n_1,0,m_1,0) \ , \\
&U_{\delta_5}(n_1,n_2,0,m_2) = U_{\delta_5}(0,0,n_2,m_2) \ .
\end{aligned} \tag{3.3.9}$$

The GSO phases for any n_i, m_i, $i = 1, 2$ are given generally by an expression of the form

$U_{\delta_5}(\boldsymbol{n},\boldsymbol{m})$

$$= \alpha\,(-1)^{n_1+n_2+m_1+m_2} + \beta_1\,(-1)^{n_1+n_2+m_1} + \beta_2\,(-1)^{n_1+n_2+m_2} + \beta_3\,(-1)^{n_1+m_1+m_2}$$

$$+\ \beta_4\,(-1)^{n_2+m_1+m_2} + \gamma_1\,(-1)^{n_1+n_2} + \gamma_2\,(-1)^{m_1+m_2} + \gamma_3\,(-1)^{n_1+m_1}$$

$$+\ \gamma_4\,(-1)^{n_2+m_2} + \gamma_5\,(-1)^{n_1+m_2} + \gamma_6\,(-1)^{n_2+m_1} + \varepsilon_1\,(-1)^{n_1} + \varepsilon_2\,(-1)^{n_2}$$

$$+\ \varepsilon_3\,(-1)^{m_1} + \varepsilon_4\,(-1)^{m_2} + \eta\ . \tag{3.3.10}$$

The phase (3.3.10) must reduce to (3.3.6)–(3.3.9) when the appropriate winding numbers are set to zero. This gives a system of equations which are enough to determine $U_{\delta_5}(\boldsymbol{n},\boldsymbol{m})$ up to a proportionality constant which may be fixed by requiring the phase to be normalised as ± 1.

One can compute all 16 phase factors in this way as functions of generic thermal winding numbers $\boldsymbol{n},\boldsymbol{m}$. After some inspection and calculation, one finds that as a function $U : \{0,1\}^2 \times \mathbb{Z} \to \{\pm 1\}$ the GSO phase (3.3.5) is given by

$$U_{\left(\begin{smallmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{smallmatrix}\right)}(\boldsymbol{n},\boldsymbol{m})$$

$$= \tfrac{1}{4}\,(-1)^{\boldsymbol{a}\cdot\boldsymbol{b}}\Big[(-1)^{n_1+n_2+m_1+m_2}\,(-1)^{a_1+a_2+b_1+b_2} - (-1)^{n_1+n_2+m_1}\,(-1)^{a_1+a_2+b_1}$$

$$-\ (-1)^{n_1+n_2+m_2}\,(-1)^{a_1+a_2+b_2} - (-1)^{n_1+m_1+m_2}\,(-1)^{a_1+b_1+b_2}$$

$$-\ (-1)^{n_2+m_1+m_2}\,(-1)^{a_2+b_1+b_2} + (-1)^{n_1+n_2}\,(-1)^{a_1+a_2} + (-1)^{m_1+m_2}\,(-1)^{b_1+b_2}$$

$$-\ (-1)^{n_1+m_1}\,(-1)^{a_1+b_1} - (-1)^{n_2+m_2}\,(-1)^{a_2+b_2} + (-1)^{n_1+m_2}\,(-1)^{a_1+b_2}$$

$$+\ (-1)^{n_2+m_1}\,(-1)^{a_2+b_1} - (-1)^{n_1}\,(-1)^{a_1} + (-1)^{n_2}\,(-1)^{a_2}$$

$$+\ (-1)^{m_1}\,(-1)^{b_1} + (-1)^{m_2}\,(-1)^{b_2} + 1\Big]\ . \tag{3.3.11}$$

For our particular calculation the Riemann surface Σ_2 is a branched covering of a torus, and the modified GSO projection is very simple since there is only one homology cycle of the cover which is wrapped around the temperature direction after the reduction to (3.2.11) given by $(\boldsymbol{n},\boldsymbol{m}) \to (0,0,r,0)$. The only even spin structure GSO phases which are non-trivial for generic r are given by

$$U_{\delta_7}(0,0,r,0) = U_{\delta_8}(0,0,r,0) = U_{\delta_9}(0,0,r,0) = U_{\delta_0}(0,0,r,0) = (-1)^r\ . \tag{3.3.12}$$

All other even spin structure phases are equal to 1.

3.3.3 Chiral Measure

Holomorphic factorization of the genus two superstring measure at zero temperature is achieved by trading the Belavin-Knizhnik obstruction (encoded through the partition function of a free chiral scalar field on Σ_2 given by $(4\pi^2\,\alpha'\,)^{-5}\,(\det\overline{\partial}_0)^{-10})$ for an integral over internal loop momenta $\boldsymbol{p}_\mu \in \mathbb{R}^2$, $\mu = 0, 1, \ldots, 9$ flowing through the $\boldsymbol{a}$ cycles of Σ_2 [68]. The resulting dependence on moduli and spin structures is intricately encoded into various sections of the twisted spinor bundles over Σ_2 [71, 72], which may be expressed in terms of modular forms associated with the Riemann surface [39]. The *chiral* measure corresponding to a fixed even spin structure δ is required to be free of global gravitational anomalies on super-moduli space before integrating out the fermionic moduli [73]. On $\mathcal{M}_2$ it may be computed explicitly to be [39]

$$\mathrm{d}\mu_2[\delta](\Omega) = \left(\frac{1}{4\pi^2\,\alpha'}\right)^2\ \mathrm{d}\Omega_{11}\wedge\mathrm{d}\Omega_{22}\wedge\mathrm{d}\Omega_{12}\ \frac{\Xi_6[\delta](\Omega)\ \Theta[\delta](\mathbf{0}|\Omega)^4}{\Psi_{10}(\Omega)}\ , \tag{3.3.13}$$

where Ψ_{10} is the modular form of weight ten defined in (3.2.39) which arises as the bosonic contribution, and

$$\Xi_6[\delta](\Omega) := \sum_{1\le k<l\le 3} \langle\nu_{i_k}|\nu_{i_l}\rangle \prod_{j=4,5,6} \Theta[\nu_{i_k}+\nu_{i_l}+\nu_{i_j}](\mathbf{0}|\Omega)^4\ . \tag{3.3.14}$$

Here we have chosen a partition $\{i_1, i_2, i_3\}\cup\{i_4, i_5, i_6\} = \{1, 2, 3, 4, 5, 6\}$ of the index set labelling the odd characteristics (3.3.2) such that $\delta = \nu_{i_1} + \nu_{i_2} + \nu_{i_3} = \nu_{i_4} + \nu_{i_5} + \nu_{i_6}$. The quantity (3.3.14) depends only on the spin structure δ and not on the actual triplet of odd characteristics used to represent δ. This follows from the fact that the odd spin structures $\nu_{i_1}, \nu_{i_2}, \nu_{i_3}$ result from a choice of worldsheet gravitino field insertion points, and the two-loop chiral measure is completely independent of these points [39]. The object $\Xi_6[\delta](\Omega)$ has modular weight 6, but it is *not* a modular form because it depends on the spin structure δ and an additional sign factor arises in its modular transformation laws [39]. As a consequence, the measure (3.3.13) is modular covariant of weight -5.

The full chiral genus two superstring measure is obtained by summing (3.3.13) over all even spin structures δ with weights provided by the phases $U_\delta(\boldsymbol{n}, \boldsymbol{m})$ computed in

Section 3.3.2 which take into account the modification of the GSO projection at finite temperature. Thus we define

$$\mathrm{d}\mu_2 \begin{bmatrix} \boldsymbol{n} \\ \boldsymbol{m} \end{bmatrix} (\Omega) = \sum_{\delta \text{ even}} U_\delta(\boldsymbol{n}, \boldsymbol{m})\ \mathrm{d}\mu_2[\delta](\Omega)\ . \tag{3.3.15}$$

The quantity $\Upsilon_8(\Omega) := \sum_{\delta \text{ even}} \Xi_6[\delta](\Omega)\, \Theta[\delta](\mathbf{0}|\Omega)^4$ is a uniquely constructed modular form of weight 8. Using the Riemann bilinear relations one can show that [39] $\Upsilon_8(\Omega) = 2\,\Psi_8(\Omega) - \frac{1}{2}\,\Psi_4(\Omega)^2$ where $\Psi_8(\Omega)$ and $\Psi_4(\Omega)$ are respectively the weight 8 and weight 4 generators of the polynomial ring of genus two modular forms. By Igusa's theorem [74], $\Psi_8(\Omega)$ is the unique modular form of weight 8 with $4\,\Psi_8(\Omega) = \Psi_4(\Omega)^2$. It follows that $\Upsilon_8(\Omega) = 0$ and thus we have the identity

$$\sum_{\delta \text{ even}} \Xi_6[\delta](\Omega)\, \Theta[\delta](\mathbf{0}|\Omega)^4 = 0\ . \tag{3.3.16}$$

Using this along with the modified GSO phases (3.3.12) corresponding to the reduced form of the period matrix (3.2.18), we can bring (3.3.15) to the form

$$\begin{aligned} \mathrm{d}\mu_2 \begin{bmatrix} 0 \\ 0 \\ r \\ 0 \end{bmatrix} (\Omega) \ &= \ \left(\frac{1}{4\pi^2\,\alpha'}\right)^2 \mathrm{d}\Omega_{11} \wedge \mathrm{d}\Omega_{22} \wedge \mathrm{d}\Omega_{12}\, \frac{\left(\mathrm{e}^{\pi\,\mathrm{i}\,r} - 1\right)}{\Psi_{10}(\Omega)} \\ &\quad \times \Big(\Xi_6[\delta_7](\Omega)\, \Theta[\delta_7](\mathbf{0}|\Omega)^4 + \Xi_6[\delta_8](\Omega)\, \Theta[\delta_8](\mathbf{0}|\Omega)^4 \\ &\qquad + \Xi_6[\delta_9](\Omega)\, \Theta[\delta_9](\mathbf{0}|\Omega)^4 + \Xi_6[\delta_0](\Omega)\, \Theta[\delta_0](\mathbf{0}|\Omega)^4\Big) \end{aligned} \tag{3.3.17}$$

As in the one-loop case, when r is even the Fermi fields are periodic and so the fermions and bosons have the same boundary conditions. These sectors are supersymmetric, and the mode expansions of both the fermion and boson fields contain zero modes. The integration over fermionic zero modes gives zero. Hence (3.3.17) vanishes, as expected in the supersymmetric sectors.

3.3.4 Free Energy

The chiral measure (3.3.15) is a modular form of weight -5. When we include both left and right moving degrees of freedom of the string theory, the non-chiral measure $\mathrm{d}\mu[{\boldsymbol{n} \atop \boldsymbol{m}}](\Omega) \wedge \overline{\mathrm{d}\mu[{\boldsymbol{n} \atop \boldsymbol{m}}](\Omega)}$ is a modular form of weight -10. The complete measure which defines a modular invariant function on moduli space $\mathcal{M}_2$ is thus

$$\mathrm{d}\mu_2 \begin{bmatrix} \boldsymbol{n} \\ \boldsymbol{m} \end{bmatrix} \big(\Omega\,, \overline{\Omega}\,\big) = \big(\det \Omega_2\big)^{-5}\ \mathrm{d}\mu_2 \begin{bmatrix} \boldsymbol{n} \\ \boldsymbol{m} \end{bmatrix} (\Omega) \wedge \overline{\mathrm{d}\mu_2 \begin{bmatrix} \boldsymbol{n} \\ \boldsymbol{m} \end{bmatrix} (\Omega)}\ . \tag{3.3.18}$$

We now substitute the full superstring measure (3.3.18) into (3.1.11) using (3.3.17), and resolve the delta-function constraint after performing the necessary reduction to (3.2.18) (including the appropriate Jacobian). The superstring free energy is thereby found to be

$$F_2 = -\frac{g_s^2}{4}\left(\frac{1}{4\sqrt{2}\,\pi\,\beta\,R}\right)^4 \sum_{N=1}^{\infty} \frac{\mathrm{e}^{-\frac{\beta\,N}{\sqrt{2}\,R}}}{N} \sum_{\substack{r\,z=N \\ r\,\mathrm{odd}}} \frac{1}{r^4} \sum_{\substack{x,y\in\mathbb{Z}/z\,\mathbb{Z} \\ y\neq 0}} \int_\Delta \frac{\mathrm{d}^2\tau}{(\tau_2)^4} \left|\Psi_{10}(\Omega)\right|^{-2}$$

$$\times \Big|\Xi_6[\delta_7](\Omega)\,\Theta[\delta_7](\mathbf{0}|\Omega)^4 + \Xi_6[\delta_8](\Omega)\,\Theta[\delta_8](\mathbf{0}|\Omega)^4 + \Xi_6[\delta_9](\Omega)\,\Theta[\delta_9](\mathbf{0}|\Omega)^4 + \Xi_6[\delta_0](\Omega)\,\Theta[\delta_0](\mathbf{0}|\Omega)^4\Big|^2 . \qquad (3.3.19)$$

The quantities (3.3.14) are worked out in Appendix B. We denote $\Theta_i(\Omega) := \Theta[\delta_i](\mathbf{0}|\Omega)$. Using the explicit expression (3.2.39), the integrand in (3.3.19) can be expanded out in terms of the ten even characteristic genus two theta-constants to get

$$\begin{aligned} F_2 \;=\; & -\frac{g_s^2}{4}\left(\frac{1}{4\sqrt{2}\,\pi\,\beta\,R}\right)^4 \sum_{N=1}^{\infty} \frac{\mathrm{e}^{-\frac{\beta\,N}{\sqrt{2}\,R}}}{N} \sum_{\substack{r\,z=N \\ r\,\mathrm{odd}}} \frac{1}{r^4} \\ & \times \sum_{\substack{x,y\in\mathbb{Z}/z\,\mathbb{Z} \\ y\neq 0}} \int_\Delta \frac{\mathrm{d}^2\tau}{(\tau_2)^4} \left| 4\left(\frac{\Theta_7\,\Theta_8\,\Theta_9\,\Theta_0}{\Theta_1\,\Theta_2\,\Theta_3\,\Theta_4\,\Theta_5\,\Theta_6}\right)^2 \right. \\ & + \left(\frac{\Theta_2\,\Theta_3\,\Theta_5\,\Theta_7}{\Theta_1\,\Theta_4\,\Theta_6\,\Theta_8\,\Theta_9\,\Theta_0}\right)^2 - \left(\frac{\Theta_1\,\Theta_4\,\Theta_6\,\Theta_7}{\Theta_2\,\Theta_3\,\Theta_5\,\Theta_8\,\Theta_9\,\Theta_0}\right)^2 + \left(\frac{\Theta_2\,\Theta_3\,\Theta_6\,\Theta_8}{\Theta_1\,\Theta_4\,\Theta_5\,\Theta_7\,\Theta_9\,\Theta_0}\right)^2 \\ & - \left(\frac{\Theta_1\,\Theta_4\,\Theta_5\,\Theta_8}{\Theta_2\,\Theta_3\,\Theta_6\,\Theta_7\,\Theta_9\,\Theta_0}\right)^2 + \left(\frac{\Theta_3\,\Theta_4\,\Theta_5\,\Theta_9}{\Theta_1\,\Theta_2\,\Theta_6\,\Theta_7\,\Theta_8\,\Theta_0}\right)^2 - \left(\frac{\Theta_1\,\Theta_2\,\Theta_6\,\Theta_9}{\Theta_3\,\Theta_4\,\Theta_5\,\Theta_7\,\Theta_8\,\Theta_0}\right)^2 \\ & \left. + \left(\frac{\Theta_3\,\Theta_4\,\Theta_6\,\Theta_0}{\Theta_1\,\Theta_2\,\Theta_5\,\Theta_7\,\Theta_8\,\Theta_9}\right)^2 - \left(\frac{\Theta_1\,\Theta_2\,\Theta_5\,\Theta_0}{\Theta_3\,\Theta_4\,\Theta_6\,\Theta_7\,\Theta_8\,\Theta_9}\right)^2 \right|^2 . \qquad (3.3.20) \end{aligned}$$

The theta-constants appearing in (3.3.20) are functions of the period matrix (3.2.18) and therefore depend on both the discrete and continuous parameters which characterize the branched covers Σ_2. Their explicit forms in terms of elliptic Jacobi-Erderlyi functions are given by the formula (3.2.35). We have not found any genus one theta-function identities which could simplify (3.3.20) and make this expression more explicit.

Note that, in contrast to the one-loop case which relied solely on the Jacobi identity (3.1.45), the modular invariance of the two-loop thermodynamic free energy does not follow from Riemann identities alone but in addition requires a special property of the ring of modular forms at genus two. The drastic difference between the summation prefactors in the bosonic case (3.2.40) and in the supersymmetric case (3.3.20) reflects the different analytic natures of the associated twist field perturbations described in Section 2.3. This difference will be encountered again in a more explicit form in Section 3.5.2.

3.4 Heterotic Strings

Let us now describe how our analysis applies to heterotic string theory. We replace the matter field action $S[X] + \overline{S[X]}$ in (3.1.1) by

$$S_{\rm het}[X,\lambda] = \frac{1}{4\pi\,\alpha'} \int\limits_{\Sigma_2} {\rm d}^2 z\ \Big(|\partial x^\mu|^2 + \psi_\mu\,\overline{\partial}\psi^\mu + \lambda_A\,\partial\lambda^A\Big)\ , \tag{3.4.1}$$

where the fermionic fields λ^A, $A = 1,\dots,32$ are Lorentz singlets. Both ψ^μ and λ^A are Majorana-Weyl fermion fields. The ghost contributions are unchanged. Thus the left-moving (holomorphic) part of the heterotic string coincides with that of the superstring whose chiral modular covariant measure is given by (3.3.17). After bosonization of λ^A, the right-moving (antiholomorphic) part coincides with that of the bosonic string of Section 3.2.7 with 16 anti-chiral bosons compactified on the Cartan torus of the heterotic gauge group G, where $G = Spin(32)/\mathbb{Z}_2$ or $G = E_8 \times E_8$. The compactified bosonic fields produce an extra winding contribution given by a theta-function of the root lattice of G, which at genus two is the unique modular form of weight eight given by [54]

$$\Psi_8(\Omega) = \sum_{\delta\ {\rm even}} \Theta[\delta](\mathbf{0}|\Omega)^{16}\ . \tag{3.4.2}$$

It follows that the two-loop anti-chiral heterotic string measure is [39]

$${\rm d}\mu_2^{\rm het}\big(\,\overline{\Omega}\,\big) = \left(\frac{1}{4\pi^2\,\alpha'}\right)^6\ {\rm d}\overline{\Omega}_{11}\wedge{\rm d}\overline{\Omega}_{12}\wedge{\rm d}\overline{\Omega}_{22}\ \frac{\overline{\Psi_8(\Omega)}}{\overline{\Psi_{10}(\Omega)}}\ . \tag{3.4.3}$$

The full modular invariant non-chiral measure is thus $(\det\Omega_2)^{-5}\ \mathrm{d}\mu_2[{}^{\boldsymbol{n}}_{\boldsymbol{m}}](\Omega)\wedge \mathrm{d}\mu_2^{\rm het}(\,\overline{\Omega}\,)$. Substituting this into (3.1.11) using (3.3.17) and (3.4.3), by proceeding as before we find that the heterotic string free energy is given by

$$\begin{aligned} F_2^{\rm het} \ &= \ \frac{g_s^2}{8}\left(\frac{1}{128\,\sqrt2\,\pi^3\,\alpha'\,\beta\,R}\right)^4\sum_{N=1}^\infty\frac{\mathrm{e}^{-\frac{\beta\,N}{\sqrt2\,R}}}{N}\ \sum_{\substack{r\,z=N\\ r\,{\rm odd}}}\frac{1}{r^4}\ \sum_{\substack{x,y\in\mathbb{Z}/z\,\mathbb{Z}\\ y\neq0}}\ \int\limits_\Delta\frac{\mathrm{d}^2\tau}{(\tau_2)^4}\ \frac{\overline{\Psi_8(\Omega)}}{\big|\Psi_{10}(\Omega)\big|^2}\\ &\qquad\times\Big(\Xi_6[\delta_7](\Omega)\,\Theta[\delta_7](\mathbf{0}|\Omega)^4+\Xi_6[\delta_8](\Omega)\,\Theta[\delta_8](\mathbf{0}|\Omega)^4\\ &\qquad\qquad+\Xi_6[\delta_9](\Omega)\,\Theta[\delta_9](\mathbf{0}|\Omega)^4+\Xi_6[\delta_0](\Omega)\,\Theta[\delta_0](\mathbf{0}|\Omega)^4\Big)\ . \end{aligned} \tag{3.4.4}$$

As in (3.3.20), this expression can be expanded into the ten even characteristic genus two theta-constants by using (3.2.39), (3.4.2) and the formulas of Appendix B to get

$$\begin{aligned} F_2^{\rm het} = \frac{g_s^2}{8}&\left(\frac{1}{128\,\sqrt2\,\pi^3\,\alpha'\,\beta\,R}\right)^4\sum_{N=1}^\infty\frac{\mathrm{e}^{-\frac{\beta\,N}{\sqrt2\,R}}}{N}\ \sum_{\substack{r\,z=N\\ r\,{\rm odd}}}\frac{1}{r^4}\ \sum_{\substack{x,y\in\mathbb{Z}/z\,\mathbb{Z}\\ y\neq0}}\ \sum_{i=0}^9\ \int\limits_\Delta\frac{\mathrm{d}^2\tau}{(\tau_2)^4}\ \frac{\big(\overline{\Theta}_i\big)^{14}}{\prod\limits_{j\neq i}\overline{\Theta}_j}\\ \times\Bigg[4&\left(\frac{\Theta_7\,\Theta_8\,\Theta_9\,\Theta_0}{\Theta_1\,\Theta_2\,\Theta_3\,\Theta_4\,\Theta_5\,\Theta_6}\right)^2+\left(\frac{\Theta_2\,\Theta_3\,\Theta_5\,\Theta_7}{\Theta_1\,\Theta_4\,\Theta_6\,\Theta_8\,\Theta_9\,\Theta_0}\right)^2-\left(\frac{\Theta_1\,\Theta_4\,\Theta_6\,\Theta_7}{\Theta_2\,\Theta_3\,\Theta_5\,\Theta_8\,\Theta_9\,\Theta_0}\right)^2\\ +&\left(\frac{\Theta_2\,\Theta_3\,\Theta_6\,\Theta_8}{\Theta_1\,\Theta_4\,\Theta_5\,\Theta_7\,\Theta_9\,\Theta_0}\right)^2-\left(\frac{\Theta_1\,\Theta_4\,\Theta_5\,\Theta_8}{\Theta_2\,\Theta_3\,\Theta_6\,\Theta_7\,\Theta_9\,\Theta_0}\right)^2+\left(\frac{\Theta_3\,\Theta_4\,\Theta_5\,\Theta_9}{\Theta_1\,\Theta_2\,\Theta_6\,\Theta_7\,\Theta_8\,\Theta_0}\right)^2\\ -&\left(\frac{\Theta_1\,\Theta_2\,\Theta_6\,\Theta_9}{\Theta_3\,\Theta_4\,\Theta_5\,\Theta_7\,\Theta_8\,\Theta_0}\right)^2+\left(\frac{\Theta_3\,\Theta_4\,\Theta_6\,\Theta_0}{\Theta_1\,\Theta_2\,\Theta_5\,\Theta_7\,\Theta_8\,\Theta_9}\right)^2-\left(\frac{\Theta_1\,\Theta_2\,\Theta_5\,\Theta_0}{\Theta_3\,\Theta_4\,\Theta_6\,\Theta_7\,\Theta_8\,\Theta_9}\right)^2\Bigg]\ , \end{aligned}$$

which can again be expressed in terms of elliptic Jacobi-Erderlyi functions by using the formula (3.2.35).

In the free string limit $g_s\to0$ the space of physical states of the heterotic sigma-model on the symmetric product orbifold (1.0.3) is naturally isomorphic to the Fock space of second quantized heterotic strings in DLCQ [38, 41]. The $(\mathbb{Z}_2)^N$ factor in this quotient space is a discrete gauge symmetry acting on twisted sector gauge fermions λ^A in the fundamental representation of G. The additional $\mathbb{Z}_2$-orbifolds are achieved by extra GSO projections on λ^A, and they are necessary to reproduce the light-cone Green-Schwarz heterotic string field theory [38, 41]. The right-moving sector is thus given by the standard $\mathbb{R}^{24}/\mathbb{Z}_2$ orbifold conformal field theory. This $\mathbb{Z}_2$-orbifold for $g_s>0$ is manifested through the decomposition of the theta-constants comprising the modular form (3.4.2) according to (3.2.35), and it can be thought of as being ultimately responsible in this instance for the fibred decomposition of the Jacobian variety (3.2.19). Let us also remark that in order to implement S-duality with Type IB superstring theory (as is necessary in formulating the heterotic matrix string theory conjecture), one should add a Wilson line which breaks the heterotic gauge group G to $SO(16) \times SO(16)$ [38, 41]. This may be achieved by adding an appropriate B-field term $\lambda^A\, B_{AB}\, \lambda^B$ to the heterotic string action (3.4.1), whose effect is to simply modify the modular form (3.4.2) in a standard way. It amounts to a shift of the imaginary part Ω_2 of the period matrix of Σ_2 and thus produces a reduction onto different tori in the right-moving sector. This (non-modular) change of the base tori can be derived directly from the corresponding Polyakov path integral [75].

3.5 Boundary Contributions

In this final section we will elucidate some arithmetic and physical aspects of the two-loop superstring free energy (3.3.20). We have seen that the pertinent genus two theta-functions (3.2.35) factorize into elliptic Jacobi-Erderlyi functions associated with the fibration (3.2.19) of the Jacobian variety of the original curve Σ_2 into two tori $\mathbb{T}^2_{\mathrm{i}\,\nu}$ and $\mathbb{T}^2_{\tau}$. But the resulting formulas for the free energies are quite involved and difficult to deal with analytically. We will now explore some regions of the moduli space $\mathcal{M}_2$ wherein this factorization simplifies drastically and some precise information can be extracted from these expressions.

3.5.1 Pinching Parameters

Let us begin with some general aspects concerning the generic relationship between genus two curves and elliptic curves. Generally, any genus two surface Σ_2 is a connected sum $\Sigma_2 = \mathbb{T}^2_{\tau_1} \# \mathbb{T}^2_{\tau_2}$ of two tori whose periods can be expressed in terms of the moduli τ_i of the tori and a complex number t. The positive number $|t| < 1$ is the radius of the disks that are excised from the two tori in order to sew them together to produce Σ_2. Let $q_i := \mathrm{e}^{\,2\pi\,\mathrm{i}\,\tau_i}$, $i = 1, 2$. Then the pinching parameters q_1, q_2, t form an alternative set of moduli for Σ_2.

The genus two period matrix Ω may be computed as a holomorphic function of the pinching parameters q_1, q_2, t [76]. For this, we use the sewing formalism to express the holomorphic one-differential $\boldsymbol{\omega}$ of Σ_2 as a power series in t with coefficients calculated from the genus one differentials $\omega^{(i)}$ of $\mathbb{T}^2_{\tau_i}$, $i = 1, 2$, and then use (3.1.4) to calculate the period matrix elements. We will need these expressions only to leading order in $t \to 0$, in which case the period matrix is given by

$$\begin{aligned} \Omega_{11} &= \tau_1 + \frac{t^2}{2\pi\,\mathrm{i}}\,\hat{E}_2(q_1) + O\left(t^4\right) \ , \\ \Omega_{22} &= \tau_2 + \frac{t^2}{2\pi\,\mathrm{i}}\,\hat{E}_2(q_2) + O\left(t^4\right) \ , \\ \Omega_{12} &= -\frac{t}{2\pi\,\mathrm{i}}\left(1 + \hat{E}_2(q_1)\,\hat{E}_2(q_2)\,t^2\right) + O\left(t^5\right) \end{aligned} \tag{3.5.1}$$

where

$$\hat{E}_2(q) = -\frac{1}{12} + 2\sum_{n=1}^{\infty}\sigma_1(n)\ q^n \tag{3.5.2}$$

is the normalized elliptic Eisenstein series, with $\sigma_1(n)$ the number of n-sheeted unbranched covers of a torus given by (3.1.32). Let us now specialize to the case where $\Sigma_2 \to \mathbb{T}^2_{\mathrm{i}\,\nu}$ is a branched covering with the reduced form (3.2.18) of its period matrix. The moduli of the two connecting tori can then be identified as $\tau_1 = (x + \mathrm{i}\,\frac{r}{\nu})/z$ and $\tau_2 = -\Omega_{22} = \tau$. The torus $\mathbb{T}^2_{\tau_1}$ in this case is an unbranched cover of the base $\mathbb{T}^2_{\mathrm{i}\,\nu}$ of degree $N = r\,z$. The radius of the connecting cylinder may be identified as $|t| = \frac{y}{z}$, which satisfies $0 < |t| < 1$ since $y \in \mathbb{Z}/z\,\mathbb{Z}$ and $y \neq 0$.

There are two classes of degenerations of the Riemann surface Σ_2 up to modular transformations. When $t \to 0$, the connecting cylinder is pinched down and Σ_2 degenerates into the two tori $\mathbb{T}^2_{\tau_1}$ and $\mathbb{T}^2_{\tau_2}$. This provides a geometric description of the moduli space $\mathcal{M}_2$ near the divisor of surfaces Σ_2 with nodes, and it corresponds to the limit in which the two branch points on $\mathbb{T}^2_{\mathrm{i}\,\nu}$ coincide (singularity type (b) in the terminology of Section 3.2.1). When $q_i \to 0$ for $i = 1$ or $i = 2$, i.e. $\tau_i \to \mathrm{i}\,\infty$, the torus $\mathbb{T}^2_{\tau_i}$ degenerates to a Riemann sphere by making its homology cycle β infinitely long, or equivalently by modular invariance shrinking the cycle to zero size (singularity type (c)). It is straightforward to see that the other boundary limits of the moduli space $\mathcal{M}_2$, determined by the positivity condition (3.2.24), can be mapped into these other two cases. Let us now examine each of these limits in some detail.

3.5.2 Factorization

In the sewing construction one may view the genus two surface Σ_2 as the disjoint union $\Sigma_2 = \mathbb{E}_1 \amalg \mathbb{A}_t \amalg \mathbb{E}_2$, where $\mathbb{A}_t = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_i \in \mathbb{B}^2 \ , \ z_1 z_2 = t\}$ for $t \neq 0$ is the annulus with outer radius 1 and inner radius t, $\mathbb{B}^2$ is the unit disk in $\mathbb{C}$, and $\mathbb{E}_i = \mathbb{T}^2_{\tau_i} \setminus \mathbb{B}^2$ with z_i local complex coordinates on $\mathbb{T}^2_{\tau_i}$. In conformal field theory, the surfaces with boundary $\mathbb{E}_i$, $i = 1,2$ define two states $\langle \mathbb{E}_1|$ and $|\mathbb{E}_2\rangle$. Within the Hamiltonian framework, we identify the annulus with a cylinder via the exponential map. The cylinder amplitude then corresponds to the operator insertion $t^{L_0}\,\bar t^{\,\overline{L}_0}$, where $L_0 + \overline{L}_0$ is the worldsheet Hamiltonian and $L_0 - \overline{L}_0$ is the momentum operator.

The genus two superstring free energy is then given symbolically by

$$F_2 = \big\langle \mathbb{E}_1 \big| \, t^{L_0}\,\bar t^{\,\overline{L}_0} \, \big| \mathbb{E}_2 \big\rangle \ . \tag{3.5.3}$$

We can insert a complete set of states into the matrix element (3.5.3) which diagonalize the Virasoro operators $L_0, \overline{L}_0$ to get

$$F_2 = \sum_I \big\langle \mathbb{E}_1 \big| \psi_I \big\rangle \, \big\langle \psi_I \big| \, t^{L_0}\,\bar t^{\,\overline{L}_0} \, \big| \psi_I \big\rangle \, \big\langle \psi_I \big| \mathbb{E}_2 \big\rangle \ . \tag{3.5.4}$$

This yields a Laurent series expansion in $|t|$. After GSO projection, the leading contribution comes from the massless vacuum states having $L_0 = \overline{L}_0 = 0$ and zero momentum, so that

$$F_2 = F^{(1)}\,F^{(2)} + O\big(|t|\big) \ , \tag{3.5.5}$$

where $F^{(i)}$ is the one-loop free energy for the torus $\mathbb{T}^2_{\tau_i}$. We should stress that the expression (3.5.5) is only meant to be symbolic. In particular, it is only valid at fixed spin structure and fixed winding numbers around the finite temperature DLCQ torus $\mathbb{T}^2_{\mathrm{i}\nu}$, in which case the leading term is actually down by a negative power of $|t|$. Summing over these quantum numbers mixes the two one-loop contributions in a non-trivial way and spoils the explicit factorization of the leading order term. We shall see this explicitly below. The higher-order terms in (3.5.5) arise from propagation of massless physical states in the long thin tube connecting the two tori [69], and in this limit the genus two free energy is related to a sum of products of one-loop tadpoles for the massless states represented as torus one-point functions.

We will now identify these one-loop string theories. Let $\delta = \binom{\boldsymbol{a}_1}{\boldsymbol{a}_2} \neq \delta_0$ be any even genus two spin structure such that $\boldsymbol{a}_i \in \{0,1\}^2$ is an even genus one spin structure on $\mathbb{T}^2_{\tau_i}$. In the limit $t \to 0$, the leading asymptotics of the genus two theta-constants are given by

$$\begin{aligned} \Theta[\delta](\mathbf{0}|\Omega) &= \theta[\boldsymbol{a}_1](0|\tau_1)\,\theta[\boldsymbol{a}_2](0|\tau_2) + O\left(t^2\right) \ , \\ \Theta[\delta_0](\mathbf{0}|\Omega) &= t\ \eta(\tau_1)^3\,\eta(\tau_2)^3 + O\left(t^3\right) \ , \end{aligned} \tag{3.5.6}$$

which implies that the cusp form (3.2.39) has the leading asymptotics

$$\Psi_{10}(\Omega) = t^2\ \eta(\tau_1)^{24}\,\eta(\tau_2)^{24} + O\left(t^4\right) \ . \tag{3.5.7}$$

It is instructive to first examine the behaviour of the bosonic free energy (3.2.40) in this limit. Notice, first of all, that since $y \in \mathbb{Z}/z\,\mathbb{Z}$ with $y \neq 0$, the limit $t \to 0$ is equivalent to taking $z \to \infty$, i.e. the limit $N \to \infty$ of branched covers with large degree. This means that we should look at the large N asymptotic tail behaviour of the series (3.2.40). One then has

$$\begin{aligned} \lim_{z\to\infty} F_2^{\rm bos} &= -g_s^2 \left(\frac{1}{4\sqrt{2}\,\pi\,\beta\,R}\right)^{12} \sum_{N=1}^\infty \mathrm{e}^{-\frac{\beta\,N}{\sqrt{2}\,R}} \sum_{r\,z=N} \left(\frac{z}{r}\right)^{12} \\ &\quad \times \sum_{\substack{x,y\in\mathbb{Z}/z\,\mathbb{Z}\\ y\neq 0}} \frac{1}{y^4}\,\mathcal{Z}_1^{\rm bos}\left(\tau'\,,\,\overline{\tau}\,'\right)\Big|_{\tau'=\frac{x+\frac{\mathrm{i}\,r}{\nu}}{z}}\ \tilde{F}_1^{\rm bos} \end{aligned} \tag{3.5.8}$$

where

$$\tilde{F}_1^{\rm bos} = \int\limits_\Delta \frac{\mathrm{d}^2\tau}{(\tau_2)^{12}}\ \mathcal{Z}_1^{\rm bos}(\tau,\overline{\tau}\,) \tag{3.5.9}$$

and $\mathcal{Z}_1^{\rm bos}(\tau,\overline{\tau}\,) =$ Tr $q^{L_0-2}\,\overline{q}^{\,\overline{L}_0-2} = |\eta(\tau\,)|^{-48}$ is the one-loop first quantized partition function on $\mathbb{T}^2_\tau$. Thus the contribution of the unramified coverings of $\mathbb{T}^2_{\mathrm{i}\,\nu}$ is the same as in the one-loop computation of Section 3.1.2, while the contribution over the auxilliary torus $\mathbb{T}^2_\tau$ resembles the second quantized one-loop bosonic partition function (Note that this is *not* the standard $SL(2,\mathbb{Z})$ modular invariant partition function, as modular invariance of the expression (3.5.8) under the genus two residual modular group $\mathcal{G} \subset Sp(4,\mathbb{Z})$ is required here). This is a twisted admixture of the operation providing the mapping from first quantization to second quantization that was given by Hecke transforms in Section 3.1.2.

To understand the algebraic meaning of the mapping in the present case, we now turn our attention to the superstring free energy (3.3.20). For this, we also need the asymptotic behaviours of the quantities (3.3.14), which from (3.5.6) and the formulas of Appendix B can be computed to be

$$\begin{aligned} \Xi_6[\delta](\Omega) &= -2^8 \left\langle \boldsymbol{a}_1 \middle| \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)\right\rangle \left\langle \boldsymbol{a}_2 \middle| \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)\right\rangle \eta(\tau_1)^{12}\,\eta(\tau_2)^{12} + O\left(t^2\right) \ , \\ \Xi_6[\delta_0](\Omega) &= -3\cdot 2^8 \ \eta(\tau_1)^{12}\,\eta(\tau_2)^{12} + O\left(t^2\right) \ . \end{aligned} \tag{3.5.10}$$

Substituting (3.5.6) and (3.5.10) into the numerator of the integrand in (3.3.19), one finds that the contributions from the spin structures δ_7, δ_8 and δ_9 sum to 0 by the Jacobi abstruse identity (3.1.45). This sum is tantamount to a partial GSO projection which removes the would be tachyonic divergence coming from (3.5.7) in the degeneration limit $t \to 0$. Only the contribution from the spin structure δ_0 remains, and (3.3.19) becomes

$$\begin{aligned} \lim_{z\to\infty} F_2 &= -\frac{g_s^2}{4}\left(\frac{1}{4\sqrt{2}\,\pi\,\beta\,R}\right)^4 \sum_{N=1}^\infty \frac{\mathrm{e}^{-\frac{\beta\,N}{\sqrt{2}\,R}}}{N} \sum_{\substack{r\,z=N\\ r\,\mathrm{odd}}} \frac{1}{r^4} \sum_{\substack{x,y\in\mathbb{Z}/z\,\mathbb{Z}\\ y\neq 0}} \int_\Delta \frac{\mathrm{d}^2\tau}{(\tau_2)^4}\left(\frac{3\pi^2}{4}\,\frac{y^2}{z^2}\right)^2 \\ &= -\frac{\sqrt{3}\,\pi^2\,g_s^2}{8}\left(\frac{1}{4\sqrt{2}\,\pi\,\beta\,R}\right)^4 \sum_{N=1}^\infty \frac{\mathrm{e}^{-\frac{\beta\,N}{\sqrt{2}\,R}}}{N} \sum_{\substack{r\,z=N\\ r\,\mathrm{odd}}} \frac{1}{r^4}\left(\tfrac{1}{5}\,z^2 - \tfrac{1}{2}\,z\right) \end{aligned} \tag{3.5.11}$$

to $O(z^{-1})$. The removal of the tachyonic divergence from $\tilde{F}_1^{\rm bos}$ in (3.5.8) has completely trivialized the partition function over the auxilliary torus and the only contribution that remains is from the unbranched cover over $\mathbb{T}^2_{\mathrm{i}\,\nu}$. The precise form of this sum is now determined by the way in which we analyse the large degree asymptotics as $N \to \infty$ of this series.

Let us first take the limit $z \to \infty$ with r finite. In this limit $\tau_1 \to 0$ and the covering torus Σ_1 shrinks to a point. Nevertheless, some remnant of the genus two covering map remains due to the fibration over the auxilliary torus $\mathbb{T}^2_\tau$ in (3.2.19). In this regime we may disregard the odd parity constraint on the sum over the divisors r in (3.5.11), and the free energy thereby becomes

$$\lim_{\substack{z\to\infty \\ r\ll z}} F_2 = -\frac{\sqrt{3}\,\pi^2\,g_s^2}{8}\left(\frac{1}{4\sqrt{2}\,\pi\,\beta\,R}\right)^4\sum_{N=1}^\infty \frac{\mathrm{e}^{-\frac{\beta\,N}{\sqrt{2}\,R}}}{N^5}\,\left(\tfrac15\,\sigma_6(N)-\tfrac12\,\sigma_5(N)\right) \tag{3.5.12}$$

where the divisor functions

$$\sigma_k(N) = \sum_{z|N} z^k \tag{3.5.13}$$

generalize the integers $\sigma_1(N)$ in (3.1.32) which count the unramified coverings Σ_1 of $\mathbb{T}^2_{\mathrm{i}\,\nu}$. The series (3.5.12) can be naturally related to the Hecke algebra as follows.

Consider the lattice $\Lambda_\tau := \mathbb{Z}\oplus\mathbb{Z}\,\tau$ such that $\mathbb{T}^2_\tau = \mathbb{C}/\Lambda_\tau$. For any integer $k\geq 2$, introduce the holomorphic Eisenstein series [57]

$$G_{2k}(\tau) := \sum_{\substack{\lambda\in\Lambda_\tau \\ \lambda\neq(0,0)}} \frac{1}{\lambda^{2k}} = 2\,\zeta(2k) + 2\,\frac{(2\pi\,\mathrm{i}\,)^k}{(2k-1)!}\sum_{n=1}^\infty \sigma_{2k-1}(n)\;q^n \tag{3.5.14}$$

with $q=\mathrm{e}^{\,2\pi\,\mathrm{i}\,\tau}$. This defines a modular form of weight $2k$. The action on (3.5.14) of the Hecke operator $\mathbf{H}_N$ defined in (3.1.48) is given by

$$\mathbf{H}_N * G_{2k}(\tau) = N^{2k-1}\sum_{\substack{\Lambda'\subset\Lambda_\tau \\ [\Lambda_\tau:\Lambda']=N}}\;\sum_{\substack{\lambda\in\Lambda' \\ \lambda\neq(0,0)}}\frac{1}{\lambda^{2k}}\ . \tag{3.5.15}$$

To work out this sum explicitly, suppose first that $N=p$ is a prime number. If $\lambda\in p\,\Lambda_\tau$, then λ lies in all sublattices Λ' of Λ_τ of index p and so contributes $\frac{\sigma_1(p)}{\lambda^{2k}} = \frac{p+1}{\lambda^{2k}}$ to the sum (3.5.15). Otherwise, λ lies in only one sublattice $\Lambda' = p\,\Lambda_\tau\oplus\mathbb{Z}\,\lambda$ and so contributes $\frac{1}{\lambda^{2k}}$. Thus

$$\mathbf{H}_p * G_{2k}(\tau) = p^{2k-1}\;G_{2k}(\tau) + p^{2k}\sum_{\substack{\lambda\in p\,\Lambda_\tau \\ \lambda\neq(0,0)}}\frac{1}{\lambda^{2k}} = p^{2k-1}\;G_{2k}(\tau)+G_{2k}(\tau) \tag{3.5.16}$$

and it follows that $G_{2k}(\tau)$ is an eigenform of $\mathbf{H}_p$ with eigenvalue $\sigma_{2k-1}(p)=1+p^{2k-1}$. In the general case, we use the prime factorization of the integer N along with the Hecke algebra property $\mathbf{H}_n\circ\mathbf{H}_m = \mathbf{H}_{n\,m}$ for $\gcd(n,m)=1$ to conclude that the Eisenstein series G_{2k} is a simultaneous eigenform of each Hecke operator $\mathbf{H}_N$ with eigenvalue $\sigma_{2k-1}(N)$. Similarly, each $\mathbf{H}_N$ has eigenforms comprised of elliptic cusp forms $\eta(\tau)^{24}\, G_{2k}(\tau)$ [57].

Let us now take the limit $z \to \infty$ with $r \sim z$. In this limit $\tau_1 \to \frac{\mathrm{i}}{\nu}$ and the surface Σ_2 factorizes into the original spacetime torus $\mathbb{T}^2_{\mathrm{i}\nu}$ (up to a modular transformation) and the auxilliary torus $\mathbb{T}^2_\tau$. The free energy (3.5.11) in this regime vanishes,

$$\lim_{\substack{z\to\infty \\ r\sim z}} F_2 = 0 \ , \tag{3.5.17}$$

to leading order. At this order supersymmetry is restored by the factorization and there are no contributions from this boundary component of the moduli space $\mathcal{M}_2$. The combinatorics of the covers in these factorizing degeneration limits are thereby accounted for by a sort of "topological" string theory which counts particular eigenvalues in the spectra of the Hecke operators. The role of the degenerate free energy as a generating function for the Hecke spectra will also persist at higher orders in the cylindrical length t. For example, the Siegel cusp form of weight ten has has the leading expansion [58]

$$\Psi_{10}(\Omega)^{-1} = t^{-2}\,\eta(\tau_1)^{-24}\,\eta(\tau_2)^{-24}\,\Big[1 + 12\,t^2\,\hat{E}(q_1)\,\hat{E}(q_2) + O\left(t^4\right)\Big] \tag{3.5.18}$$

as $t \to 0$.

3.5.3 Collapsing Homology Cycles

Let us now look at the limit $q_1 \to 0$ in which the handle with homology cycles a_1, b_1 degenerates. In this non-separating degeneration limit, the surface Σ_2 becomes the auxilliary torus $\mathbb{T}^2_\tau$. If $\boldsymbol{a} \in \{0,1\}^2$ is any even genus one characteristic, then the even characteristic genus two theta-constants generally have a power series expansion around $q_1 = 0$ given by

$$\begin{aligned}
\Theta\big[{}^{\boldsymbol{a}}_{00}\big](\mathbf{0}|\Omega) &= \sum_{n=-\infty}^{\infty} (q_1)^{n^2}\ \theta[\boldsymbol{a}]\big(-\tfrac{n\,t}{2\pi\,\mathrm{i}}\big|\tau_2\big)\ , \\
\Theta\big[{}^{\boldsymbol{a}}_{01}\big](\mathbf{0}|\Omega) &= \sum_{n=-\infty}^{\infty} (-1)^n\,(q_1)^{n^2}\ \theta[\boldsymbol{a}]\big(-\tfrac{n\,t}{2\pi\,\mathrm{i}}\big|\tau_2\big)\ , \\
\Theta\big[{}^{\boldsymbol{a}}_{10}\big](\mathbf{0}|\Omega) &= \sum_{n=-\infty}^{\infty} (q_1)^{(n+\frac12)^2}\ \theta[\boldsymbol{a}]\big(-\tfrac{(n+\frac12)\,t}{2\pi\,\mathrm{i}}\big|\tau_2\big)\ ,
\end{aligned}$$

$$\Theta\big[\delta_0\big](\mathbf{0}|\Omega) \;=\; \sum_{n=-\infty}^{\infty} \mathrm{i}\,(-1)^n\,(q_1)^{(n+\frac12)^2}\;\theta_1\big(-\tfrac{(n+\frac12)\,t}{2\pi\,\mathrm{i}}\big|\tau_2\big)\ . \tag{3.5.19}$$

It follows that the cusp form (3.2.39) has the leading asymptotics

$$\Psi_{10}(\Omega) = -(q_1)^2\,\eta(\tau_2)^{18}\,\theta_1\big(-\tfrac{t}{4\pi\,\mathrm{i}}\big|\tau_2\big)^2 + O\left((q_1)^2\right)\ . \tag{3.5.20}$$

After some algebra using the formulas of Appendix B, one thereby finds that leading behavior of the free energy (3.3.19) is given by

$$\begin{aligned}
&\lim_{q_1\to0}\,F_2 = \\
&-\frac{g_s^2}{32}\left(\frac{1}{4\sqrt2\,\pi\,\beta\,R}\right)^4\sum_{N=1}^{\infty}\mathrm{e}^{-\frac{\beta\,N}{\sqrt2\,R}}\sum_{\substack{r\,z=N\\ r\,\mathrm{odd}}}\frac{1}{r^5}\sum_{\substack{y\in\mathbb{Z}/z\,\mathbb{Z}\\ y\neq0}}\int_\Delta\frac{\mathrm{d}^2\tau}{(\tau_2)^4}\,\frac{1}{\big|\eta(\tau)\big|^{36}\,\big|\theta_1\big(\frac{y}{2\,z}\big|\tau\big)\big|^4}\\
&\times\Big|\theta_4(0|\tau)^8\,\Big[\theta_4\big(\tfrac{y}{2\,z}\big|\tau\big)^4\,\theta_1\big(\tfrac{y}{2\,z}\big|\tau\big)^4+\theta_2\big(\tfrac{y}{2\,z}\big|\tau\big)^4\,\theta_3\big(\tfrac{y}{2\,z}\big|\tau\big)^4\Big]\\
&+\,\theta_2(0|\tau)^8\,\Big[\theta_4\big(\tfrac{y}{2\,z}\big|\tau\big)^4\,\theta_3\big(\tfrac{y}{2\,z}\big|\tau\big)^4+\theta_2\big(\tfrac{y}{2\,z}\big|\tau\big)^4\,\theta_1\big(\tfrac{y}{2\,z}\big|\tau\big)^4\Big]\\
&-\,\theta_3(0|\tau)^8\,\Big[\theta_1\big(\tfrac{y}{2\,z}\big|\tau\big)^4\,\theta_3\big(\tfrac{y}{2\,z}\big|\tau\big)^4+\theta_2\big(\tfrac{y}{2\,z}\big|\tau\big)^4\,\theta_4\big(\tfrac{y}{2\,z}\big|\tau\big)^4\Big]\Big|^2\ .
\end{aligned} \tag{3.5.21}$$

The elliptic modular integrals in (3.5.21) are finite.

The degeneration limit $q_1 \to 0$ corresponds to the shrinking limit $\nu \to 0$ of the original spacetime torus $\mathbb{T}^2_{\mathrm{i}\,\nu}$. There are two ways in which we can make the parameter (3.1.12) vanish. Taking $\beta \to \infty$ gives the zero temperature limit of the free energy, which is proportional to the vacuum energy. Since $N \geq 1$, all terms in the series are exponentially damped and thus the vacuum energy vanishes, as expected since this limit simply corresponds to the restoration of supersymmetry at zero temperature. On the other hand, taking $R \to \infty$ decompactifies the light cone and sends the exponential factors to 1 in (3.5.21). Apart from an overall factor, the free energy is then independent of temperature, except for its dependence on the winding number r. In this case the strings effectively propagate on a $\mathbb{Z}_2$ orbifold of flat space [77] defined by the antiperiodic fermion boundary conditions, which is presumably a subsector of the symmetric orbifold superconformal field theory on $\mathbb{R}^8$ for each N. This string theory is non-supersymmetric and hence has a non-vanishing vacuum energy corresponding to contributions from physical tachyons [69]. In each of these decompactification limits, the discrete data of the branched cover should assemble themselves into a continuum limit which restores the two complex dimensions of the moduli space $\mathcal{M}_2$ [60].

Let us now consider the non-separating degeneration limit $q_2 \to 0$ in which the branched cover Σ_2 becomes an unramified covering of the original spacetime torus $\mathbb{T}^2_{\mathrm{i}\,\nu}$ (up to a modular transformation). This corresponds to the contributions from the $\tau \to \mathrm{i}\,\infty$ region of the elliptic modular integral in (3.3.19). We may use the same asymptotic formulas (3.5.19) and (3.5.20) with q_1, τ_2 replaced by q_2, τ_1. The terms $\Xi_6[\delta_i]\,\Theta[\delta_i](\mathbf{0}|\Omega)^4$ for $i = 7, 8$ have leading terms of order q_2. These two terms thus give a contribution to the integration over moduli space which has a simple pole at $q_2 = 0$. This divergence arises from the tachyon traversing the a_2 cycle of the elliptic component $\mathbb{T}^2_\tau$ of the degeneration [69, 39]. However, the sum $\Xi_6[\delta_7]\,\Theta[\delta_7](\mathbf{0}|\Omega)^4 + \Xi_6[\delta_8]\,\Theta[\delta_8](\mathbf{0}|\Omega)^4$ is found to vanish to this order and thus removes the pole. This corresponds to a partial GSO projection in the Neveu-Schwarz sector of the genus one component $\mathbb{T}^2_\tau$ which eliminates the tachyon. The contributions from the remaining spin structures δ_0 and δ_9 correspond to Ramond states propagating in $\mathbb{T}^2_\tau$ and are of order $(q_2)^2$, yielding no poles.

Working out each of the four contributions to (3.3.19) up to order $(q_2)^2$ leads after some algebra to the free energy

$$\lim_{q_2\to 0} F_2 = -\frac{g_s^2}{64}\left(\frac{1}{4\sqrt{2}\,\pi\,\beta\,R}\right)^4 \sum_{N=1}^\infty \frac{\mathrm{e}^{-\frac{\beta\,N}{\sqrt{2}\,R}}}{N} \sum_{\substack{r\,z=N \\ r\ \mathrm{odd}}} \frac{1}{r^4} \sum_{\substack{x,y\in\mathbb{Z}/z\,\mathbb{Z} \\ y\neq 0}} \left|\mathcal{Z}_1^\infty(\zeta|\tau_1)\right|^2\Bigg|_{\substack{\zeta=\frac{y}{z} \\ \tau_1=\frac{x+\mathrm{i}\,\frac{r}{\nu}}{z}}} \tag{3.5.22}$$

where

$$\begin{aligned}
\mathcal{Z}_1^\infty(\zeta|\tau_1) \;=\; & \frac{1}{\eta(\tau_1)^{18}\,\theta_1\big(\frac{\zeta}{2}\big|\tau_1\big)^2}\Bigg[2\,\theta_2(0|\tau_1)^8\Big(\theta_1\big(\tfrac{\zeta}{2}\big|\tau_1\big)^4\,\theta_2\big(\tfrac{\zeta}{2}\big|\tau_1\big)^4 + \theta_3\big(\tfrac{\zeta}{2}\big|\tau_1\big)^4\,\theta_4\big(\tfrac{\zeta}{2}\big|\tau_1\big)^4\Big) \\
& - \theta_3(0|\tau_1)^8\Big(\theta_1\big(\tfrac{\zeta}{2}\big|\tau_1\big)^4\,\theta_3\big(\tfrac{\zeta}{2}\big|\tau_1\big)^4 + \theta_2\big(\tfrac{\zeta}{2}\big|\tau_1\big)^4\,\theta_4\big(\tfrac{\zeta}{2}\big|\tau_1\big)^4\Big) \\
& + \theta_4(0|\tau_1)^8\Big(\theta_1\big(\tfrac{\zeta}{2}\big|\tau_1\big)^4\,\theta_4\big(\tfrac{\zeta}{2}\big|\tau_1\big)^4 + \theta_2\big(\tfrac{\zeta}{2}\big|\tau_1\big)^4\,\theta_3\big(\tfrac{\zeta}{2}\big|\tau_1\big)^4\Big) \\
& - 8\,\eta(\tau_1)^3\,\theta_1\big(\tfrac{\zeta}{2}\big|\tau_1\big)^4\Big(\theta_2(0|\tau_1)\,\theta_3(0|\tau_1)\,\theta_4(\zeta|\tau_1) + \theta_2(0|\tau_1)\,\theta_4(0|\tau_1)\,\theta_3(\zeta|\tau_1) \\
& \qquad + \theta_3(0|\tau_1)\,\theta_4(0|\tau_1)\,\theta_2(\zeta|\tau_1)\Big)
\end{aligned}$$

$$- 8\,\eta(\tau_1)^3\,\theta_2\big(\tfrac{\zeta}{2}\big|\tau_1\big)^4\,\Big(\theta_2(0|\tau_1)\,\theta_3(0|\tau_1)\,\theta_4(\zeta|\tau_1) + \theta_2(0|\tau_1)\,\theta_4(0|\tau_1)\,\theta_3(\zeta|\tau_1)$$

$$- \theta_3(0|\tau_1)\,\theta_4(0|\tau_1)\,\theta_2(\zeta|\tau_1)\Big)\Bigg] \qquad (3.5.23)$$

and we have dropped an irrelevant overall numerical constant in (3.5.22) arising from the remaining modular integration over $\tau_2 \in \Delta$. As before, the non-vanishing of this boundary contribution is due to the presence of physical tachyons. This free energy is a natural extension of the one-loop result of Section 3.1.2, illustrating the appropriate modification for the action of the Hecke algebra at two-loops.

Chapter 4

Twist Field Correlators on Symmetric Orbifolds

4.1 Correlation Functions on Permutation Orbifolds

In this section we will discuss some general aspects of permutation orbifolds of conformal field theories, and in particular the case of two-dimensional sigma models on symmetric product orbifolds of flat space. We first describe the general structure of the partition functions of these models, and then explain the construction of various classes of correlation functions including those of twist field operators. For the moment we treat only bosonic sigma models explicitly in order to highlight the essential details, defering a more detailed analysis of the supersymmetric and heterotic cases to Section 4.4. We also explain how these orbifold theories can be interpreted as string field theories.

4.1.1 Permutation Orbifolds

When a two-dimensional conformal field theory has a discrete symmetry, one can consider the orbifold theory arising from quotienting with respect to the symmetry. The simplest example is the free boson on the circle $\mathbb{S}^1$. Its action $\frac{1}{4\pi\,\alpha'}\int_\Sigma \mathrm{d}^2z\ \|\partial X\|^2$ is invariant under the reflection $X \to -X$. The quotient of the target space is the well-known geometric orbifold $\mathbb{S}^1/\mathbb{Z}_2$, and the coordinate field X can have non-trivial monodromy when encircling a non-contractible cycle of the worldsheet Σ. If the radius of the circle is equal to the fundamental string length $\ell_s = \sqrt{\alpha'}$, then the resulting orbifold conformal field theory is the "square" of the critical Ising model [78].

Permutation orbifolds represent a large class of orbifolds where the parent conformal field theory (whose quotient is taken) has physical Hilbert space $\mathcal{H}$ with a discrete symmetry. This concept was first introduced in [79], and used for the construction of a $\mathbb{Z}_2$ orbifold of the $E_8 \times E_8$ heterotic string in [80]. One of their main applications is to the second quantization of string theory [4], and they have recently been argued [81] to describe new physical string theories at multiples of the critical dimension. A permutation orbifold of an arbitrary conformal field theory $\mathcal{C}$, by any finite symmetry group G regarded as a subgroup of a symmetric group of some degree, is a consistent conformal field theory. All of its important quantities (central charge, conformal weights, genus one characters, modular S and T matrices, genus one partition function, *etc.*) were worked out originally for cyclic groups in [82], and then generalized to arbitrary finite groups in [25]. These formulae express a given quantity as a combinatorial expansion, depending on the twist group G, of the same quantity in the parent theory.

Highest weight states in a permutation orbifold $\mathcal{C} \wr G := (\mathcal{C})^{\otimes N}/G$ correspond to orbits of a subgroup $G < S_N$ of the symmetric group of degree N acting on the N-fold tensor product of states in the parent theory $\mathcal{C}$.[1] In the case that $\mathcal{C}$ admits a sigma model description with embedding coordinate field $X \in M$, there is a corresponding sigma model description of $\mathcal{C} \wr G$ on the geometric orbifold M^N/G [31]. One introduces N identical coordinate fields $X^a = X$, $a = 1, \dots, N$ on the worldsheet Σ and allows for G-twisting of them along non-trivial cycles. For example, on the torus $\Sigma = \mathbb{T}^2$ with modulus τ, the boundary conditions of the N coordinate fields are labelled by two commuting permutations $P, Q \in G < S_N$ such that

$$X^a(z+1) = X^{P(a)}(z) \qquad \text{and} \qquad X^a(z+\tau) = X^{Q(a)}(z) \ , \tag{4.1.1}$$

where in general $g(a)$ denotes the image of the label a under the permutation $g \in G$. For a non-trivial pair (P, Q), these boundary conditions are called twisted sectors

[1] There is an additional label corresponding to the irreducible character of the double of the stabilizer of the orbit. See [25] for the precise definition.

of the theory. Two pairs (P,Q) and $(g\,P\,g^{-1}, g\,Q\,g^{-1})$ with $g \in S_N$ correspond to the same twisted sector, since we can get from one to the other by relabelling the coordinate fields $a \to g(a)$.

In general, a twisted sector is given by an equivalence class of homomorphisms from the fundamental group $\pi_1(\Sigma)$ of the worldsheet to the twist group G. (Since $\pi_1(\mathbb{T}^2) = \mathbb{Z} \oplus \mathbb{Z}$, on the torus one specifies a homomorphism by choosing the image in G of the two commuting generators.) Two homomorphisms Φ, Φ' define the same twisted sector, and are said to be equivalent, if they are related by conjugation as $\Phi'(-) = g\,\Phi(-)\,g^{-1}$ for some $g \in S_N$. The geometric interpretation is provided by the fact that every equivalence class $[\Phi]$ of homomorphisms $\Phi : \pi_1(\Sigma) \to S_N$ determines an unramified cover $\hat{\Sigma}$ of degree N over the Riemann surface Σ. The coordinate label a corresponds to the label of a sheet and Φ is called the monodromy homomorphism of the covering. Conjugation of homomorphisms corresponds to relabelling of the sheets. In the case of the torus $\Sigma = \mathbb{T}^2$ with the boundary conditions (4.1.1), and with the subgroup generated by the pair of permutations P, Q acting transitively on the set of coordinate labels $a = 1, \ldots, N$, one can define a single new field $\mathcal{X}(z)$ which generates all of the fields $X^a(z)$ through the identifications

$$\mathcal{X}(z + m + n\,\tau) = X^{P^m\,Q^n(a)}(z) \tag{4.1.2}$$

with $n, m \in \mathbb{Z}$ and a fixed choice of a. This field is single-valued on a torus which is a cover the original torus $\mathbb{T}^2$, whose modular parameter can be determined from the doubly periodic function $\mathcal{X}$ on $\mathbb{T}^2$.

The modular invariant partition function of the permutation orbifold is determined entirely by the above data. It is given by [25] [2]

$$Z^G(\tau) = \frac{1}{|G|} \sum_{\Phi:\pi_1(\Sigma)\to G} \Big(\prod_{\xi\in\mathcal{O}(\Phi)} Z\big(\tau^{\xi}\big) \Big) \tag{4.1.3}$$

where the product runs over the orbits ξ of the image $\Phi(\pi_1(\Sigma))$ in G and $Z(\tau^\xi)$ is the modular invariant partition function of the parent conformal field theory on the connected component, corresponding to ξ, of the cover of Σ given by the homomorphism Φ. (The covering space $\hat{\Sigma}$ is connected if and only if $\Phi(\pi_1(\Sigma))$ acts transitively

[2] It is also expressible as a sesquilinear expansion in the Virasoro characters $\mathrm{Tr}_{\mathcal{H}}(q^{L_0 - c/24})$, whose form is known in permutation orbifolds [25, 30].

in S_N). The summation over Φ defines the projection onto G-invariant states and ensures modular invariance of the partition function. We will now explain how to determine (4.1.3) in practice.

The complex structure τ of the worldsheet $\Sigma = \Sigma_\tau$ is encoded by a monomorphism $u : \pi_1(\Sigma) \to I$, where I is the isometry group of the universal cover U of Σ. For genus $g > 1$ the latter space is a two-dimensional hyperbolic space, say the upper half plane $U = \mathbb{U}$, and $I = PSL(2, \mathbb{R})$. The surface Σ equipped with a complex structure can be presented as the quotient $\Sigma_\tau = \mathbb{U}/u(\pi_1(\Sigma))$ and its complex structure inherited from $\mathbb{U}$ is encoded by the uniformizing group $u(\pi_1(\Sigma))$. Given a monodromy homomorphism Φ, the fundamental group of the corresponding cover $\hat{\Sigma}$ is isomorphic to the stabilizer subgroup $H_a = \pi_1(\Sigma)_\xi := \{\gamma \in \pi_1(\Sigma) \mid \Phi(\gamma)(a) = a\}$ with fixed $a \in \xi$ (represented by closed loops based at sheet a). Its index is equal to the length of the orbit $[\pi_1(\Sigma) : H_a] = |\xi|$, which is the number of sheets of the corresponding connected component of $\hat{\Sigma}$. Thus the monodromy homomorphism determines the topology of the covering space $\hat{\Sigma}$. We can now define the uniformizing group (and hence the complex structure τ^ξ) of the cover $\hat{\Sigma}$ to be given by $u(H_a)$ (*i.e.*, $\hat{\Sigma}_{\tau^\xi} = \mathbb{U}/u(H_a)$), which is a subgroup of $u(\pi_1(\Sigma))$ in accordance with the expected property $\pi_1(\hat{\Sigma}) < \pi_1(\Sigma)$. Note that the representative of the orbit $a \in \xi$ can be arbitrarily chosen. This is because $H_a = \gamma\, H_{a'}\, \gamma^{-1}$ with $\gamma \in \pi_1(\Sigma)$ for any $a, a' \in \xi$ and conjugate subgroups of $\pi_1(\Sigma)$ give rise to isometric quotients, hence determining equivalent surfaces. The homomorphism u is not unique, as it can be composed with a modular transformation, but the partition function is modular invariant which makes the formula (4.1.3) well defined.

The expression (4.1.3) is an example of the typical structure of a quantity defined on a Riemann surface Σ in a permutation orbifold. It is given by a combinatorial expansion (depending only on G) over the same quantity in the parent theory $\mathcal{C}$ defined on all of those surfaces which cover Σ whose monodromy group is a subgroup of G. Its direct applicability is limited somewhat by the Riemann-Hurwitz formula for the genus $\hat{g}$ of the unramified cover $\hat{\Sigma}$ given by

$$\hat{g} = N\,(g-1) + 1 \ . \tag{4.1.4}$$

This implies that, unless $g = 1$, we would need to know the partition functions of the parent theory on surfaces of genera higher than g in order to write down the genus g partition function of the orbifold.

The case $g = 1$ is, however, much simpler. The universal cover of the torus $\mathbb{T}^2$ is $U = \mathbb{C}$ and $I = \{T_c \mid c \in \mathbb{C}\}$ is the group of translations $T_c : z \mapsto z + c$ of the complex plane. We saw above that specifying a homomorphism Φ amounts to assigning commuting elements $P, Q \in G$ for the generators (α, β) of $\pi_1(\mathbb{T}^2) = \mathbb{Z} \oplus \mathbb{Z}$. The stabilizer subgroup H_a of any representative of an orbit $a \in \xi$ can be characterized by three positive integers s, m, r such that r is the smallest positive integer satisfying $P^r(a) = a$, $0 \leq s < r$ and H_a is generated by $\alpha^r, \alpha^s \beta^m$. Then the index of this subgroup is given by $|\xi| = r\, m$. The image of H_a under the isomorphism $u : (\alpha, \beta) \mapsto (T_1, T_\tau)$ determines a subgroup $< T_r\, , \, T_{s+m\,\tau} >$ and the corresponding quotient of $\mathbb{C}$ is the torus with Teichmüller parameter given by

$$\tau^\xi = \frac{s + m\,\tau}{r}\ . \tag{4.1.5}$$

The fact that the finite index subgroups of the group $\mathbb{Z} \oplus \mathbb{Z}$ are all isomorphic to the group itself implies that all unramified covers of the torus are tori. In sigma model language, the path integral over the multi-valued fields X^a on the torus $\mathbb{T}^2$ is constructed by calculating the path integral over the single-valued field $\mathcal{X}$ on the covering torus and summing over every possible $\mathcal{X}$ constructed by different choices of the commuting pair $P, Q \in G$. For example, the genus one partition function of the S_3 orbifold is given by [29]

$$\begin{aligned} Z^{S_3}(\tau) \ &= \ \tfrac16\, Z(\tau)^3 + \tfrac12\, Z(\tau)\, \Big(Z(2\tau) + Z\big(\tfrac{\tau}{2}\big) + Z\big(\tfrac{\tau+1}{2}\big)\Big) \\ &\quad + \tfrac13\, \Big(Z(3\tau) + Z\big(\tfrac{\tau}{3}\big) + Z\big(\tfrac{\tau+1}{3}\big) + Z\big(\tfrac{\tau+2}{3}\big)\Big)\ . \end{aligned} \tag{4.1.6}$$

Note that the individual terms in (4.1.6) are not modular invariant, but their sum is.

4.1.2 Symmetric Products

Permutation orbifolds whose twist group G is the full symmetric group S_N are called symmetric products $\mathrm{Sym}^N(\mathcal{C}) := (\mathcal{C})^{\otimes N}/S_N$. In this case the formula (4.1.3) takes into account all N-sheeted coverings. Starting from a fixed parent theory $\mathcal{C}$ and a given worldsheet genus g, the generating function of partition functions for all N can be written in a closed form thanks to a combinatorial identity due to Bántay [28]. This identity translates the sum over homomorphisms in (4.1.3) to a sum over finite index subgroups of the group $\Gamma = \pi_1(\Sigma)$ and is given by

$$1+\sum_{N=1}^{\infty}\frac{1}{N!}\sum_{\Phi:\Gamma\to S_N}\Big(\prod_{\xi\in\mathcal{O}(\Phi)}\mathcal{Z}(\Gamma_\xi)\Big)=\exp\Big(\sum_{H<\Gamma}\frac{\mathcal{Z}(H)}{[\Gamma:H]}\Big)\ , \tag{4.1.7}$$

where Γ_ξ is the stabilizer of the orbit ξ and $[\Gamma : H]$ denotes the index of the subgroup H in Γ. The formula (4.1.7) holds generally for any finitely generated group Γ and any conjugation invariant function $\mathcal{Z}$ (*i.e.*, $\mathcal{Z}(\gamma\, H\, \gamma^{-1}) = \mathcal{Z}(H)$ for all $\gamma \in \Gamma$) from the set of finite index subgroups of Γ to a commutative ring R.

The proof of (4.1.7) is instructive. A given term $\prod_\xi \mathcal{Z}(\Gamma_\xi)$ in the sum on the left-hand side of (4.1.7) depends only on the equivalence class of the homomorphism Φ. An equivalence class can be written as

$$[\Phi]=\bigoplus_{k=1}^{N} n_k\,\phi_k\ , \tag{4.1.8}$$

where ϕ_k is a transitive equivalence class whose orbits all have length k and $n_k \geq 0$ is its integer multiplicity with $\sum_k n_k = N$. One can then rewrite the product $\prod_\xi \mathcal{Z}(\Gamma_\xi) = \prod_k \mathcal{Z}(\Gamma_k)^{n_k}$, where Γ_k is the stabilizer subgroup of an arbitrary representative of the image of ϕ_k in S_N. The cardinality of the equivalence class $[\Phi]$ can be determined as follows. The total number of possible elements to conjugate with is $|S_N| = N!$, but not all of these give inequivalent homomorphisms Φ. The permutations which exchange the orbits that have the same S_N-action do not change $[\Phi]$, so we have to divide by their number which is $n_k!$. Finally, we have to divide out the number of cosets $\gamma\,\Gamma_k$ with $\gamma\,\Gamma_k\,\gamma^{-1} = \Gamma_k$, which is the index $\gamma_k = [N_\Gamma(\Gamma_k) : \Gamma_k]$ of the stabilizer Γ_k in its normalizer subgroup $N_\Gamma(\Gamma_k)$. Thus $|[\Phi]| = N!/\prod_k n_k!\,\gamma_k^{n_k}$.

One can now rewrite the left-hand side of (4.1.7) as

$$\begin{aligned}1+\sum_{N=1}^{\infty}\frac{1}{N!}\sum_{\substack{\{n_k\}\\ n_1+\cdots+n_N=N}}\frac{N!}{\prod\limits_{k=1}^{N} n_k!\,\gamma_k^{n_k}}\Big(\prod_{k=1}^{N}\mathcal{Z}(\Gamma_k)^{n_k}\Big) &= \prod_{k=1}^{\infty}\Big(\sum_{n_k=0}^{\infty}\frac{\mathcal{Z}(\Gamma_k)^{n_k}}{n_k!\,\gamma_k^{n_k}}\Big)\\ &= \prod_{k=1}^{\infty}\exp\Big(\frac{\mathcal{Z}(\Gamma_k)}{\gamma_k}\Big)\ .\end{aligned} \tag{4.1.9}$$

Note that here a summation over conjugacy classes of index k subgroups is implicitly assumed. The final step consists in rewriting the product of exponentials as the exponential of a sum over k, and then translating the latter summation into a sum over index k subgroups. There are $[\Gamma : N_\Gamma(\Gamma_k)]$ distinct subgroups in the conjugacy class of Γ_k (as $\gamma\,\Gamma_k\,\gamma^{-1} \neq \Gamma_k$ if $\gamma \notin N_\Gamma(\Gamma_k)$), so we need to divide by this number if we wish to sum over all index k subgroups. Then the resulting factor in the denominator

$$\gamma_k\left[\Gamma : N_\Gamma(\Gamma_k)\right] \;=\; \left[N_\Gamma(\Gamma_k) : \Gamma_k\right]\left[\Gamma : N_\Gamma(\Gamma_k)\right] \;=\; [\Gamma : \Gamma_k] \;=\; k \tag{4.1.10}$$

is precisely the index of Γ_k in Γ and we have arrived at (4.1.7).

Let us now apply the identity (4.1.7) to the uniformizing group $\Gamma = u(\pi_1(\Sigma))$ of a compact Riemann surface $\Sigma = \Sigma_\tau$ with the definition

$$\mathcal{Z}(H) := Z\big(\tau^H\big)\ \kappa^{[\Gamma:H]} \tag{4.1.11}$$

where $Z(\tau^H)$ is the modular invariant partition function of $\mathcal{C}$ defined on the surface $\Sigma_{\tau^H} = U/H$, with U the universal cover of Σ_τ, and κ is a formal variable which is determined by physical constants in applications. The result is the grand canonical partition function

$$Z^{\mathrm{Sym}}(\tau,\kappa) \;:=\; 1+\sum_{N=1}^\infty \kappa^N\ Z^{S_N}(\tau) \;=\; \exp\Big(\sum_{N=1}^\infty \kappa^N\ \mathcal{H}_N Z(\tau)\Big)\ , \tag{4.1.12}$$

where $Z^{S_N}(\tau)$ is the partition function for the S_N orbifold given by the formula (4.1.3) and the operator $\boldsymbol{H}_N$ is defined on modular invariant functions by

$$\boldsymbol{H}_N Z(\tau) = \frac{1}{N}\sum_{[\Gamma:H]=N} Z\big(\tau^H\big)\ . \tag{4.1.13}$$

Note that the product over the orbits ξ in (4.1.7) gives a sum for the power of κ equal to $\sum_\xi\,[\Gamma:\Gamma_\xi] = \sum_\xi\,|\xi| = N$. This generating function is a sum over all possible (finite-sheeted) covers of the surface Σ that the parent conformal field theory $\mathcal{C}$ is defined on, and its logarithm gives the restricted sum over connected covers. The operator defined by (4.1.13) yields a sum over subgroups $H < \pi_1(\Sigma)$ of index N, and in the case of the torus $\Sigma = \mathbb{T}^2$ it coincides with the Hecke operator (2.1.20) acting on the partition function of the parent theory by

$$\boldsymbol{H}_N Z(\tau) = \frac{1}{N}\sum_{r\,m=N}\ \sum_{s\in\mathbb{Z}/r\,\mathbb{Z}} Z\big(\tfrac{s+m\,\tau}{r}\big)\ . \tag{4.1.14}$$

4.1.3 Sigma Models at One-Loop

Our primary example of a permutation orbifold in this thesis will be that of sigma models on symmetric products of flat space $\mathbb{R}^d$ at one-loop order in string perturbation theory. Let us describe this example explicitly in the case of a single boson X in $\mathbb{R}$. The path integral of the sigma model conformal field theory on a symmetric product is gotten by considering the grand canonical partition function

$$Z^{\rm Sym}(\tau,\kappa) = 1 + \sum_{N=1}^\infty \kappa^N \sum_{\substack{P,Q\in S_N\\ P\,Q=Q\,P}} \frac{1}{N!} \int_{(P,Q)} \mathcal{D}X^1\cdots\mathcal{D}X^N \ \exp\Big(-\sum_{a=1}^N I(X^a)\Big) \ , \tag{4.1.15}$$

where

$$I(X) = \frac{1}{4\pi\,\alpha'} \int_{\mathbb{T}}^2 {\rm d}^2 z \ \frac{1}{2\,{\rm i}\,\tau_2}\, \partial X(z)\,\overline{\partial}X(z) \tag{4.1.16}$$

is the bosonic Polyakov action and $z=\sigma^1-\tau\,\sigma^2$, $\sigma^1,\sigma^2\in[0,1]$ are complex coordinates on the torus with respect to the complex structure $\tau=\tau_1+{\rm i}\,\tau_2$, $\tau_1\in\mathbb{R}, \tau_2>0$. The sum over commuting pairs of permutations, specifying monodronomy homomorphisms $\Phi:\pi_1(\mathbb{T}^2)\to S_N$, is taken over worldsheet instantons of the field theory labelled by the boundary conditions (4.1.1). Note that any metric on the torus can be written as

$${\rm d}s^2 = {\rm e}^{\,2\phi(z)}\ |{\rm d}z|^2 \tag{4.1.17}$$

where the scalar field $\phi(z)$ on $\mathbb{T}^2$ is an arbitrary conformal factor.

From the general formulas (4.1.12) and (4.1.14) above it follows that the partition function (4.1.15) is given by the combinatorial formula

$$Z^{\rm Sym}(\tau,\kappa) = \exp\Big(\sum_{N=1}^\infty \kappa^N \sum_{r\,m=N} \ \sum_{s\in\mathbb{Z}/r\,\mathbb{Z}} \frac{1}{N}\, \mathfrak{z}\big(\tfrac{s+m\,\tau}{r}\big)\Big) \ , \tag{4.1.18}$$

where

$$\mathfrak{z}(\tau) = \int \mathcal{D}X \ \ {\rm e}^{\,-I(X)} \tag{4.1.19}$$

is the sigma model partition function on the torus with target space $\mathbb{R}$. This gives a sum of the partition function on a particular torus $\mathbb{T}^2$ over the discrete set of covering tori. The Gaussian integral (4.1.19) can be evaluated in terms of a Quillen norm as

$$\mathfrak{z}(\tau) = \left(\frac{\mathrm{vol}(\mathbb{T}^2)\ \det{}'\,\Delta}{4\pi^2\,\alpha'}\right)^{-1/2} \tag{4.1.20}$$

where Δ is the scalar Laplacian operator on $\mathbb{T}^2$ with respect to the torus metric (4.1.17), $\mathrm{vol}(\mathbb{T}^2)$ is the volume of the surface $\mathbb{T}^2$ in (4.1.17), and $\det{}'\,\Delta$ denotes the determinant of Δ with zero modes excluded. At genus one, this determinant has a natural holomorphic splitting and $\mathfrak{z}(\tau)$ is a section of the determinant line bundle $\underline{\det}\,(\,\overline{\partial}\,)^{-1/2}\otimes\underline{\det}\,(\partial)^{-1/2}$ over the moduli space of complex structures on $\mathbb{T}^2$. The determinant of the Dolbeault operator $\overline{\partial}$ is the automorphic form on Teichmüller space given by

$$\det{}'\,\overline{\partial} = \mathrm{e}^{\,S_{\rm L}(\phi)/24\pi}\ \eta(\tau)^2\ , \tag{4.1.21}$$

where $S_{\rm L}(\phi)$ is the Liouville action and $\eta(\tau) = \mathrm{e}^{\,\pi\,\mathrm{i}\,\tau/12}\prod_{n\in\mathbb{N}}(1-\mathrm{e}^{\,2\pi\,\mathrm{i}\,n\,\tau})$ is the Dedekind function. The partition function (4.1.19) is thus given explicitly by

$$\mathfrak{z}(\tau) = \mathrm{e}^{\,-S_{\rm L}(\phi)/24\pi}\ \Big(\frac{1}{4\pi^2\,\alpha'}\int_{\mathbb{T}^2}\mathrm{d}^2z\ \mathrm{e}^{\,\phi(z)}\Big)^{-1/2}\ \frac{1}{\big|\eta(\tau)\big|^2}\ . \tag{4.1.22}$$

By replacing $\mathfrak{z}(\tau)$ with $\mathfrak{z}(\tau)^d$ in (4.1.18) we get the corresponding result for the parent conformal field theory of a free boson on the target space $\mathbb{R}^d$. Moreover, the combinatorial formula (4.1.18) is completely generic and holds for any sigma model partition function on the torus. For example, we may simply replace $\mathfrak{z}(\tau)$ by the appropriate superstring or heterotic string partition functions at one-loop (with some modifications that we discuss in Section 4.4).

The formula (4.1.12) can also be used to compute any correlation function of fields which are unaffected by the orbifolding. These are the operators which are symmetric under permutations of the indices of the scalar field X. Given any function f, we use the notation $\mathrm{Tr}\,f(X) := \sum_a f(X^a)$ for such an operator refering to a diagonal matrix of the N independent fields X^a. The (normalized) correlation function is defined by

$$\big\langle\,\mathrm{Tr}\,f(X)\,\big\rangle^{\rm Sym}(\tau,\kappa) \tag{4.1.23}$$

$$:= \frac{1}{Z^{\rm Sym}(\tau,\kappa)}\Big(1+\sum_{N=1}^\infty\frac{\kappa^N}{N!}\sum_{\substack{P,Q\in S_N\\ P\,Q=Q\,P}}\ \int_{(P,Q)}\mathcal{D}X^1\cdots\mathcal{D}X^N\ \ \mathrm{Tr}\,f(X)\ \mathrm{e}^{\,-\,\mathrm{Tr}\,I(X)}\Big)\ .$$

Rather than trying to determine the combinatorics of this amplitude directly, we will calculate instead the generating function

$$Z_\zeta^{\rm Sym}(\tau,\kappa) ~:=~ \big\langle\, {\rm e}^{\,\zeta\,{\rm Tr}\,f(X)}\,\big\rangle^{\rm Sym}(\tau,\kappa) ~=~ \sum_{n=0}^\infty\,\big\langle\big(\,{\rm Tr}\,f(X)\,\big)^n\big\rangle^{\rm Sym}(\tau,\kappa)~\frac{\zeta^n}{n!} ~. \tag{4.1.24}$$

Then we can get the correlation function (4.1.23) by differentiation as

$$\big\langle\,{\rm Tr}\,f(X)\,\big\rangle^{\rm Sym}(\tau,\kappa) = \left.\frac{\partial Z_\zeta^{\rm Sym}(\tau,\kappa)}{\partial\zeta}\right|_{\zeta=0} ~. \tag{4.1.25}$$

The generating function (4.1.24) is just the symmetric product partition function of the sigma model conformal field theory with a shifted action

$$I_\zeta(X) = I(X) - \zeta\,f(X) \tag{4.1.26}$$

and the normalization $Z_{\zeta=0}^{\rm Sym}(\tau,\kappa)=1$. It can thus be calculated by using the combinatorial formulae (4.1.12) and (4.1.14) as above, with the result

$$Z_\zeta^{\rm Sym}(\tau,\kappa) = \frac{1}{Z^{\rm Sym}(\tau,\kappa)}~\exp\Big(\,\sum_{N=1}^\infty\,\frac{\kappa^N}{N}~\sum_{r\,m=N}~\sum_{s\in\mathbb{Z}/r\,\mathbb{Z}}\,\mathfrak{z}_\zeta\big(\tfrac{s+m\,\tau}{r}\big)\Big) \tag{4.1.27}$$

where

$$\mathfrak{z}_\zeta(\tau) = \int\,\mathcal{D}X~~{\rm e}^{-I_\zeta(X)} \tag{4.1.28}$$

is the sigma model partition function on the torus with respect to the modified action (4.1.26). To carry out the differentiation in (4.1.25), we first calculate

$$\left.\frac{\partial\,\mathfrak{z}_\zeta(\tau)}{\partial\zeta}\right|_{\zeta=0} = \big\langle f(X)\big\rangle(\tau) \tag{4.1.29}$$

where the (unnormalized) expectation values are calculated as Gaussian moments with respect to the original action (4.1.16). Combining these results along with the elementary identity $\frac{\rm d}{{\rm d}\zeta}\,{\rm e}^{\,F(\zeta)} = F'(\zeta)~~{\rm e}^{\,F(\zeta)}$ gives finally

$$\big\langle\,{\rm Tr}\,f(X)\,\big\rangle^{\rm Sym}(\tau,\kappa) = \sum_{N=1}^\infty\,\frac{\kappa^N}{N}~\sum_{r\,m=N}~\sum_{s\in\mathbb{Z}/r\,\mathbb{Z}}\,\big\langle f(X)\big\rangle\big(\tfrac{s+m\,\tau}{r}\big) ~. \tag{4.1.30}$$

The correlation function of the symmetric operator ${\rm Tr}\,f(X)$ in the symmetric product is thus likewise expressed in terms of the correlation function of the operator $f(X)$ on all unramified covering spaces over the base torus $\mathbb{T}^2$. These formulae have natural extensions to higher loops, but in those instances they require knowledge of the correlation functions of $f(X)$ on all higher genus Riemann surfaces.

4.1.4 Twist Fields

A twist field $\sigma_P(w)$ in a generic permutation orbifold $\mathcal{C} \wr G$ is a primary field that creates the vacuum state of a twisted sector at a point $w \in \Sigma$. In a sigma model conformal field theory, its insertion results in non-trivial local monodromy

$$X^a\big((z-w)\ \mathrm{e}^{2\pi\,\mathrm{i}}\big)\,\sigma_P(w) = X^{P(a)}(z)\,\sigma_P(w) \tag{4.1.31}$$

where the permutation P is an element of the twist group $G < S_N$. Its effect is to thus make the local field X multi-valued about the insertion point $w \in \Sigma$. If $P = (n)$ consists of a single cycle of length $n > 1$, then the corresponding twist field $\sigma_{(n)}(w)$ permutes n copies of $\mathcal{C}$ in a $\mathbb{Z}_n$-twisted sector and is a primary field with conformal weight [12]

$$\Delta_{(n)} = \tfrac{d}{24}\left(n - \tfrac{1}{n}\right) \tag{4.1.32}$$

for a d-dimensional boson. The corresponding fields $X^{a_i}(z)$, $i = 1, \dots, n$ can then be glued together into one field $\mathcal{X}(z)$ which is identified with a long string of length n.

In the general case, we have seen that twisted sectors are in one-to-one correspondence with conjugacy classes of G. The conjugacy class $[P]$ of an element $P \in S_N$ can be decomposed into combinations of cyclic permutations as $[P] = \prod_n (n)^{N_n}$ with $N_n \geq 0$ and $\sum_n n\,N_n = N$. For a bosonic sigma model in d dimensions, the corresponding twist field has conformal dimension

$$\Delta_P \ = \ \sum_{n=1}^{N} N_n\,\Delta_{(n)} \ = \ \frac{d}{24}\left(N - \sum_{n=1}^{N}\frac{N_n}{n}\right) \ . \tag{4.1.33}$$

An S_N-invariant twist field creating the twisted sector $[P]$ of the permutation orbifold is defined by averaging over all twist fields in the conjugacy class of P to get

$$\sigma_{[P]}(w) = \frac{1}{N!}\sum_{g\in S_N}\sigma_{g\,P\,g^{-1}}(w) \ . \tag{4.1.34}$$

In this chapter we will be primarily interested in correlation functions $\langle\sigma_{[P_1]}(w_1)\cdots\sigma_{[P_k]}(w_k)\rangle^G$ of twist field operators in the permutation orbifold $\mathcal{C} \wr G$. These averages are difficult to calculate directly within a path integral formalism, because the twist fields are non-local operators. However, since these correlation functions are the vacuum functionals with twisted boundary conditions due to (4.1.31), it is natural to extend the covering surface principle as in [24]–[10] and compute them via a generalization of the permutation orbifold partition function (4.1.3) on a Riemann surface Σ of genus $g > 0$. Whenever we have twist fields inserted at k distinct points $\underline{w} := \{w_1, \dots, w_k\}$ of the worldsheet, a twisted sector is given by a conjugacy class of homomorphisms $\Phi : \pi_1(\Sigma_{\underline{w}}) \to G < S_N$ where $\Sigma_{\underline{w}} := \Sigma \setminus \underline{w}$ is the marked Riemann surface with the k twist field insertion points deleted. It is restricted by admissibility criteria which require that the images of the generators γ_i of $\pi_1(\Sigma_{\underline{w}})$ which are contractible to w_i must be simple cycles of length $\nu_i > 1$ if a $\mathbb{Z}_{\nu_i}$ twist field $\sigma_{(\nu_i)}(w_i)$ is inserted at w_i. Each such homomorphism Φ determines a cover of the worldsheet Σ on which a single new field $\mathcal{X}(z)$, defined by a formula analogous to (4.1.2), is single-valued. Namely, after going around a curve γ which is closed on the marked worldsheet $\Sigma_{\underline{w}}$, one sews the fields $X^{\Phi(\gamma)(a)}(z)$ into $\mathcal{X}(z)$. Thus, the contribution to the correlation function from the worldsheet instanton sector determined by the homomorphism Φ is the free partition function on the cover of Σ determined by Φ.

While the sum arising in the orbifold partition function (4.1.3) is only over unramified covers $\hat{\Sigma}$ of Σ, the twist field correlation functions involve sums over branched covers $\hat{\Sigma}_{\underline{\hat{w}}}$ where $\underline{\hat{w}} := f^{-1}(\underline{w})$ is the set of pre-images of the set $\underline{w}$ under the covering map $f : \hat{\Sigma} \to \Sigma$. The Riemann-Hurwitz formula for the genus $\hat{g}$ of the covering space with the given monodromy homomorphism is the general one for covers with ramification given by

$$\hat{g} \;=\; N\,(g-1) + 1 + \tfrac{B}{2} \qquad \text{with} \quad B \;=\; \sum_{i=1}^{k} (\nu_i - 1) \ , \tag{4.1.35}$$

where ν_i is the ramification index given by the length of the cycle of the i-th primary twist field. As before, we have to take into account those homomorphisms Φ whose image does not act transitively on the coordinate labels $a = 1, \dots, N$. In this case the simple cycle condition for fixed length ν_i has to hold for each orbit ξ. This ensures that the genus of the connected component of the cover determined by the action of $\Phi(\pi_1(\Sigma_{\underline{w}}))$ on each orbit ξ is equal to $\hat{g}$. We may now write down a formula analogous to (4.1.3) for the normalized k-point correlation function of twist field operators given by

$$\Big\langle \prod_{i=1}^{k} \sigma_{[P_i]}(w_i) \Big\rangle^{G} = \frac{1}{|G|} \sum_{\Phi:\pi_1(\Sigma_{\underline{w}})\to G} \frac{1}{Z^G(\tau)} \Big(\prod_{\xi\in\mathcal{O}(\Phi)} Z\big(\tau^{\xi,\underline{w}}\big) \Big) , \qquad (4.1.36)$$

where $\tau^{\xi,\underline{w}}$ is the complex structure of the covering surface determined by the worldsheet modulus τ, the stabilizer $\pi_1(\Sigma_{\underline{w}})_\xi$, and the branch point loci $\underline{w}$.

There are three crucial differences between the formulae (4.1.36) and (4.1.3). Firstly, the twist field correlation functions are not expressed in terms of correlation functions but instead in terms of partition functions. Secondly, there is a restriction on the admissible homomorphisms Φ to ensure that they have the prescribed monodromy around the punctures, *i.e.*, $\Phi(\gamma_i)$ has to be a simple cycle of length ν_i in each orbit. Thirdly, while the uniformization theorem provided us with a computational recipe for obtaining the Teichmüller coordinate τ^ξ in terms of τ via knowledge of Φ, it does not apply to the twist field k-point functions. The reason is that τ parametrizes the uniformizing group of the compact Riemann surface Σ, which is isomorphic to $\pi_1(\Sigma)$, while the domain of the monodromy homomorphism Φ is $\pi_1(\Sigma_{\underline{w}})$ which differs from the domain of the isomorphism from the abstract group $\pi_1(\Sigma)$ to the uniformizing group $u(\pi_1(\Sigma))$. Therefore, the complex structure of the ramified cover $\hat{\Sigma}_{\underline{\hat{w}}}$ is a function of that of the base space Σ, the locations $\underline{w}$ of the branch points, and the monodromy homomorphism Φ.

We are also interested in twist field correlation functions on symmetric products. In order to apply a version of (4.1.7) we need to pass the constraint, which is imposed on the admissible homomorphisms Φ in (4.1.36), to the definition of the function $\mathcal{Z}(H)$. Let us specialize the discussion to the torus $\Sigma = \mathbb{T}^2$ for definiteness. In this case, the genus of the covering surface $\hat{\Sigma}$ is $\hat{g}$ whenever its branching number is $B = 2(\hat{g}-1)$. A standard presentation of the fundamental group of the marked torus is given by

$$\Gamma \ := \ \pi_1(\mathbb{T}_{\underline{w}}) \ = \ <\alpha,\beta,\gamma_1,\dots,\gamma_k \ \big| \ [\alpha,\beta]\,\gamma_1\cdots\gamma_k = 1> \ . \qquad (4.1.37)$$

To each N-sheeted cover of $\mathbb{T}^2$ there corresponds a conjugacy class of subgroups of Γ of index N [83], which is the stabilizer of the monodromy homomorphism Φ acting in S_N. Note that the group (4.1.37) is isomorphic to the free group on $k+1$ generators $\alpha,\beta,\gamma_1,\dots,\gamma_{k-1}$, and any subgroup of a free group is also free. This is consistent with the fact [83] that the stabilizer subgroup is isomorphic to $\pi_1(\hat{\Sigma}_{\underline{\hat{w}}}) < \pi_1(\mathbb{T}^2_{\underline{w}})$. To decide when a given finite index subgroup $H < \Gamma$ corresponds to a stabilizer subgroup of an admissible homomorphism Φ in (4.1.36), we proceed as follows. Let $\hat{\imath} : \hat{\Sigma}_{\underline{\hat{w}}} \hookrightarrow \hat{\Sigma}$ be the natural inclusion of surfaces. The induced homomorphism $\hat{\imath}_* : \pi_1(\hat{\Sigma}_{\underline{\hat{w}}}) \to \pi_1(\hat{\Sigma})$ is then the natural forgetful map. Since $H \cong \pi_1(\hat{\Sigma}_{\underline{\hat{w}}})$, a formal criterion for the admissibility of a finite index subgroup $H < \pi_1(\mathbb{T}^2_{\underline{w}})$ is given by

$$H/\ker\big(\hat{\imath}_*\big) \cong \pi_1\big(\hat{\Sigma}\big) \ . \tag{4.1.38}$$

We can use (4.1.38) to check whether a given subgroup H is admissible. If the quotient is defined and it yields a group isomorphic to $\pi_1(\hat{\Sigma})$, then H is admissible. This property does not depend on the conjugacy class of H in $\pi_1(\mathbb{T}^2_{\underline{w}})$. We can thus give an implicit definition for the function appearing in (4.1.7) as

$$\mathcal{Z}(H) := \begin{cases} \dfrac{Z\big(\tau^{H,\underline{w}}\big)}{Z^{S_N}(\tau)}\ \kappa^{[\Gamma:H]} & \text{if } H \text{ satisfies (4.1.38)} \ , \\ 0 & \text{otherwise} \ . \end{cases} \tag{4.1.39}$$

We may then apply the formula (4.1.7) to get the generating function of twist field correlation functions.

In the following we will apply this formalism to study the perturbation of the sigma model conformal field theory, on the symmetric product of $\mathbb{R}^d$, by an irrelevant operator of conformal dimension $\frac{3}{2}$. For this, we introduce the bosonic Dijkgraaf-Verlinde-Verlinde (DVV) interaction vertex [3, 38] which is defined with respect to the $\mathbb{Z}_2$ twist field $\sigma_{ab}(w)$ corresponding to the transposition in S_N that interchanges the fields X^a and X^b while leaving all others invariant. These twist fields generate the elementary joining and splitting of strings in the symmetric product, and they can be built out of standard $\mathbb{Z}_2$ orbifold twist operators [18, 24]. Then the translationally invariant vertex operator is defined by

$$V_{\rm bos} = -\frac{\lambda\, N}{\mathrm{vol}(\mathbb{T}^2)} \int_{\mathbb{T}}^2 \mathrm{d}\mu(z) \sum_{1\le a<b\le N} \sigma_{ab}(z) \ , \tag{4.1.40}$$

where λ is a coupling constant proportional to the string coupling g_s. In contrast to the originally proposed genus zero case [3, 12, 38], we will find that the DVV vertex operator at genus one needs to be defined using a non-constant measure $\mathrm{d}\mu(z) =$

$\mathrm{d}^2z/\mu(z)$ on the torus $\mathbb{T}^2$. It will be determined explicitly in the ensuing sections (as will the coupling constant λ) by modular invariance requirements. When $d=24$, the twist field $\sigma_{ab}(w)$ is a primary field of conformal weight $\frac{3}{2}$. Starting from the one-loop action (4.1.16), the interacting symmetric product sigma model is defined by the action

$$I_{\rm int}^{S_N}(X) = \ {\rm Tr}\, I(X) + V_{\rm bos} \tag{4.1.41}$$

with ${\rm Tr}\, I(X) = \sum_a \ I(X^a)$.

In this chapter we will compute the leading order effect of this perturbation. Using translational invariance of the sigma model path integral to move one of the branch points to the origin $z=0$, we are thus interested in computing the translationally invariant correlator

$$\big\langle \,{}^\circ_\circ\, V_{\rm bos}\, V_{\rm bos}\, {}^\circ_\circ\, \big\rangle^{S_N} = \frac{\lambda^2\, N^2}{{\rm vol}(\mathbb{T}^2)\, \mu(0)}\ \sum_{a_i<b_i}\ \int_{\mathbb{T}^2} {\rm d}\mu(z)\ \big\langle\sigma_{a_1b_1}(z)\,\sigma_{a_2b_2}(0)\big\rangle^{S_N}\ . \tag{4.1.42}$$

The computation of the two-point functions in (4.1.42) specializes the above discussion to the case $g=1$, $\hat g=2$, and $k=2$. There are two simple branch points with ramification indices $\nu_1=\nu_2=2$ and $\Gamma=\pi_1(\mathbb{T}^2\setminus\{z,0\})$. Then the logarithm of the generating function (4.1.7) with the definition (4.1.39) is given by a sum over the modular invariant vacuum amplitudes on all connected N-sheeted genus two covers $\hat\Sigma$ with two fixed simple branch points. In this case the first quantized modular invariant partition function for the parent theory is the two-loop version of (4.1.20) on $\hat\Sigma$ (with vanishing Liouville field $\phi=0$ for simplicity) given by [67, 54]

$$\mathfrak{z}^{(2)}(\tau) = \frac{\big(\det({\rm Im}\,\tau)\big)^{3-d/2}}{\big(4\pi^2\,\alpha'\,\big)^{-d/2}\,\big|\Psi_{10}(\tau)\big|^2}\ , \tag{4.1.43}$$

where d is the spacetime dimension ($d=26$ for the critical bosonic string). Here $\Psi_{10}(\tau)$ is the genus two parabolic modular form of weight ten with no zeroes or singularities (the Igusa cusp form), defined on the Siegel half-space $\mathbb{U}^2=\{\tau \mid {\rm Im}(\tau_{11})> 0\ ,\ {\rm Im}(\tau_{22})>0\ ,\ \det({\rm Im}\,\tau)>0\}$ of 2×2 Riemann period matrices τ with the boundary component $\mathbb{U}\times\mathbb{U}$ consisting of diagonal matrices removed. It can be expressed in terms of the ten genus two theta-constants $\Theta\big({\boldsymbol a \atop \boldsymbol b}\big)(\tau) := \Theta\big({\boldsymbol a \atop \boldsymbol b}\big)(0,0|\tau)$ with even binary characteristics $\boldsymbol{a} = (a_1, a_2), \boldsymbol{b} = (b_1, b_2) \in \mathbb{Z}^2/2\mathbb{Z}^2$ as

$$\Psi_{10}(\tau) = 2^{-12} \prod_{\boldsymbol{a}\cdot\boldsymbol{b}\equiv 0 \bmod 2} \Theta\big(\begin{smallmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{smallmatrix}\big)(\tau)^2 \ . \tag{4.1.44}$$

4.1.5 Thermodynamics of DLCQ Strings

In the genus one case $\Sigma = \mathbb{T}^2$, the logarithm of the right-hand side of (4.1.12) coincides with the free energy of second quantized string theory on the target space $M \times \mathbb{S}^1 \times \mathbb{R}$ when the parent theory is the corresponding conformal field theory on the spacetime M in the free string limit $g_s \to 0$ [4, 5]. The matching is provided by identifying the modulus of the worldsheet and that of the spacetime torus, where the second compact direction is timelike and is generated by the trace taken in computing the free energy amplitude. Its radius is identified with the inverse temperature β. In discrete light cone quantization (DLCQ), the light cone Hamiltonian and momentum are given by

$$H = P^+ \qquad \text{and} \qquad P^- = N/R \tag{4.1.45}$$

where R is the radius of the compactified light-like direction $x^+ \in \mathbb{S}^1$ and $N \in \mathbb{N}_0$. The thermodynamic free energy $F^{(1)}_{\rm DLCQ}$ is then defined by

$$\mathrm{e}^{\,-\beta\, F^{(1)}_{\rm DLCQ}} = \mathrm{Tr}\ \mathrm{e}^{\,-\frac{\beta}{\sqrt2}\,(P^+ + P^-)} = \sum_{N=0}^\infty \mathrm{e}^{\,-\beta\, N/\sqrt2\, R}\ \mathrm{Tr}_{\,\mathcal{H}_N}\, \mathrm{e}^{\,-\beta\, P^+/\sqrt2} \ , \tag{4.1.46}$$

where $\mathcal{H}_N$ denotes the sector of the physical Hilbert space with definite total light cone momentum $P^- = N/R$. The trace over this subspace can be computed by using the mass-shell relation $P^+ = H^\perp/P^-$, where $H^\perp$ is the Hamiltonian for the transverse degrees of freedom along M.

In this way one arrives at the expression (4.1.12) with the definition (4.1.14) and $\kappa := \mathrm{e}^{\,-\beta/\sqrt2\, R}$. The Teichmüller parameter of the base torus $\mathbb{T}^2$ on which the string bits live is

$$\tau^\bullet := \frac{4\pi\,\mathrm{i}\,\alpha'}{\sqrt2\,\beta\,R} \ . \tag{4.1.47}$$

The qualitative reason for the equivalence is that the second quantized vacuum amplitude is given by the integral of the conformal field theory partition function over the moduli space of complex structures, but the only contributing surfaces at one-loop order are those which arise by winding the string around the compact directions. In other words, only the discretized moduli space of unramified covers of the torus $\mathbb{T}^2$ is summed over and taking the logarithm eliminates the disconnected covers.

When $M = \mathbb{R}^{24}$ one finds that the DLCQ partition function for bosonic string theory coincides exactly with the partition function of the symmetric product in the limit $N \to \infty$, with the length n_i of a long string identified with the light cone momentum $P_i^- = n_i/R$ for $i = 1, \dots, 24$. Checking the equivalence of perturbative bosonic string dynamics and the corresponding interacting symmetric product of $\mathbb{R}^{24}$ beyond the free string limit $g_s \to 0$ requires computing the thermal free energy in DLCQ at higher genus and the appropriate amplitudes in the permutation orbifold perturbed by the DVV interaction vertex (4.1.40). The former amplitudes truncate to sums over branched covers of the spacetime torus $\mathbb{T}^2$ arising in the null compactification at finite temperature [6], while the local structure of the operator $V_{\rm bos}$ matches nicely with the cubic string interaction vertices in light cone Green-Schwarz string field theory [14]–[15]. In this setting the string interactions are generated by sewing together torus worldsheets along branch cuts.

On the DLCQ side, the next-to-leading order contribution is the two-loop free energy which was computed in (3.2.40) with the result

$$F^{(2)}_{\rm DLCQ}\big(\tau^\bullet\,,\,\kappa\big) \;=\; -g_s^2\,\left|\frac{\tau^\bullet}{32\pi^2\,\alpha'}\right|^{12}\,\sum_{N=2}^\infty\,\frac{\kappa^N}{N^2}\,\sum_{r\,m=N}\,\Big(\frac{r}{m}\Big)^{10} \times \sum_{\substack{s,t\in\mathbb{Z}/r\,\mathbb{Z}\\ t\neq0}}\,\int_\triangle\,\frac{{\rm d}^2\tau^\#}{\big(\tau_2^\#\big)^{12}}\,\big|\Psi_{10}\big(\tau_{r,m,s,t}(\tau^\bullet,\tau^\#)\big)\big|^{-2}~. \qquad (4.1.48)$$

This thermal string amplitude is just the weighted integral over a fundamental modular domain of the genus two bosonic string partition function (4.1.43) with respect to the modular invariant integration measure on the space of 2×2 Riemann period matrices with diagonal matrices excluded, but with integration domain restricted to the partially discretized moduli space of genus two simple branched covers $\hat\Sigma$ of the torus $\mathbb{T}^2$ with modulus $\tau^\bullet$. The integers appearing in (4.1.48) can be assembled into the 2×4 matrix

$$\mathsf{M} = \begin{pmatrix} 0 & 0 & -m & 0 \\ r & 0 & -s & -t \end{pmatrix} \qquad (4.1.49)$$

which determines a homology basis for the cover in which the push-forward f_* : $H_1(\hat{\Sigma},\mathbb{Z}) \to H_1(\mathbb{T}^2,\mathbb{Z})$, induced by the holomorphic covering map $f : \hat{\Sigma} \to \mathbb{T}^2$, is given on a basis of canonical homology cycles $\hat{\alpha}_i, \hat{\beta}_i$, $i = 1,2$ for $\hat{\Sigma}$ by

$$f_*\big(\hat{\alpha}_1\,,\,\hat{\alpha}_2\,,\,\hat{\beta}_1\,,\,\hat{\beta}_2\big) = (\alpha,\beta)\ \mathsf{M} \tag{4.1.50}$$

with respect to a canonical homology basis (α,β) of the base torus. It specifies the way in which the cycles of the cover $\hat{\Sigma}$ wind around the cycles of $\mathbb{T}^2$. The period matrix $\tau \in \mathbb{U}^2 \setminus (\mathbb{U}\times\mathbb{U})$ of the cover in this basis is given by the normal form

$$\tau_{r,m,s,t}\big(\tau^\bullet\,,\,\tau^\#\big) = \begin{pmatrix} -\frac{s+m/\tau^\bullet}{r} & -\frac{t}{r} \\ -\frac{t}{r} & \tau^\# \end{pmatrix} \tag{4.1.51}$$

with $\tau^\# \in \mathbb{U}$, and the integration in (4.1.48) is taken over the standard fundamental domain $\triangle \subset \mathbb{U}$ for the action of the genus one modular group $SL(2,\mathbb{Z})$ on $\tau^\#$.

The diagonal elements of the period matrix (4.1.51) naturally capture the modulus of the degree $N = r\,m$ unramified cover of the base torus $\mathbb{T}^2$ of modulus $(\tau^\bullet)^{-1}$, along with a second torus of modulus $\tau^\#$. The key feature of the homology basis in which we have expressed the genus two amplitude (4.1.48) is that the genus two theta functions appearing in (4.1.44) admit reduction to genus one theta functions on these two tori, due to the rational-valued off-diagonal entries of (4.1.51). Hence the $\tau^\bullet$-dependence of the two-loop free energy is expressible in terms of elliptic functions, analogously to the one-loop case. Recall that the elliptic Jacobi theta function with characteristics $a,b \in \mathbb{Z}/2\mathbb{Z}$ is defined by

$$\theta\big(\tbinom{a}{b}\big)(z|\tau) = \sum_{n\in\mathbb{Z}} \exp\Big(\pi\,\mathrm{i}\,\tau\big(n+\tfrac{a}{2}\big)^2 + 2\pi\,\mathrm{i}\,\big(n+\tfrac{a}{2}\big)\,\big(z+\tfrac{b}{2}\big)\Big) \tag{4.1.52}$$

along with the Erdélyi notation

$$\begin{aligned} \theta_1(z|\tau) &= \theta\big(\tbinom{1}{1}\big)(z|\tau) \qquad \text{and} \qquad \theta_2(z|\tau) = \theta\big(\tbinom{1}{0}\big)(z|\tau)\ , \\ \theta_3(z|\tau) &= \theta\big(\tbinom{0}{0}\big)(z|\tau) \qquad \text{and} \qquad \theta_4(z|\tau) = \theta\big(\tbinom{0}{1}\big)(z|\tau)\ . \end{aligned} \tag{4.1.53}$$

Then one has the decompositions (3.2.35)

$$\Theta\big(\tbinom{\boldsymbol{a}}{\boldsymbol{b}}\big)\big(\tau_{r,m,s,t}(\tau^\bullet,\tau^\#)\big) \;=\; \frac{\mathrm{e}^{\pi\,\mathrm{i}\,a_2\,b_2/2}}{N\,\sqrt{-\,\mathrm{i}\,\tau^\#}}\ \sum_{n=0}^{N-1} (-1)^{b_2\,n}\ \theta\big(\tbinom{a_1}{b_1}\big)\Big(\big(n+\tfrac{a_2}{2}\big)\,\tfrac{m\,t}{N}\,\Big|\,\tfrac{m\,s+m^2/\tau^\bullet}{N}\Big)$$

$$\times\ \theta_j\big(\tfrac{n+a_2/2}{N}\,\big|\,-\tfrac{1}{N^2\,\tau^{\#}}\big) \qquad (4.1.54)$$

where $j = 2$ (resp. $j = 3$) when the integer $a_1\,m\,t + b_2\,N$ is odd (resp. even). For notational ease, this formula is written after performing a projective rotation $\tau_{r,m,s,t}(\tau^{\bullet},\tau^{\#}) \to -\tau_{r,m,s,t}(\tau^{\bullet},-\tau^{\#})$ along with a reflection in the modulus $\tau^{\#}$.

In this thesis we shall present a detailed comparison between the free energy (4.1.48) and the integrated (with respect to the branch point loci) two-point correlation function (4.1.42) of twist fields corresponding to transpositions, which requires the generalization of the combinatorial identity (4.1.12) to coverings with two simple branch points as explained in Section 4.1.4 above. While the auxilliary genus one surface of modulus $\tau^{\#}$ above is anticipated *a posteriori* on general grounds from the Weierstrass-Poincaré reduction theory for branched covers [**?**], its geometrical significance has been hithereto unclear. In the following we will identify this torus explicitly, which among other things will provide the transformation from the branch point loci to the modulus $\tau^{\#}$ required to match the expressions (4.1.42) and (4.1.48), as well as the measure $\mathrm{d}\mu(z)$ and coupling constant λ required to define the DVV vertex operator (4.1.40) on an elliptic curve.

4.2 $\mathbb{Z}_2$ Orbifolds

The purpose of this section is to establish the equivalence of the two-point function for the DVV vertex operator in the symmetric product $\mathbb{R}^{24} \wr \mathbb{Z}_2$ with the $N = 2$ contribution to the genus two free energy (4.1.48) of the bosonic DLCQ string. For the former calculation we will exploit the known formulae [27] for the multi-loop partition functions and twist field correlation functions on the geometric orbifold $\mathbb{S}^1/\mathbb{Z}_2$. For the latter computation we connect the form of the total reduced free energy (4.1.48) to the theory of Prym varieties for generic genus two covers of the torus $\mathbb{T}^2$ of modulus $\tau^{\bullet}$. By a theorem due to Mumford [84], the only coverings that generate Prym varieties are double covers with at most two branch points, and our case of genus two covers over an elliptic curve. Our proof puts the covering surface principle sketched in Section 4.1.4 on more solid ground, and provides a non-trivial explicit check for the computation of twist field correlation functions through two rather distinct methods.

4.2.1 Target Space vs. Permutation Orbifold

For later use, we begin by elucidating the correspondence between the sigma model conformal field theories on the geometric orbifold $\mathbb{R}^{24}/\mathbb{Z}_2$ and on the permutation orbifold $\mathbb{R}^{24}\wr\mathbb{Z}_2$. For this, let us consider the $\mathbb{S}^1/\mathbb{Z}_2$ target space orbifold of a free boson X compactified on a circle $\mathbb{S}^1$ of radius R, where the group action is the reflection involution $X \mapsto -X$. On the other hand, the permutation orbifold $\mathbb{S}^1 \wr \mathbb{Z}_2$ is defined on the tensor product of the $\mathbb{S}^1$ conformal field theory with itself. Labelling the two copies of the boson X by X^a, $a = 1, 2$, the group action of the permutation orbifold is given by $X^1 \mapsto X^2$, $X^2 \mapsto X^1$. This can be compared to the geometric orbifold group action by introducing new coordinate fields $X^\pm = X^1 \pm X^2$, so that the $\mathbb{Z}_2$ permutation group now acts as $X^\pm \mapsto \pm X^\pm$. It follows that the permutation orbifold is equivalent to the target space orbifold plus an independent free boson X^+ on $\mathbb{S}^1$. The partition functions of the two theories are thus related by

$$Z^{\mathbb{Z}_2}(\tau, R) = \mathfrak{z}(\tau, R)\ Z_{\rm orb}(\tau, R)\ , \tag{4.2.1}$$

where $\mathfrak{z}(\tau, R)$ denotes the partition function of the compactified scalar field X^+ and $Z_{\rm orb}(\tau, R)$ that of the $\mathbb{S}^1/\mathbb{Z}_2$ theory.

It is instructive to check the identity (4.2.1) explicitly at one-loop order in the decompactified circle theory. The amplitude for the boson X^+ on $\mathbb{S}^1$ is given by the worldsheet instanton sum

$$\mathfrak{z}(\tau, R) \ = \ \mathfrak{z}(\tau)\ \mathfrak{z}^{\rm cl}(\tau, R) \ := \ \frac{\sqrt{4\pi^2\,\alpha'}}{\sqrt{\tau_2}\,\big|\eta(\tau)\big|^2}\ \sum_{m,m'\in\mathbb{Z}} \frac{R}{\sqrt{\alpha'}}\ \exp\Big(-\frac{\pi\,R^2\,\big|m\,\tau - m'\,\big|^2}{\alpha'\,\tau_2}\Big)\ , \tag{4.2.2}$$

where $\mathfrak{z}(\tau)$ is the modular invariant amplitude (4.1.22) for the free boson on the real line (so that $\mathfrak{z}^{\rm cl}(\tau, R = \infty) = 1$) and henceforth we set the Liouville field $\phi = 0$. The sum in (4.2.2) runs over classical solutions with the given winding numbers around the generating cycles of a canonical homology basis. For the partition function of the target space orbifold, we note that the oscillator part $\mathfrak{z}(\tau)$ of the partition function (4.2.2) is independent of the radius R. A monodromy homomorphism Φ for an unramified double cover of a genus one surface is characterized by a binary pair $(\varepsilon, \delta) \in (\mathbb{Z}/2\mathbb{Z})^2$, where 0 (resp. 1) labels periodic (resp. antiperiodic) global monodromy around the canonical homology cycles (α, β) of the base. In the twisted sectors, the $\mathbb{Z}_2$ action $X \mapsto -X$ kills non-trivial instantons at one-loop (as a consequence of the Riemann-Roch theorem), while the quantum parts may be computed by equating the $\mathbb{Z}_2$-twisted partition function at $R = \sqrt{\alpha'}$ with that of the untwisted $\mathbb{S}^1$ theory at the self-dual radius $R = 1/\sqrt{\alpha'}$ which coincides with the multi-critical Ashkin-Teller model. The result is [27]

$$Z_{\rm orb}(\tau, R) = \tfrac{1}{2}\, \mathfrak{z}(\tau, R) + \left|\frac{\eta(\tau)}{\theta_2(\tau)}\right| + \left|\frac{\eta(\tau)}{\theta_3(\tau)}\right| + \left|\frac{\eta(\tau)}{\theta_4(\tau)}\right| \ , \tag{4.2.3}$$

where we have denoted the Jacobi-Erdélyi theta constants by $\theta_i(\tau) := \theta_i(0|\tau)$. Finally, the vacuum amplitude of the $\mathbb{Z}_2$ permutation orbifold can be determined from the formula (4.1.3) as

$$Z^{\mathbb{Z}_2}(\tau, R) = \tfrac{1}{2}\Big(\mathfrak{z}(\tau, R)^2 + \mathfrak{z}(2\tau, R) + \mathfrak{z}\big(\tfrac{\tau}{2}\,,\, R\big) + \mathfrak{z}\big(\tfrac{\tau+1}{2}\,,\, R\big)\Big) \ . \tag{4.2.4}$$

Clearly the contributions to both sides of the formula (4.2.1) from the untwisted sector match. For the contributions from the twisted sectors, we use the identities $\theta_3(\tau + 1) = \theta_4(\tau)$ and

$$\theta_2(\tau)\, \theta_3(\tau)\, \theta_4(\tau) = 2\eta(\tau)^3 \tag{4.2.5}$$

to derive the elliptic function relation

$$\begin{aligned} &\frac{1}{\big|\theta_2(\tau)\,\eta(\tau)\big|} + \frac{1}{\big|\theta_3(\tau)\,\eta(\tau)\big|} + \frac{1}{\big|\theta_4(\tau)\,\eta(\tau)\big|} \\ &= \left|\frac{\theta_3(\tau)\,\theta_3(\tau+1)}{2\eta(\tau)^4}\right| + \frac{1}{\big|\theta_3(\tau)\,\eta(\tau)\big|} + \frac{1}{\big|\theta_3(\tau+1)\,\eta(\tau)\big|} \\ &= \frac{1}{2\big|\eta(2\tau)\big|^2} + \frac{1}{\big|\eta(\frac{\tau}{2})\big|^2} + \frac{1}{\big|\eta(\frac{\tau+1}{2})\big|^2} \ , \end{aligned} \tag{4.2.6}$$

where in the last line we substituted the identity $\theta_3(\tau) = \eta(\frac{\tau+1}{2})^2/\eta(\tau + 1)$ and used $|\eta(\tau + 1)| = |\eta(\tau)|$. This equation establishes the $R \to \infty$ limit of the formula (4.2.1), for each twisted sector, which easily generalizes to $\mathbb{Z}_2$ orbifolds of $\mathbb{R}^d$ by taking appropriate powers.

4.2.2 DLCQ Strings on Double Covers

We now turn to the explicit form of the $N = 2$ part of the genus two bosonic DLCQ free energy (4.1.48) which is given explicitly by

$$\mathcal{F}_2(\tau^\bullet) = -\frac{g_s^2}{16}\left|\frac{\tau^\bullet}{16\pi^2\,\alpha'}\right|^{12}\sum_{s=0,1}\int_\triangle\frac{\mathrm{d}^2\tau^\#}{\left(\tau_2^\#\right)^{12}}\left|\Psi_{10}\left(\tau_s(\tau^\bullet,\tau^\#)\right)\right|^{-2}, \tag{4.2.7}$$

where the corresponding period matrices read

$$\tau_s\left(\tau^\bullet\,,\,\tau^\#\right) \;:=\; \tau_{r=2,m=1,s,t=1}\left(\tau^\bullet\,,\,\tau^\#\right) \;=\; \begin{pmatrix} -\frac{1}{2\tau^\bullet}-\frac{s}{2} & -\frac12 \\ -\frac12 & \tau^\# \end{pmatrix}\,. \tag{4.2.8}$$

By modular invariance it suffices to restrict to the $s = 0$ contribution. To see this, we define the $SL(2,\mathbb{Z})$ modular transformation $\tilde\tau^\# = \tau^\#/(2\tau^\# + 1)$. Then the period matrices $\tau_1(\tau^\bullet,\tau^\#)$ and $\tau_0(\tau^\bullet,\tilde\tau^\#)$ are related by the $Sp(4,\mathbb{Z})$ modular transformation

$$\tau_0\left(\tau^\bullet\,,\,\tilde\tau^\#\right) = \left(A\,\tau_1(\tau^\bullet,\tau^\#)+B\right)\left(C\,\tau_1(\tau^\bullet,\tau^\#)+D\right)^{-1} \tag{4.2.9}$$

given by the matrix

$$g \;=\; \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{pmatrix} \;=:\; \begin{pmatrix} A & B \\ C & D \end{pmatrix}\,. \tag{4.2.10}$$

Since the integration over $\tau^\#$ in (4.2.7) runs over a fundamental domain $\triangle$ for $SL(2,\mathbb{Z})$, we can compensate the omission of the $s = 1$ term by simply doubling the $s = 0$ contribution.

Let us now simplify the integrand of (4.2.7) by working out explicitly the product of theta constants appearing in the genus two modular form (4.1.44). Starting from the reduction (4.1.54) with $N = 2$, one has $j = 2$ when $a_1 = 1$ and $j = 3$ when $a_1 = 0$, and hence

$$\begin{aligned}\Theta\left({\boldsymbol{a}\atop\boldsymbol{b}}\right)\left(\tau_0(\tau^\bullet,\tau^\#)\right) \;&=\; \frac{\mathrm{e}^{\pi\,\mathrm{i}\,a_2\,b_2/2}}{2\sqrt{-\,\mathrm{i}\,\tau^\#}}\Big(\theta\left({a_1\atop b_1}\right)\left(\tfrac{a_2}{4}\,\big|\,\tfrac{1}{2\tau^\bullet}\right)\theta\left({a_1\atop 0}\right)\left(\tfrac{a_2}{4}\,\big|\,-\tfrac{1}{4\tau^\#}\right) \\ &\quad+(-1)^{b_2}\,\theta\left({a_1\atop b_1}\right)\left(\tfrac{a_2}{4}+\tfrac12\,\big|\,\tfrac{1}{2\tau^\bullet}\right)\theta\left({a_1\atop 0}\right)\left(\tfrac{a_2}{4}+\tfrac12\,\big|\,-\tfrac{1}{4\tau^\#}\right)\Big)\,.\end{aligned} \tag{4.2.11}$$

Using the property

$$\theta\binom{a}{b}\big(z+\tfrac{1}{2}\,\big|\,\tau\big) = (-1)^{a\,b}\;\theta\binom{a}{b+1}(z|\tau) \tag{4.2.12}$$

where $b+1$ is understood modulo 2, one can now write down the product of the even genus two theta constants in (4.1.44). To simplify the formulae somewhat, in the ensuing calculations we will use the shorthand notations $\theta_i^\bullet := \theta_i(0|\frac{1}{2\tau^\bullet})$, $\tilde{\theta}_i^\bullet := \theta_i(\frac{1}{4}|\frac{1}{2\tau^\bullet})$, $\theta_i^\# := \theta_i(0|-\frac{1}{4\tau^\#})$ and $\tilde{\theta}_i^\# := \theta_i(\frac{1}{4}|-\frac{1}{4\tau^\#})$.

Then the modular form (4.1.44) can be expressed as

$$\Psi_{10}\big(\tau_0(\tau^\bullet,\tau^\#)\big) = \frac{\mathcal{A}^2\;\mathcal{B}^2}{2^{32}\,\big(\tau^\#\big)^{10}} \tag{4.2.13}$$

where

$$\begin{aligned}\mathcal{A} &= \big(\theta_3^\bullet\,\theta_3^\# + \theta_4^\bullet\,\theta_4^\#\big)\,\big(\theta_2^\bullet\,\theta_2^\# + \theta_1^\bullet\,\theta_1^\#\big)\,\big(\theta_4^\bullet\,\theta_3^\# + \theta_3^\bullet\,\theta_4^\#\big) \\ &\quad\times\,\big(\theta_3^\bullet\,\theta_3^\# - \theta_4^\bullet\,\theta_4^\#\big)\,\big(\theta_4^\bullet\,\theta_3^\# - \theta_3^\bullet\,\theta_4^\#\big)\,\big(\theta_2^\bullet\,\theta_2^\# - \theta_1^\bullet\,\theta_1^\#\big)\;, \end{aligned} \tag{4.2.14}$$

$$\mathcal{B} = \big(\tilde\theta_3^\bullet\,\tilde\theta_3^\# + \tilde\theta_4^\bullet\,\tilde\theta_4^\#\big)\,\big(\tilde\theta_2^\bullet\,\tilde\theta_2^\# + \tilde\theta_1^\bullet\,\tilde\theta_1^\#\big)\,\big(\tilde\theta_4^\bullet\,\tilde\theta_3^\# + \tilde\theta_3^\bullet\,\tilde\theta_4^\#\big)\,\big(\tilde\theta_1^\bullet\,\tilde\theta_2^\# + \tilde\theta_2^\bullet\,\tilde\theta_1^\#\big)\;. \tag{4.2.15}$$

The products (4.2.14) can be immediately simplified by noticing that $\theta_1^\bullet = \theta_1(0|\frac{1}{2\tau^\bullet}) = 0$ (and similarly $\theta_1^\# = 0$). One finds

$$\mathcal{A} = \theta_2^{\bullet\,2}\,\theta_2^{\#\,2}\,\big(\theta_3^{\bullet\,2}\,\theta_4^{\bullet\,2}\,(\theta_3^{\#\,4}+\theta_4^{\#\,4}) - \theta_3^{\#\,2}\,\theta_4^{\#\,2}\,(\theta_3^{\bullet\,4}+\theta_4^{\bullet\,4})\big)\;, \tag{4.2.16}$$

$$\begin{aligned}\mathcal{B} &= \tilde\theta_1^\bullet\,\tilde\theta_2^\bullet\,\tilde\theta_3^\bullet\,\tilde\theta_4^\bullet\,\big(\tilde\theta_1^{\#\,2}+\tilde\theta_2^{\#\,2}\big)\,\big(\tilde\theta_3^{\#\,2}+\tilde\theta_4^{\#\,2}\big) \\ &\quad+\tilde\theta_1^\#\,\tilde\theta_2^\#\,\tilde\theta_3^\#\,\tilde\theta_4^\#\,\big(\tilde\theta_1^{\bullet\,2}+\tilde\theta_2^{\bullet\,2}\big)\,\big(\tilde\theta_3^{\bullet\,2}+\tilde\theta_4^{\bullet\,2}\big) \\ &\quad+\tilde\theta_1^\bullet\,\tilde\theta_2^\bullet\,\tilde\theta_3^\#\,\tilde\theta_4^\#\,\big(\tilde\theta_1^{\#\,2}+\tilde\theta_2^{\#\,2}\big)\,\big(\tilde\theta_3^{\bullet\,2}+\tilde\theta_4^{\bullet\,2}\big) + \tilde\theta_1^\#\,\tilde\theta_2^\#\,\tilde\theta_3^\bullet\,\tilde\theta_4^\bullet\,\big(\tilde\theta_1^{\bullet\,2}+\tilde\theta_2^{\bullet\,2}\big)\,\big(\tilde\theta_3^{\#\,2}+\tilde\theta_4^{\#\,2}\big)\;. \end{aligned} \tag{4.2.17}$$

Using (4.2.12) and the parity properties of the theta functions, one notices that $\tilde\theta_1^\bullet = -\tilde\theta_2^\bullet$ and $\tilde\theta_3^\bullet = \tilde\theta_4^\bullet$. We may thus simplify (4.2.17) further to

$$\mathcal{B} \;=\; -16\,\tilde\theta_1^\bullet\,\tilde\theta_2^\bullet\,\tilde\theta_3^\bullet\,\tilde\theta_4^\bullet\,\tilde\theta_1^\#\,\tilde\theta_2^\#\,\tilde\theta_3^\#\,\tilde\theta_4^\# \;=\; -4\,\theta_2^{\bullet\,2}\,\theta_3^\bullet\,\theta_4^\bullet\,\theta_2^{\#\,2}\,\theta_3^\#\,\theta_4^\# \tag{4.2.18}$$

where the second equality is a consequence of the identity for products of theta functions with identical modulus given by

$$\theta_1(2z|\tau)\,\theta_2(0|\tau)\,\theta_3(0|\tau)\,\theta_4(0|\tau) = 2\,\theta_1(z|\tau)\,\theta_2(z|\tau)\,\theta_3(z|\tau)\,\theta_4(z|\tau)\;, \tag{4.2.19}$$

applied with $z=\frac{1}{4}$.

The next step consists in using the modulus doubling identities

$$\begin{aligned}\theta_2(0|\tau)^2 &= 2\,\theta_2(0|2\tau)\,\theta_3(0|2\tau)\ ,\\ \theta_3(0|\tau)\,\theta_4(0|\tau) &= \theta_4(0|2\tau)^2\ ,\\ \theta_3(0|\tau)^2+\theta_4(0|\tau)^2 &= 2\,\theta_3(0|2\tau)^2\end{aligned} \tag{4.2.20}$$

along with the Jacobi abstruse identity

$$\theta_3(0|\tau)^4-\theta_4(0|\tau)^4=\theta_2(0|\tau)^4 \tag{4.2.21}$$

on both $\theta_i^\bullet$ and $\theta_i^\#$. After introducing the notations $\bar\theta_i^\bullet := \theta_i(0|\frac{1}{\tau^\bullet})$ and $\bar\theta_i^\# := \theta_i(0|-\frac{1}{2\tau^\#})$ we find

$$\mathcal{A}\ \mathcal{B}=-128\,\bar\theta_2^{\bullet\,2}\,\bar\theta_3^{\bullet\,2}\,\bar\theta_4^{\bullet\,2}\,\bar\theta_2^{\#\,2}\,\bar\theta_3^{\#\,2}\,\bar\theta_4^{\#\,2}\left(\bar\theta_4^{\bullet\,4}\,(\bar\theta_2^{\#\,4}+\bar\theta_3^{\#\,4})-\bar\theta_4^{\#\,4}\,(\bar\theta_2^{\bullet\,4}+\bar\theta_3^{\bullet\,4})\right) \tag{4.2.22}$$

We now undo the projective rotation $\tau_0\to-\tau_0$ and the reflection $\tau^\#\to-\tau^\#$ that were used to write (4.1.54), in order to use theta functions which are convergent on the standard domain of genus one moduli $\tau_2>0$. This affects only $\bar\theta_i^\bullet$, because its modulus changes as $\theta_i(0|\frac{1}{\tau^\bullet})\to\theta_i(0|-\frac{1}{\tau^\bullet})$. The reflection of the off-diagonal elements of the period matrix (4.1.51) which flips the sign of the argument of θ_i via (4.1.54) is easily checked to have no effect on the product (4.2.22).

The final transformation we perform on the product (4.2.22) is a modular S transformation on both $\bar\theta_i^\bullet$ and $\bar\theta_i^\#$ given by

$$\begin{aligned}\theta_2\big(0\,\big|-\tfrac{1}{\tau}\big) &= \sqrt{-\mathrm{i}\,\tau}\ \theta_4(0|\tau)\ ,\\ \theta_3\big(0\,\big|-\tfrac{1}{\tau}\big) &= \sqrt{-\mathrm{i}\,\tau}\ \theta_3(0|\tau)\ ,\\ \theta_4\big(0\,\big|-\tfrac{1}{\tau}\big) &= \sqrt{-\mathrm{i}\,\tau}\ \theta_2(0|\tau)\ .\end{aligned} \tag{4.2.23}$$

Then we can write the modular form (4.2.13) as

$$\Psi_{10}\qquad\big(\tau_0(\tau^\bullet,\tau^\#)\big)=\big(\tau^\bullet\big)^{10}\,\eta\big(\tau^\bullet\big)^{12}\,\eta\big(2\tau^\#\big)^{12} \tag{4.2.24}$$

$$\times \Big(\theta_2(2\tau^\#)^4 \left(\theta_4(\tau^\bullet)^4 + \theta_3(\tau^\bullet)^4\right) - \theta_2(\tau^\bullet)^4 \left(\theta_4(2\tau^\#)^4 + \theta_3(2\tau^\#)^4\right)\Big)^2$$

where we have used (4.2.5). Substituting into (4.2.7) and using (4.2.21) we arrive at our final form for the two-loop DLCQ free energy given by

$$\begin{aligned} \mathcal{F}_2\big(\tau^\bullet\big) &= -\frac{g_s^2}{8\left(16\pi^2\,\alpha'\right)^{12}}\,\frac{\big|\eta(\tau^\bullet)\big|^{-24}}{\big|\tau^\bullet\big|^8} \\ &\quad \times \int_\triangle \frac{\mathrm{d}^2\tau^\#}{\big(\tau_2^\#\big)^{12}}\,\left|\frac{\eta\big(2\tau^\#\big)^{-6}}{\theta_3\big(\tau^\bullet\big)^4\,\theta_4\big(2\tau^\#\big)^4 - \theta_4\big(\tau^\bullet\big)^4\,\theta_3\big(2\tau^\#\big)^4}\right|^4 . \end{aligned} \tag{4.2.25}$$

4.2.3 Prym Varieties

Our next goal is to determine the genus one modulus $\tau^\#$ explicitly in terms of the branch point loci on the base torus $\mathbb{T}^2$. This modulus arose generically from the algebraic Weierstrass-Poincaré reduction of the period matrix τ of the covering surface $\hat{\Sigma}$ to the normal form (4.1.51), which is a consequence of the fact that the genus two Riemann period matrix in this instance satisfies a Hopf condition (3.1.19). We will now elucidate the geometrical significance of this modulus for a generic genus two cover over $\mathbb{T}^2$ of degree $N = r\,m$, and then show how in the case of double covers this geometrical realization determines it explicitly as a function of branch points on the worldsheet $\mathbb{T}^2$.

Let $f : \hat{\Sigma} \to \mathbb{T}^2$ be a holomorphic map. Let ω_i, $i = 1,2$ be the canonical, normalized abelian holomorphic differentials on $\hat{\Sigma}$ with the periods

$$\oint_{\hat{\alpha}_i} \omega_j = \delta_{ij} \qquad \text{and} \qquad \oint_{\hat{\beta}_i} \omega_j = \tau_{ij} \ . \tag{4.2.26}$$

On the base elliptic curve $\mathbb{T}^2$ the holomorphic one-form is $\mathrm{d}z$ with the periods $\oint_\alpha \mathrm{d}z = 1$ and $\oint_\beta \mathrm{d}z = \tau^\bullet$. The two sets of differentials are related by the pull-back homomorphism $f^* : H^{1,0}(\mathbb{T}^2,\mathbb{C}) \to H^{1,0}(\hat{\Sigma},\mathbb{C})$ through

$$f^*(\mathrm{d}z) = h_1\,\omega_1 + h_2\,\omega_2 \tag{4.2.27}$$

for some complex numbers h_i. These numbers can be determined by integrating the relation (4.2.27) over a canonical homology basis of $H_1(\hat{\Sigma},\mathbb{Z})$ using (4.1.50), and with respect to the basis specified by (4.1.49) they are given by

$$h_1 = r\,\tau^\bullet \qquad \text{and} \qquad h_2 = 0 \ . \tag{4.2.28}$$

Let $\mathrm{Jac}(\hat{\Sigma}) := H^{1,0}(\hat{\Sigma},\mathbb{C})/H^{1,0}(\hat{\Sigma},\Lambda_\tau)$ be the principally polarized Jacobian variety of $\hat{\Sigma}$, where $\Lambda_\tau = \mathbb{Z}^2 \oplus \tau\,\mathbb{Z}^2$ is the lattice of rank four induced by the period matrix τ of $\hat{\Sigma}$. It can be identified with the Picard group $\mathrm{Pic}^0(\hat{\Sigma})$ of isomorphism classes of flat line bundles over $\hat{\Sigma}$, in correspondence with degree zero divisors, and it is isomorphic to the complex two-dimensional torus $\mathbb{C}^2/\Lambda_\tau$. There is an embedding of $\hat{\Sigma}$ into $\mathrm{Jac}(\hat{\Sigma})$ provided by the Abel map $\mathfrak{A} : \hat{z} \mapsto \int^{\hat{z}} (\omega_1, \omega_2)$, which also provides the mapping from divisors to the Jacobian variety. The theta divisor is the analytic subvariety of the Jacobian defined by the equation $\Theta\big(\begin{smallmatrix}\mathbf{0}\\ \mathbf{0}\end{smallmatrix}\big)(z_1, z_2|\tau) = 0$. On the base, the Jacobian torus can instead be identified with the elliptic curve $\mathbb{T}^2$ itself and one has $\mathrm{Jac}(\mathbb{T}^2) \cong \mathbb{T}^2$.

It follows from a general property of finite morphisms between smooth projective curves [84] that the holomorphic map $f : \hat{\Sigma} \to \mathbb{T}^2$ can be factorized by means of a commutative triangle

$$\begin{array}{ccc} \hat{\Sigma} & \overset{g}{\longrightarrow} & \Sigma_1 \\ & {\scriptstyle f}\searrow & \downarrow {\scriptstyle f_1} \\ & & \mathbb{T}^2 \end{array} \tag{4.2.29}$$

where $f_1 : \Sigma_1 \to \mathbb{T}^2$ is an unramified cover. The induced pullback morphisms on the Jacobian tori have the properties that $\ker(f^*) \cong \ker(f_1^*)$ and $g^* : \Sigma_1 \to \mathrm{Jac}(\hat{\Sigma})$ is injective. This accounts for the first diagonal entry in the period matrix (4.1.51). The complimentary subvariety to $\mathrm{im}(f^*) \cong \mathbb{T}^2$ in the Jacobian torus $\mathbb{C}^2/\Lambda_\tau$ is gotten from the norm morphism

$$\Omega_f \,:\, \mathrm{Jac}\big(\hat{\Sigma}\big) \longrightarrow \mathbb{T}^2 \qquad \text{with} \quad \Omega_f(z_1, z_2) \;:=\; h_1\, z_1 + h_2\, z_2 \tag{4.2.30}$$

which takes the divisor class D of degree zero by applying f to each point of the divisor. The kernel of this morphism is a principally polarized subvariety of $\mathrm{Jac}(\hat{\Sigma})$ called the Prym variety of the cover and in the present case it is a complex one-dimensional torus $\mathbb{C}/(\mathbb{Z} \oplus \Pi\,\mathbb{Z})$ whose period Π is called the Prym modulus. In the basis defined by (4.1.49), from (4.2.28) it follows that the kernel of (4.2.30) in $\mathbb{C}^2$ consists of all points of the form $(z_1, z_2) = (\frac{m}{r}, z)$ with $m \in \mathbb{Z}$ and $z \in \mathbb{C}$. Passing to the quotient $\mathbb{C}^2/\Lambda_\tau$ using (4.1.51) truncates to points $(0, z)$ with the identifications

$z \sim z + \frac{m_1}{r} + \tau^{\#} \, m_2$ for any $m_1, m_2 \in \mathbb{Z}$. It follows that the Prym modulus in this basis is given by

$$\Pi = r \; \tau^{\#} \tag{4.2.31}$$

and we have explicitly identified the second elliptic modulus in (4.1.51). Using the factorization (4.2.29) one shows [84] that the induced theta divisor on $\ker(\Omega_f)$ is r times the theta divisor defining its principal polarization, and hence that $\ker(\Omega_f)$ is a Prym-Tyurin variety.

So far everything we have said holds generally for any N-sheeted genus two cover of the torus $\mathbb{T}^2$. When $N = 2$, wherein only the $r = 2$ term contributes in (4.1.48), the Prym variety possesses a special characterization [85] which enables one to make this construction much more explicit. Consider the element of the symplectic group $Sp(4, \mathbb{Z})$ given by

$$g \;=\; \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \;=:\; \begin{pmatrix} A & B \\ C & D \end{pmatrix} \; . \tag{4.2.32}$$

It induces the change in basis of $H_1(\hat{\Sigma}, \mathbb{Z})$ represented by

$$\mathsf{M} \;=\; \mathsf{M}' \begin{pmatrix} D^\top & B^\top \\ C^\top & A^\top \end{pmatrix} \qquad \text{with} \quad \mathsf{M}' \;=\; \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \; , \tag{4.2.33}$$

and the genus two modular transformation

$$\tau_0\big(\tau^\bullet \, , \, \tau^{\#}\big) = \big(A \, \tau_0'(\tau^\bullet, \tau^{\#}) + B\big) \, \big(C \, \tau_0'(\tau^\bullet, \tau^{\#}) + D\big)^{-1} \tag{4.2.34}$$

with

$$\tau_0'\big(\tau^\bullet \, , \, \tau^{\#}\big) = \frac{1}{2} \begin{pmatrix} \Pi + \tau^\bullet & \Pi - \tau^\bullet \\ \Pi - \tau^\bullet & \Pi + \tau^\bullet \end{pmatrix} \tag{4.2.35}$$

where we have used (4.2.31) with $r = 2$. From (4.1.50) it follows that

$$f_*\big(\hat{\alpha}_1\big) \;=\; -f_*\big(\hat{\alpha}_2\big) \;=\; \alpha \qquad \text{and} \qquad f_*\big(\hat{\beta}_1\big) \;=\; -f_*\big(\hat{\beta}_2\big) \;=\; \beta \; . \tag{4.2.36}$$

Integrating both sides of (4.2.27) in this basis thus gives $h'_1 = -h'_2 = 1$, and hence

$$f^*(\mathrm{d}z) = \omega_1 - \omega_2 \ . \tag{4.2.37}$$

What makes the instance of a double cover $f : \hat{\Sigma} \to \mathbb{T}^2$ special is that it has a canonical conformal automorphism $\iota : \hat{\Sigma} \to \hat{\Sigma}$, satisfying $f \circ \iota = f$, which is the involution permuting the sheets of the cover. It uniquely determines the covering with $\mathbb{T}^2 = \hat{\Sigma}/\iota$. From (4.2.36) it follows that

$$\iota\big(\hat{\alpha}_1\big) \ = \ -\iota\big(\hat{\alpha}_2\big) \qquad \text{and} \qquad \iota\big(\hat{\beta}_1\big) \ = \ -\iota\big(\hat{\beta}_2\big) \ , \tag{4.2.38}$$

and hence that

$$\iota^*(\omega_1) = -\omega_2 \ . \tag{4.2.39}$$

The holomorphic one-form

$$\nu = \omega_1 + \omega_2 \tag{4.2.40}$$

is called the Prym differential and it is the unique holomorphic differential on the two-sheeted cover $\hat{\Sigma}$ which is odd under the defining involution with $\iota^*(\nu) = -\nu$. It follows from (4.2.37)–(4.2.40) and the form (4.2.35) of the period matrix in this basis that the Prym period is determined by

$$\Pi = \oint_{\hat{\beta}_1} \nu \ . \tag{4.2.41}$$

The Prym differential ν is normalized with respect to the $\hat{\alpha}_1$ cycle, while it has vanishing periods around $\hat{\alpha}_1 - \hat{\alpha}_2$ and $\hat{\beta}_1 - \hat{\beta}_2$. At the level of Jacobian varieties, the Prym variety $\ker(\Omega_f)$ is isomorphic to the subvariety of $\mathrm{Jac}(\hat{\Sigma})$ consisting of degree zero divisor classes which are odd under the involution ι. Note that from (4.2.37) it follows that the embedding $f^* : \mathbb{T}^2 \hookrightarrow \mathrm{Jac}(\hat{\Sigma})$ is isomorphic to the subvariety invariant under ι.

Similarly to the even holomorphic one-form (4.2.37), the Prym differential (4.2.40) may be given explicitly as the pull-back $\nu = f^*(\mathrm{pr}(w_1, w_2))$ of a multiplicative differential $\mathrm{pr}(w_1, w_2) = \mathrm{pr}(z; w_1, w_2)\ \mathrm{d}z$ on the base elliptic curve $\mathbb{T}^2$ with modulus $\tau^\bullet$. It is required to have a square root cut singularity about each of the branch points

$w_1, w_2 \in \mathbb{T}^2$ of the cover and to have global periodicity under $z \to z + m + n\,\tau^\bullet$ for any $m, n \in \mathbb{Z}$. This uniquely determines the multiplicative differential on $\mathbb{T}^2$ in terms of Jacobi-Erdélyi elliptic functions as

$$\mathrm{pr}(z; w_1, w_2) = \frac{\theta_1\big(z - \frac{w_1+w_2}{2} \,\big|\, \tau^\bullet\big)}{\sqrt{\theta_1\big(z - w_1 \,\big|\, \tau^\bullet\big)\, \theta_1\big(z - w_2 \,\big|\, \tau^\bullet\big)}} \ . \tag{4.2.42}$$

The Prym modulus (4.2.41) may then be written as

$$\tau^\# \ = \ \tfrac{1}{2}\,\Pi \ = \ \frac{1}{2}\ \frac{\displaystyle\oint_\beta \mathrm{pr}(w_1, w_2)}{\displaystyle\oint_\alpha \mathrm{pr}(w_1, w_2)} \ , \tag{4.2.43}$$

thereby determining the desired explicit dependence of the elliptic modulus $\tau^\#$ on the branch point loci. As expected, $\Pi \to \tau^\bullet$ in the unramified limit $w_1 \to w_2$ wherein the branch cut on $\mathbb{T}^2$ closes up. It follows from (4.2.35) that this limit corresponds to approaching a separating boundary component of moduli space, wherein the genus two Riemann surface $\hat{\Sigma}$ degenerates into two copies of the base torus $\mathbb{T}^2$.

Thus far we have not accounted for global monodromy Φ of the covering map $f : \hat{\Sigma} \to \mathbb{T}^2$, *i.e.*, the above formulas are written in the untwisted sector $(\varepsilon, \delta) = (0, 0)$. For each twisted sector $(\varepsilon, \delta) \in (\mathbb{Z}/2\mathbb{Z})^2$ there is a holomorphic Prym form $\nu_{\varepsilon,\delta}$ which is odd under the involution ι and which has non-vanishing periods only around the $(\hat{\alpha}_1, \hat{\beta}_1)$ cycles of the homology group $H_1(\hat{\Sigma}, \mathbb{Z})$. They project onto multiplicative differentials $\mathrm{pr}_{\varepsilon,\delta}(w_1, w_2)$ on $\mathbb{T}^2$ which have square root cut singularities about the branch points $w_1, w_2 \in \mathbb{T}^2$. The Prym form corresponding to the characteristic (ε, δ) can be gotten from the untwisted one via a crossing transformation of the branch points

$$w_1 \ \longrightarrow \ w_1 + \delta + \varepsilon\,\tau^\bullet \qquad \text{and} \qquad w_2 \ \longrightarrow \ w_2 \tag{4.2.44}$$

to get

$$\mathrm{pr}_{\varepsilon,\delta}(z; w_1, w_2) = \mathrm{pr}\big(z\,;\, w_1 + \delta + \varepsilon\,\tau^\bullet, w_2\big) \tag{4.2.45}$$

with $\mathrm{pr}_{0,0}(w_1, w_2) = \mathrm{pr}(w_1, w_2)$. The corresponding Prym modulus is defined by

$$\Pi_{\varepsilon,\delta} = \frac{\displaystyle\oint_\beta \mathrm{pr}_{\varepsilon,\delta}(w_1, w_2)}{\displaystyle\oint_\alpha \mathrm{pr}_{\varepsilon,\delta}(w_1, w_2)} \tag{4.2.46}$$

with $\Pi_{0,0} = \Pi$.

These constructions of Prym varieties and Prym differentials have natural generalizations to double covers $\hat{\Sigma}$ of a genus g surface Σ with $k = 2n$ branch points ($n = 0, 1$), with genus $\hat{g} = 2g + n - 1$ determined by the Riemann-Hurwitz formula (4.1.35). In this case the Prym variety is a complex torus of dimension $g + n - 1$. By the Riemann-Roch theorem, there are exactly $g + n - 1$ independent holomorphic one-forms which are odd under the automorphism ι and which form a basis for the Prym differentials. The remaining g even ones on $\hat{\Sigma}$ are preimages of the holomorphic differentials on the base space Σ. A further generalization exists to more general abelian automorphism groups of a cover. The action of the group on $H^{1,0}(\hat{\Sigma}, \mathbb{C})$ is then always diagonal on a suitable basis of holomorphic differentials and the subspace corresponding to a non-trivial set of eigenvalues are pull-backs of multiplicative elliptic differentials, whose multiplicative factors are given by these eigenvalues. This is exploited implicitly in the computation of $\mathbb{Z}_N$ orbifold twist field amplitudes in [21].

4.2.4 Correlation Functions of Twist Field Operators

We now come to the computation of the two-point function $\langle\sigma(z)\,\sigma(0)\rangle^{\mathbb{Z}_2}$ of $\mathbb{Z}_2$ twist fields $\sigma(z) = \sigma_{12}(z)$ in the $\mathbb{R}^{24} \wr \mathbb{Z}_2$ permutation orbifold. We begin by discussing some general aspects concerning global monodromy in the covering surface construction of Section 4.1.4. Recall that the sum appearing in the correlation function (4.1.36) of interest (computed with the amplitude (4.1.43)) is restricted to the set of admissible monodromy homomorphisms Φ such that each connected component of the corresponding cover $\hat{\Sigma}$ of the base torus $\mathbb{T}^2$ is a surface of genus two. This is ensured by the requirement that the monodromy of the generators of $\pi_1(\mathbb{T}^2_{\underline{w}})$ encircling the punctures be a simple transposition in each orbit $\xi \in \mathcal{O}(\Phi)$. The period matrix $\tau^{\xi,\underline{w}}$ depends on the monodromy only via its stabilizer subgroups, which are the finite index subgroups $H < \pi_1(\mathbb{T}^2_{\underline{w}})$ obeying the admissibility criterion (4.1.38). Consider the stabilizer subgroup $H = H_a$ of a given sheet a corresponding to a transitive homomorphism $\Phi : \pi_1(\mathbb{T}^2_{\underline{w}}) \to S_N$. Since it is isomorphic to $\pi_1(\hat{\Sigma}_{\underline{\hat{w}}})$ and since there are $2N - 2$ preimages of the two branch points of $\mathbb{T}^2_{\underline{w}}$, it is a group freely generated by $2N + 1$ elements. The kernel of the forgetful homomorphism $\hat{\imath}_* : \pi_1(\hat{\Sigma}_{\underline{\hat{w}}}) \to \pi_1(\hat{\Sigma})$ is given by the normal closure

$$\widehat{N}_H\big(\hat{\gamma}_1,\dots,\hat{\gamma}_{2N-2}\big) = < h\,\hat{\gamma}_1\,h^{-1},\dots,h\,\hat{\gamma}_{2N-2}\,h^{-1} \bigm| h\in H > \tag{4.2.47}$$

of the generators $\hat{\gamma}_i$ encircling the ramification points.

When $N=2$ the generators $\hat{\gamma}_i$ are easily determined. Let us use the presentation $\pi_1(\mathbb{T}^2_{\underline{w}}) = <\alpha,\beta,\gamma>$. The generators of $\pi_1(\hat{\Sigma}_{\underline{\hat{w}}})$ encircling the ramification points are the (pullbacks of the) squares of the generators of $\pi_1(\mathbb{T}^2_{\underline{w}})$ which encircle the punctures. For $N=2$, the preimages of the punctures are precisely the ramification points, and hence one has

$$\ker\big(\hat{\imath}_*\big) = \widehat{N}_H\big(\gamma^2\,,\,([\alpha,\beta]\,\gamma)^2\big)\ . \tag{4.2.48}$$

There are four homomorphisms with the prescribed monodromy representing the four twisted sectors $(\varepsilon,\delta)\in(\mathbb{Z}/2\mathbb{Z})^2$, and all of them are transitive. There are correspondingly exactly four admissible subgroups H of index two. Since $\mathbb{Z}_2$ is an abelian group, conjugacy classes of homomorphisms contain only one element. Their precise forms and the corresponding stabilizers can be determined explicitly.

The simplest example is provided by the admissible homomorphism Φ_1 which sends γ to the transposition (1 2) and α,β both to the identity. Its stabilizer H_1 is freely generated by the words $\alpha,\beta,\alpha\,\gamma\,\alpha^{-1},\beta\,\gamma\,\beta^{-1},\gamma^2$. We then seek a presentation of the generators $\hat{\alpha}_1,\hat{\alpha}_2,\hat{\beta}_1,\hat{\beta}_2,\hat{\gamma}$ of $\pi_1(\hat{\Sigma}_{\underline{\hat{w}}})$ such that the quotient by the relations $\gamma^2=([\alpha,\beta]\,\gamma)^2=1$ yields the group $\pi_1(\hat{\Sigma})$ with $[\hat{\alpha}_1,\hat{\beta}_1]\,[\hat{\alpha}_2,\hat{\beta}_2]\in\widehat{N}_{H_1}(\gamma^2,([\alpha,\beta]\,\gamma)^2)$. For the case at hand, one sees that the assignments $\hat{\alpha}_1=\alpha,\hat{\beta}_1=\beta,\hat{\alpha}_2=\alpha\,\gamma\,\alpha^{-1},\hat{\beta}_2=\beta\,\gamma\,\beta^{-1},\hat{\gamma}=\gamma^2$ suffice. This determines the homomorphism of fundamental groups $\hat{\imath}_*\circ\tilde{f}_*^{-1}$, where $\tilde{f}$ is the restriction of the covering map to the marked surfaces. Since the abelianization of $\pi_1(\mathbb{T}^2_{\underline{w}})$ factors through this map, the powers of α,β in the canonical homology generators $\hat{\alpha}_1,\hat{\alpha}_2,\hat{\beta}_1,\hat{\beta}_2$ gives the map (4.1.50). This yields the covering homology matrix

$$\mathsf{M}_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \tag{4.2.49}$$

which obeys the Hopf condition. Reduction of this matrix via an $Sp(4,\mathbb{Z})$ modular transformation as in Section 4.2.3 above yields the normal form (4.1.49) with $r=2,m=t=1,s=0$. The other three admissible homomorphisms are similarly treated.

However, the above formalism is sensitive only to the induced homomorphism f_* between homology groups rather than homotopy groups, and it is difficult to proceed further with the explicit construction of the modular invariant amplitude (4.1.36). We will return to this issue in some more detail in the next section. Here we shall compute the twist field correlation function using results of [27] where the correlation functions are computed for a free boson X in the geometric orbifold $\mathbb{S}^1/\mathbb{Z}_2$ using the covering space method explained in Section 4.1.4. The two-point correlation function on the torus $\mathbb{T}^2$ with twist field insertions may be computed from the path integral over field configurations $\hat{X}$ on the double cover $\hat{\Sigma}$ which are odd under the canonical involution with $\hat{X} \circ \iota = -\hat{X} \bmod 2\pi R$. As in Section 4.2.1 above, in each twisted sector (ε, δ) the amplitude is a product of a radius independent quantum piece and a classical piece. The instanton configurations on the worldsheet $\hat{\Sigma}$ that contribute to the classical part of the correlation function are analogous to the untwisted ones used in Section 4.2.1 above. In the homology basis specified by (4.2.33), the boundary conditions of the boson $\hat{X}$ in the given twisted sector are characterized by the Prym differential $\nu_{\varepsilon,\delta}$. The classical contribution is then completely analogous to that in (4.2.2) with the period τ equal to the Prym modulus $\Pi_{\varepsilon,\delta}$.

The quantum contributions may be computed by equating the two-loop orbifold amplitude with that of the circle theory at the self-dual radius as before, with the additional observation that the twist fields in this correspondence are equivalent to magnetic vertex operators [27]. At this radius the momentum lattices appearing in the classical partition sums can be built up from a finite number of square sublattices. A term by term comparison of the chiral blocks gives an expression for the ratio of a twisted determinant to the untwisted determinant $\mathfrak{z}(\tau^\bullet)$ as the modulus squared of a holomorphic function of the positions of the branch points on $\mathbb{T}^2$. In this way the normalized twist field two-point function on $\mathbb{T}^2$ with the twist characteristic (ε, δ) in the $\mathbb{Z}_2$ target space orbifold of the compactified boson X can be written as [27]

$$\big\langle \sigma(z)\,\sigma(0)\big\rangle_{\rm orb}^{\varepsilon,\delta} = \mathfrak{z}\big(\tau^\bullet\big)\ \Big|c\big(\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\big)\Big|^{-2}\ \mathfrak{z}^{\rm cl}(\Pi_{\varepsilon,\delta}, R)\ , \tag{4.2.50}$$

where

$$c\big(\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\big) = E(z)^{1/8}\ \frac{\theta\big(\begin{smallmatrix}a\\ b\end{smallmatrix}\big)(0|\Pi_{\varepsilon,\delta})}{\sqrt{\theta\big(\begin{smallmatrix}a+\varepsilon\\ b+\delta\end{smallmatrix}\big)\big(\frac{z}{2}\,\big|\,\tau^\bullet\big)\ \theta\big(\begin{smallmatrix}a\\ b\end{smallmatrix}\big)\big(0\,\big|\,\tau^\bullet\big)}}\ . \tag{4.2.51}$$

Here we have used translation invariance to fix one of the twist field insertion points at the origin, and $(a,b)\neq(1,1)$ is a fixed arbitrary characteristic. The quantity $E(z)$ is the prime form of the elliptic curve $\mathbb{T}^2$ given by

$$E(z)=\frac{\theta_1\big(z\,\big|\,\tau^\bullet\big)}{\theta_1'\big(0\,\big|\,\tau^\bullet\big)} \tag{4.2.52}$$

with $\theta_1'(z|\tau):=\frac{\partial}{\partial z}\theta_1(z|\tau)$, and it is the doubly periodic elementary solution of the Laplace equation on the torus. The independence of the expression (4.2.51) on the choice of characteristic (a,b) is the mathematical statement of the Schottky relations [85] (see Section 4.2.5 below).

We can now write down the desired amplitude in the permutation orbifold $\mathbb{R}^{24}\wr\mathbb{Z}_2$. For this, we redefine the independent bosons X_i^a, $i=1,\dots,24$, $a=1,2$ to $X_i^\pm=X_i^1\pm X_i^2$ as in Section 4.2.1 above. Since the $\mathbb{Z}_2$ permutation group acts on the 24 bosons simultaneously, both the global and local monodromy of the fields X_i^+ are trivial, and the twist operators act as the identity on these fields. The path integral over X_i^+ thus leads simply to an overall factor $\mathfrak{z}(\tau^\bullet,R)^{24}$. On the other hand, the twist operators act as a $\mathbb{Z}_2$ twist field simultaneously on all sigma model fields X_i^-. It follows that the correct prescription is to raise the geometric $\mathbb{Z}_2$ orbifold twist field correlation function in each sector to the power 24, and then sum over the twisted sectors. The X_i^+ contribution is cancelled in the suitably normalized correlation function by the same factors coming from the partition function (4.2.1). One should then take the decompactification limit $R\to\infty$, wherein $\mathfrak{z}^{\rm cl}(\Pi_{\varepsilon,\delta},R=\infty)=1$ as before. This gives the two-point function

$$\big\langle\sigma(z)\,\sigma(0)\big\rangle^{\mathbb{Z}_2}=\lim_{R\to\infty}\,\frac12\,\sum_{(\varepsilon,\delta)\in(\mathbb{Z}/2\mathbb{Z})^2}\Big(\big\langle\sigma(z)\,\sigma(0)\big\rangle_{\rm orb}^{\varepsilon,\delta}\Big)^{24}\ . \tag{4.2.53}$$

Substituting (4.1.22) and (4.2.50)–(4.2.52), and using the identity

$$\theta_1'(0|\tau)=-2\pi\,\eta(\tau)^3\ , \tag{4.2.54}$$

then leads to the explicit formula

$$\begin{aligned}\big\langle\sigma(z)\,\sigma(0)\big\rangle^{\mathbb{Z}_2}&=\frac12\left(\frac{4\sqrt2\,\pi^{5/2}\,\alpha'}{\tau_2^\bullet}\right)^{12}\left|\frac{\theta\big({a\atop b}\big)\big(0\,\big|\,\tau^\bullet\big)^4}{\theta_1\big(z\,\big|\,\tau^\bullet\big)\,\eta\big(\tau^\bullet\big)^5}\right|^6\\&\quad\times\sum_{(\varepsilon,\delta)\in(\mathbb{Z}/2\mathbb{Z})^2}\left|\frac{\theta\big({a+\varepsilon\atop b+\delta}\big)\big(\frac z2\,\big|\,\tau^\bullet\big)}{\theta\big({a\atop b}\big)\big(0\,\big|\,\Pi_{\varepsilon,\delta}\big)^2}\right|^{24}\ .\end{aligned} \tag{4.2.55}$$

4.2.5 DLCQ Free Energy = DVV Correlator

We will now prove the main result of this section, establishing the equivalence

$$\mathcal{F}_2\big(\tau^\bullet\big) = \frac{4\lambda^2}{\tau_2^\bullet\,\mu(0)}\int_{\mathbb{T}}^2 \mathrm{d}\mu(z)\,\big\langle\sigma(z)\,\sigma(0)\big\rangle^{\mathbb{Z}_2} \tag{4.2.56}$$

between the DLCQ free energy on the double cover $\hat{\Sigma}\to\mathbb{T}^2$ given by (4.2.25) and the translationally invariant correlator (4.1.42) of the DVV vertex operator determined by the twist field two-point function (4.2.55) on $\mathbb{R}^{24}\wr\mathbb{Z}_2$. We begin by observing that the right-hand side of the formula (4.2.56) is independent of the twist characteristic (ε,δ) in (4.2.55). This follows from the fact that one can get any twisted sector from the untwisted one $(\varepsilon,\delta)=(0,0)$ by a crossing transformation (4.2.44). Crossing symmetry of the orbifold theory, along with modular invariance at genus one, is the remnant of genus two modular invariance on the covering space [27]. One can check this invariance explicitly by showing that the z-dependent part of the correlation function (4.2.50) transforms under the crossing transformation (4.2.44) precisely by changing $(0,0)\to(\varepsilon,\delta)$, just like the Prym modulus according to (4.2.45).

Next we examine the change of integration variables from the modulus $\tau^\#$ in (4.2.25) to the branch point location in (4.2.56). For this, we require the Jacobian $|\mathrm{d}\tau^\#/\mathrm{d}z|^2$. The explicit dependence of the Prym modulus Π on the branch point loci is given by the formula (4.2.43) with $w_1=z, w_2=0$, but this is not convenient for computing the requisite derivative $\mathrm{d}\Pi/\mathrm{d}z$. Instead, it is more useful to use the *implicit* dependence of the Prym modulus on the branch point z dictated by the Schottky relations. For zero characteristics $(\varepsilon,\delta)=(0,0)$, they are given by

$$\frac{\sqrt{\theta_i\big(\frac{z}{2}\,\big|\,\tau^\bullet\big)\,\theta_i\big(0\,\big|\,\tau^\bullet\big)}}{\theta_i(0|\Pi)} = \frac{\sqrt{\theta_j\big(\frac{z}{2}\,\big|\,\tau^\bullet\big)\,\theta_j\big(0\,\big|\,\tau^\bullet\big)}}{\theta_j(0|\Pi)}\ . \tag{4.2.57}$$

By separating the explicit z and Π dependences for $i=4$ and $j=2$, we can write (4.2.57) as

$$\frac{\theta_2(0|\Pi)}{\theta_4(0|\Pi)} = \sqrt{\frac{\theta_2\big(0\,\big|\,\tau^\bullet\big)\,\theta_2\big(\frac{z}{2}\,\big|\,\tau^\bullet\big)}{\theta_4\big(0\,\big|\,\tau^\bullet\big)\,\theta_4\big(\frac{z}{2}\,\big|\,\tau^\bullet\big)}}\ . \tag{4.2.58}$$

Taking the total derivative of the relation (4.2.58) with respect to z yields

$$\frac{\partial}{\partial\Pi}\left(\frac{\theta_2(0|\Pi)}{\theta_4(0|\Pi)}\right)\frac{\mathrm{d}\Pi}{\mathrm{d}z} = \frac{\mathrm{d}}{\mathrm{d}z}\sqrt{\frac{\theta_2\big(0\,\big|\,\tau^\bullet\big)\,\theta_2\big(\frac{z}{2}\,\big|\,\tau^\bullet\big)}{\theta_4\big(0\,\big|\,\tau^\bullet\big)\,\theta_4\big(\frac{z}{2}\,\big|\,\tau^\bullet\big)}}\ . \tag{4.2.59}$$

We can transform the Π derivative by using the heat equation

$$\frac{\partial\theta_i(z|\Pi)}{\partial\Pi}+\frac{\mathrm{i}}{4\pi}\frac{\partial^2\theta_i(z|\Pi)}{\partial z^2}=0 \tag{4.2.60}$$

to get the form

$$\frac{\partial}{\partial\Pi}\left(\frac{\theta_2(0|\Pi)}{\theta_4(0|\Pi)}\right)=-\frac{\mathrm{i}}{4\pi\,\theta_4(0|\Pi)^2}\frac{\partial}{\partial w}\left(\theta_4(w|\Pi)^2\,\frac{\partial}{\partial w}\frac{\theta_2(w|\Pi)}{\theta_4(w|\Pi)}\right)\Bigg|_{w=0}\,. \tag{4.2.61}$$

We may then use the identity for the derivative of a ratio of theta functions given by

$$\frac{\partial}{\partial w}\left(\frac{\theta_2(w|\Pi)}{\theta_4(w|\Pi)}\right)=-\pi\,\theta_3(0|\Pi)^2\,\frac{\theta_1(w|\Pi)\,\theta_3(w|\Pi)}{\theta_4(w|\Pi)^2} \tag{4.2.62}$$

to arrive at

$$\frac{\partial}{\partial\Pi}\left(\frac{\theta_2(0|\Pi)}{\theta_4(0|\Pi)}\right)=\frac{\mathrm{i}}{4}\,\frac{\theta_3(0|\Pi)^3\,\theta_1'(0|\Pi)}{\theta_4(0|\Pi)^2}\,. \tag{4.2.63}$$

The differentiation on the right-hand side of (4.2.59) is an easy exercise. This calculation can be repeated starting from the Schottky relation (4.2.57) with $i=4$ and $j=3$. The final result is identical to that above with the replacements $\theta_2\leftrightarrow\theta_3$ of theta functions everywhere. In this way we can finally write

$$\begin{aligned}\left|\frac{\mathrm{d}\Pi}{\mathrm{d}z}\right|^2 &= \pi^2\left|\frac{\sqrt{\theta_2\big(0\,|\,\tau^\bullet\big)\,\theta_2\big(\tfrac{z}{2}\,|\,\tau^\bullet\big)}}{\theta_2(0|\Pi)}\,\frac{\sqrt{\theta_3\big(0\,|\,\tau^\bullet\big)\,\theta_3\big(\tfrac{z}{2}\,|\,\tau^\bullet\big)}}{\theta_3(0|\Pi)}\right| \\ &\quad\times\left|\frac{\theta_2\big(0\,|\,\tau^\bullet\big)^2\,\theta_3\big(0\,|\,\tau^\bullet\big)^2}{\theta_4\big(0\,|\,\tau^\bullet\big)}\,\frac{\theta_1\big(\tfrac{z}{2}\,|\,\tau^\bullet\big)^2}{\theta_4\big(\tfrac{z}{2}\,|\,\tau^\bullet\big)^3}\,\frac{\theta_4(0|\Pi)^4}{\theta_1'(0|\Pi)\,\sqrt{\theta_2(0|\Pi)\,\theta_3(0|\Pi)}}\right|\end{aligned} \tag{4.2.64}$$

To compare (4.2.64) with the elliptic functions appearing in the expressions (4.2.25) and (4.2.55) for $(\varepsilon,\delta)=(0,0)$, we exploit the identity (4.2.19) and the Schottky relations (4.2.57) again to write

$$\left|\frac{\mathrm{d}\Pi}{\mathrm{d}z}\right|^2=\pi^2\left|\frac{\theta_1\big(\tfrac{z}{2}\,|\,\tau^\bullet\big)^3}{\theta_1\big(z\,|\,\tau^\bullet\big)\,\theta_1'(0|\Pi)^2}\prod_{i=1,2}\frac{\sqrt{\theta\big(\begin{smallmatrix}a_i\\ b_i\end{smallmatrix}\big)\big(\tfrac{z}{2}\,|\,\tau^\bullet\big)\,\theta\big(\begin{smallmatrix}a_i\\ b_i\end{smallmatrix}\big)\big(0\,|\,\tau^\bullet\big)}}{\theta\big(\begin{smallmatrix}a_i\\ b_i\end{smallmatrix}\big)(0|\Pi)}\right|\,, \tag{4.2.65}$$

where $(a_i,b_i)\in\{(0,0)\,,\,(0,1)\,,\,(1,0)\}$ are arbitrary characteristics which we will choose conveniently. We can now use the identities (4.2.5), (4.2.19) and (4.2.54) along with

$$\theta_3\big(\tfrac{z}{2}\,|\,\tau^\bullet\big)^2\,\theta_4\big(0\,|\,\tau^\bullet\big)^2-\theta_4\big(\tfrac{z}{2}\,|\,\tau^\bullet\big)^2\,\theta_3\big(0\,|\,\tau^\bullet\big)^2=-\theta_1\big(\tfrac{z}{2}\,|\,\tau^\bullet\big)^2\,\theta_2\big(0\,|\,\tau^\bullet\big)^2 \tag{4.2.66}$$

to expand the expression (4.2.65) into

$$\left|\frac{\mathrm{d}\Pi}{\mathrm{d}z}\right|^2 = \frac{1}{2^{18}}\left|\frac{\eta\big(\tau^\bullet\big)^{-42}}{\theta_1\big(z\,\big|\,\tau^\bullet\big)^6}\prod_{i=1}^{8}\frac{\sqrt{\theta\big({a_i\atop b_i}\big)\big(\frac z2\,\big|\,\tau^\bullet\big)\,\theta\big({a_i\atop b_i}\big)\big(0\,\big|\,\tau^\bullet\big)}}{\theta\big({a_i\atop b_i}\big)(0|\Pi)}\right|$$

$$\times\left|\theta_2\big(\tfrac z2\,\big|\,\tau^\bullet\big)\,\theta_3\big(\tfrac z2\,\big|\,\tau^\bullet\big)\,\theta_4\big(\tfrac z2\,\big|\,\tau^\bullet\big)\,\theta_3\big(0\,\big|\,\tau^\bullet\big)^2\,\theta_4\big(0\,\big|\,\tau^\bullet\big)^2\right.$$

$$\left.\times\Big[\theta_3\big(\tfrac z2\,\big|\,\tau^\bullet\big)^2\,\theta_4\big(0\,\big|\,\tau^\bullet\big)^2-\theta_4\big(\tfrac z2\,\big|\,\tau^\bullet\big)^2\,\theta_3\big(0\,\big|\,\tau^\bullet\big)^2\Big]\right|^4 . \qquad (4.2.67)$$

We have again used (4.2.57) to infer that every term of the product in (4.2.67) is independent of the chosen characteristic (a_i, b_i).

Let us now substitute (4.2.67) into the integral (4.2.25), recalling that $\Pi = 2\tau^\#$. We can again exploit the freedom in choice of characteristics (a_i, b_i) to combine the theta functions in (4.2.67) with the ones $\theta_i(0|\Pi) =: \theta\big({a_i\atop b_i}\big)(0|\Pi)$ and $\theta_i(0|\tau^\bullet) =: \theta\big({a_i\atop b_i}\big)(0|\tau^\bullet)$ appearing in (4.2.25) by re-expressing Dedekind functions as theta functions using (4.2.5). The simplification effectively amounts to replacing each factor $\theta_i(0|\Pi)$ with $\sqrt{\theta_i(\frac z2|\tau^\bullet)\,\theta_i(0|\tau^\bullet)}$. We can use this trick to cancel the difference of theta functions appearing in the integrand of (4.2.25) by simply doing this replacement for every term, and remembering that there are in total 40 factors of $\theta_i(0|\Pi)$ in each term of the expansion of the fourth power of the difference.

In this way, it is straightforward to see after some inspection that the free energy (4.2.25) may be written in terms of an integral over the branch point location on the torus $\mathbb{T}^2$ as

$$\mathcal{F}_2\big(\tau^\bullet\big) = \frac{g_s^2}{\big(32\pi^2\,\alpha'\big)^{12}}\,\frac{\big|\eta(\tau^\bullet)\big|^{-30}}{4\,\big|\tau^\bullet\big|^8}$$

$$\times\int_{\mathbb{T}}^2\frac{\mathrm{d}^2z}{\big(\mathrm{Im}\,\Pi(z)\big)^{12}}\left|\frac{1}{\theta_1(z|\tau^\bullet)^6}\prod_{i=1}^{48}\frac{\sqrt{\theta\big({a_i\atop b_i}\big)\big(\frac z2\,\big|\,\tau^\bullet\big)\,\theta\big({a_i\atop b_i}\big)\big(0\,\big|\,\tau^\bullet\big)}}{\theta\big({a_i\atop b_i}\big)\big(0\,\big|\,\Pi(z)\big)}\right| \qquad (4.2.68)$$

It is now clear that with (4.2.55) the DLCQ free energy function (4.2.68) can be expressed in the form (4.2.56) if we choose the measure

$$\mathrm{d}\mu(z) = \frac{\mathrm{d}^2z}{\mu(z)} \qquad \text{with} \quad \mu(z) = \Big(\frac{2\pi^2\,\alpha'}{\tau_2^\bullet}\,\mathrm{Im}\,\Pi(z)\Big)^{d/2} \qquad (4.2.69)$$

where $d = 24$ is the spacetime dimension of the permutation orbifold. Using $\Pi(0) =$

$\tau^\bullet$, the coupling constant λ is then given by

$$\lambda = \frac{4g_s}{\pi^3 \left|512\,\tau^\bullet\right|^4}\,\sqrt{\tau_2^\bullet}\ . \tag{4.2.70}$$

Note that the coupling (4.2.70) has the correct infrared behaviour $\lambda \to 0$ as $\tau_2^\bullet \to \infty$ to ensure that the interacting sigma model approaches a conformal fixed point in the infrared limit.

From the genus two perspective the origin of the measure (4.2.69) is clear. It arises from the $Sp(4,\mathbb{Z})$ modular invariant integration over the moduli space of genus two branched covering maps $f : \hat{\Sigma} \to \mathbb{T}^2$. From the genus one perspective it is a consequence of the conformal anomaly, implying that the local twist field correlation functions depend on the coordinatization chosen on the Riemann surface $\mathbb{T}^2$. For the twist field operators the natural choice is the coordinate z of $\mathbb{T}^2$, but to induce the modular invariant interactions of strings in the symmetric product a non-trivial integration measure (4.2.69) must be adapted. We will see this explicitly in the next section when we study the action of the mapping class group of the punctured torus $\mathbb{T}^2_{\underline{w}}$.

4.3 Nonabelian Orbifolds

In this section we address some issues surrounding the extensions of the results of the previous section to S_N orbifolds with $N > 2$. At this stage, however, we have not succeeded in making the construction as explicit as for the $\mathbb{Z}_2$ orbifold. The main technical obstruction is the combined noncommutativity of the twist group S_N and the fundamental group $\pi_1(\mathbb{T}^2_{\underline{w}})$ of the punctured torus. For twist group $\mathbb{Z}_2$ the image of the latter group under a given monodromy homomorphism Φ is of course an abelian group, enabling explicit constructions. But these constructions become ambiguous and inconsistent in the nonabelian case, as one must deal with the full nonabelian homotopy group and not just its abelianization to the homology group. We are not aware of any direct computation of the twist field correlation functions in these specific instances. In the following we will highlight some of the main technical issues surrounding these calculations in the higher degree permutation orbifolds, and in particular to what extent the DLCQ free energy (4.1.48) can be used to provide an explicit representative for the DVV correlator (4.1.42) using the combinatorial formula (4.1.36). One of the outcomes of this analysis will be a more precise, general description of the measure $\mathrm{d}\mu(z)$ required in the definition of the vertex operator (4.1.40).

4.3.1 Uniformization Construction

Let us recall the general construction of Section 4.1.4. A correlation function involving twist fields alone in any permutation orbifold is defined through the generalized partition function (4.1.36). It gives a twist field correlation function on a worldsheet Σ as a sum over twisted sectors, each characterized by a conjugacy class of monodromy homomorphisms. One term is given by the partition function of the covering space $\hat{\Sigma}$ determined by Hurwitz data, comprising the monodromy, the complex structure of the worldsheet Σ and the insertion points of the twist field operators. The issue is how to determine the covering space and its complex structure in terms of the Hurwitz data. The monodromy in the case of k distinct insertion points on the worldsheet is a homomorphism $\Phi : \pi_1(\Sigma_{\underline{w}}) \to G < S_N$, and the general Riemann-Hurwitz formula (4.1.35) for ramified coverings gives the genus $\hat{g}$ of the covering space. Determining the topological type of the cover is analogous to the unramified case. The fundamental group of the marked cover $\hat{\Sigma}_{\underline{\hat{w}}}$ is given by a stabilizer subgroup $H_a < \pi_1(\Sigma_{\underline{w}})$. The index a is the label of a sheet, which is permuted by the twist group $G < S_N$, and different choices of a result in conjugate subgroups of $\pi_1(\Sigma_{\underline{w}})$ corresponding to different choices of pre-image of the base point of $\pi_1(\Sigma_{\underline{w}})$ as the base point of $\pi_1(\hat{\Sigma}_{\underline{\hat{w}}})$.

However, it is much more difficult to determine the complex structure of the cover. Recall that the prescription for the unramified case was to choose a uniformizing homomorphism $u : \pi_1(\Sigma) \to U$ such that $\Sigma_\tau = U/u(\pi_1(\Sigma))$. Then one needs to restrict u to the stabilizer subgroup of $\pi_1(\Sigma)$ corresponding to the monodromy homomorphism Φ. But the domain of the monodromy is $\pi_1(\Sigma_{\underline{w}})$ for the ramified case, which is a group distinct from $\pi_1(\Sigma)$. Hence it is not straightforward to extend this uniformization method to the case of branched coverings. Consider the commutative diagram

$$\begin{array}{ccc} \hat{\Sigma}_{\underline{\hat{w}}} & \xrightarrow{\hat{\imath}} & \hat{\Sigma} \\ \tilde{f} \downarrow & & \downarrow f \\ \Sigma_{\underline{w}} & \xrightarrow[\imath]{} & \Sigma \end{array} \tag{4.3.1}$$

where the maps $\imath$ and $\hat{\imath}$ are the canonical inclusions (filling in the deleted points), and $\tilde{f}$ is the restriction of the covering map f to the punctured surfaces. Passing to the corresponding pushforwards, this diagram induces a commutative diagram of fundamental groups given by

$$\begin{array}{ccc} \pi_1(\hat{\Sigma}_{\underline{\hat{w}}}) & \xrightarrow{\hat{\imath}_*} & \pi_1(\hat{\Sigma}) \\ \tilde{f}_* \downarrow & & \downarrow f_* \\ \pi_1(\Sigma_{\underline{w}}) & \xrightarrow[\imath_*]{} & \pi_1(\Sigma) \end{array} . \tag{4.3.2}$$

Let $\mathcal{T}(k, g)$ denote the Teichmüller space of genus g Riemann surfaces with k punctures. Let $\mathcal{M}(k, g)$ be the mapping class group of the (marked) Riemann surface $\Sigma_{\underline{w}}$ acting on $\mathcal{T}(k, g)$. One seeks maps which fit into the commutative diagram

$$\begin{array}{ccc} \mathcal{T}(\hat{k}\,,\,\hat{g}) & \longrightarrow & \mathcal{T}(0, \hat{g}) \\ \downarrow & & \downarrow \\ \mathcal{T}(k, g) & \longrightarrow & \mathcal{T}(0, g) \end{array} \tag{4.3.3}$$

associated to the covering and the inclusions such that the vertical arrow on the left is given by the surjective map $U/u(\pi_1(\hat{\Sigma}_{\underline{\hat{w}}})) \cong U/u(H_a) \to U/u(\pi_1(\Sigma_{\underline{w}}))$, where u is a uniformizing map of punctured surfaces. In this way one can incorporate the information from the monodromy contained in the admissible finite index subgroup

H_a. Note that the corresponding complex dimensions of the spaces involved in (4.3.3) map as

$$\begin{array}{ccc} 3\hat{g}-3+\hat{k} & \longrightarrow & 3\hat{g}-3 \\ \downarrow & & \downarrow \\ 3g-3+k & \longrightarrow & 3g-3 \end{array} \qquad (4.3.4)$$

for $g > 0$ (except for $\dim_{\mathbb{C}} \mathcal{T}(0,1) = 1$).

The problem rests in the construction of the horizontal arrows of (4.3.3). Since the pushforward $\imath_*$ is a group homomorphism, the image of an element of a uniformizing group $u(\pi_1(\Sigma_{\underline{w}})) < PSL(2,\mathbb{R})$, which we identify with the complex structure given by $\mathbb{U}/u(\pi_1(\Sigma_{\underline{w}})) \in \mathcal{T}(k,g)$, is a coset and thus not an element in $PSL(2,\mathbb{R})$. Thus even though the quotient of the uniformizing group of the marked surface by the normal closure of the parabolic generators is isomorphic to $\pi_1(\Sigma)$ (by the admissibility constraint), it is not a subgroup of $PSL(2,\mathbb{R})$. The same remarks apply to the map $\hat{\imath}$ inducing the top horizontal arrow in (4.3.3). Therefore it is not possible to apply the method of uniformization which worked for the unramified case, and the forgetful maps (*i.e.*, the horizontal arrows in (4.3.3)) need to be constructed by hand.

Let us specialize to our main problem of interest, where the base space is the torus $\Sigma = \mathbb{T}^2$ with $k = 2$ simple branch points. For an N-sheeted cover of genus $\hat{g} = 2$ there are $\hat{k} = 2N - 2$ preimages of these branch points, so that two of the N preimages of a generic point of the base coincide for a branch point. The main obstacle in constructing the map $\imath_{\mathcal{T}} : \mathcal{T}(2,1) \to \mathcal{T}(0,1)$ rests in the fact that a flat torus admits a complete euclidean metric, whereas a punctured torus admits a complete hyperbolic metric. Thus in order to apply uniformization one needs to construct a map between the space of flat tori and the space of hyperbolic tori. Let us assume that the branch points are distinguished points of the flat metric on $\mathbb{T}^2$. Using the automorphism group of the torus we may fix the location of one of the branch points at the origin. Then one requires a bijection $\mathcal{T}(2,1) \to \mathcal{T}(0,1) \times \mathbb{U}$, where the second branch point z varies in the complex upper half plane $\mathbb{U}$. This must be done in such a way that a lift of the mapping class group $\mathcal{M}(0,1) = SL(2,\mathbb{Z})$ to $\mathcal{M}(2,1)$ acts equivariantly on

$\mathcal{T}(2,1)$ with respect to this bijection.

An element of $\mathcal{T}(2,1)$ is a twice punctured hyperbolic torus. Using the uniformizing homomorphism $u : \pi_1(\mathbb{T}^2_{\underline{w}}) \to PSL(2,\mathbb{R})$, it can be characterized as a discrete Fuchsian group

$$u\big(\pi_1(\mathbb{T}^2_{\underline{w}})\big) = < \alpha,\beta,\gamma \in PSL(2,\mathbb{R}) \,\big|\, |\operatorname{tr}\alpha\,| > 2 \ , \ |\operatorname{tr}\beta\,| > 2 \ ,$$
$$|\operatorname{tr}\gamma\,| = |\operatorname{tr}[\alpha,\beta]\,\gamma\,| = 2 > \ . \tag{4.3.5}$$

The hyperbolic generators α,β correspond to translation along a canonical homology basis of the unmarked torus, while γ and $[\alpha,\beta]\,\gamma$ are the parabolic generators corresponding to the punctures.[3] Then the complex structure is given by $\mathbb{U}/u(\pi_1(\mathbb{T}^2))$. The subgroup (4.3.5) contains three real parameters for each generator, two trace relations for parabolicity and a conjugation symmetry which eliminates three parameters, hence the real dimension of $\mathcal{T}(2,1)$ is $3\cdot 3-2-3=4$, as anticipated.

The space $\mathcal{T}(0,1)\times\mathbb{U}$ is coordinatized by ordered pairs (τ,z), where τ is a genus one modulus and z is a distinguished point on $\mathbb{T}^2$. The mapping class group $\mathcal{M}(0,1)\cong SL(2,\mathbb{Z})$ of the flat torus acts on these pairs through the generators

$$T\,:\,(\tau,z)\ \longmapsto\ (\tau+1,z) \qquad \text{and} \qquad S\,:\,(\tau,z)\ \longmapsto\ \big(-\tfrac{1}{\tau}\,,\,\tfrac{z}{\tau}\big) \tag{4.3.6}$$

obeying $S^4 = (T\,S)^3\,S^2 = 1$. The modular S-transformation here is defined via analytic continuation along a clockwise oriented path around the origin in the complex z-plane. A lift of these generators to the mapping class group $\mathcal{M}(2,1)$ of the twice punctured hyperbolic torus is presented in [86] as an action on the generators of (4.3.5) by

$$\tilde{T}\,:\,\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}\ \longmapsto\ \begin{pmatrix}\alpha\\ \beta\,\alpha\\ \gamma\end{pmatrix} \qquad \text{and} \qquad \tilde{S}\,:\,\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}\ \longmapsto\ \begin{pmatrix}\beta^{-1}\\ \alpha\\ \beta^{-1}\,\gamma\,\beta\end{pmatrix}\ . \tag{4.3.7}$$

This lift of $SL(2,\mathbb{Z})$ is not unique. In fact, the modular group $\mathcal{M}(2,1)$ is an extension of $SL(2,\mathbb{Z})$ by $\mathcal{B}(2,1)/\Gamma_{\mathcal{M}(2,1)}$, where $\Gamma_{\mathcal{M}(2,1)}$ is the center of $\mathcal{M}(2,1)$ and $\mathcal{B}(2,1)$

[3] We could have equivalently used an independent parabolic generator γ' with the relation $[\alpha,\beta]\,\gamma\,\gamma'=1$.

denotes the two-stranded braid group of the torus [86]. Equivariance of the bijection $\imath_{\mathcal{T}} : \mathcal{T}(2,1) \to \mathcal{T}(0,1)$ with respect to these actions is then the statement

$$\imath_{\mathcal{T}} \circ \tilde{T} \ = \ T \circ \imath_{\mathcal{T}} \qquad \text{and} \qquad \imath_{\mathcal{T}} \circ \tilde{S} \ = \ S \circ \imath_{\mathcal{T}} \ . \tag{4.3.8}$$

We have not succeeded in constructing explicitly the required modular equivariant bijections, and it is not possible to write an algebraic formula [87]. One could try to surpass this problem by working directly with the hyperbolic presentation of the tori, and the known bijection between the Fenchel-Nielsen coordinates of Teichmüller space and the Fuchsian coordinates parametrizing the uniformizing group [87]. But there is a great deal of ambiguity in this procedure which prevents an explicit construction, and there is no canonical way to identify the modular parameters of the torus itself and those corresponding to the branch points.

4.3.2 Homology Construction

Given the technical difficulties encountered above, we now turn to an alternative approach to determining the complex structure of the cover via the push-forward induced on homology groups $f_* : H_1(\hat{\Sigma}, \mathbb{Z}) \to H_1(\Sigma, \mathbb{Z})$, which is provided by the abelianization of the diagram (4.3.2) for the fundamental groups. If a canonical basis is fixed both in the homology group of the base and that of the cover, then this map is given by a $2g \times 2\hat{g}$ matrix $\mathsf{M}^\top$. This matrix can then be used to determine the period matrix of the cover in terms of the period matrix of the base and some additional parameters (3.1.18). For sufficiently low genus, the period matrix τ uniquely characterizes the complex structure. We will go through this construction in detail for the relevant case of the genus two cover $f : \hat{\Sigma} \to \mathbb{T}^2$ for the two point function of twist fields corresponding to simple branch points. In this case the complex structure on $\hat{\Sigma}$ is determined by a canonical map $H^{1,0}(\hat{\Sigma}, \mathbb{C}) \otimes H_1(\hat{\Sigma}, \mathbb{Z}) \to \mathbb{C}$.

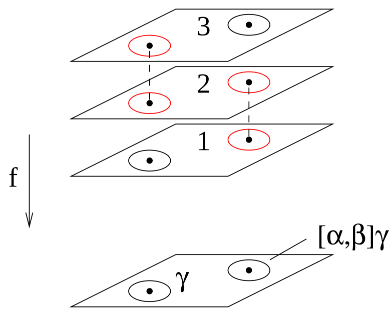
Let us see first how the matrix representation M of f_* can be determined and compared to the construction of Section 4.1.5. The main difference from the $N = 2$ case studied at the beginning of Section 4.2.4 is that for $N > 2$ the preimages of the punctures are no longer just the ramification points, since there are $2N - 2 > 2$ preimages of the branch points. Let $\pi_1(\mathbb{T}^2_{\underline{w}}) = < \alpha, \beta, \gamma >$, the free group on three generators such that $\ker(\imath_*)$ is the normalizer $N_{\pi_1(\mathbb{T}^2_{\underline{w}})}(\gamma, [\alpha, \beta]\,\gamma)$. In other words, the image of α and β are the standard generators of $\pi_1(\mathbb{T}^2)$, whereas γ and $[\alpha, \beta]\,\gamma$ correspond to simple closed curves which are contractible to the branch points. The stabilizer $H_a < \pi_1(\mathbb{T}^2_{\underline{w}})$ corresponding to a monodromy homomorphism Φ is a subgroup of index N in the case of an N-sheeted cover. It can be presented in terms of $4 + (2N - 2) - 1$ words from $\pi_1(\mathbb{T}^2_{\underline{w}})$ which freely generate the group H_a. By identifying H_a with $\pi_1(\hat{\Sigma}_{\underline{\hat{w}}})$, this presentation gives the homomorphism $\tilde{f}_*$ explicitly. There are $N - 2$ independent elements from H_a which are conjugate to γ in $\pi_1(\mathbb{T}^2_{\underline{w}})$, and another $N - 2$ elements which are conjugate to $[\alpha, \beta]\,\gamma$. There is one further element conjugate to γ^2 and another one conjugate to $([\alpha, \beta]\,\gamma)^2$. This is because the $N - 2$ generators of $\pi_1(\hat{\Sigma}_{\underline{\hat{w}}})$ corresponding to simple closed curves contractible to $N - 2$ preimages of a branch point project to the simple closed curve contractible to the branch point, whereas the other two generators project to curves with winding number two about each of the branch points.

The normalizer of these $2N - 2$ generators in H_a is the subgroup $\ker(\hat{\imath}_*)$. One then seeks $4 + 2N - 3$ generating elements such that $2N - 3$ are in $\ker(\hat{\imath}_*)$ and also the commutator product $[\hat{\alpha}_1, \hat{\beta}_1]\,[\hat{\alpha}_2, \hat{\beta}_2]$ of a suitably chosen remaining four. In other words, $\hat{\alpha}_i$, $\hat{\beta}_i$ are representatives of the cosets that project to a canonical homology basis of $\pi_1(\hat{\Sigma})$ under the map $\hat{\imath}_*$. Due to the commutativity of the diagram (4.3.2) and the abelianization, the entry M_{ij} of the 2×4 homology covering matrix is the sum of powers of the i-th generator of $H_1(\mathbb{T}^2, \mathbb{Z})$ (α or β) appearing in the expression of the j-th generator of $H_1(\hat{\Sigma}, \mathbb{Z})$ ($\hat{\alpha}_1$, $\hat{\beta}_1$, $\hat{\alpha}_2$ or $\hat{\beta}_2$). In this way, the two-point function may be computed by summing over admissible finite index subgroups $H_a < \Gamma = \pi_1(\mathbb{T}^2_{\underline{w}})$.

Let us look at an explicit example of how this works. For $N = 3$, there are 16 conjugacy classes of transitive monodromy homomorphisms, each class containing 6 homomorphisms. Accordingly, there are 16 conjugacy classes of admissible index three subgroups of Γ, each class having $[\Gamma : N_\Gamma(\Gamma_k)] = 3$ representatives. Consider the admissible monodromy homomorphism Φ_1 given by

$$\Phi_1 \,:\, \alpha \longmapsto (2\;3)\ , \quad \beta \longmapsto (1\;2) \quad \text{and} \quad \gamma \longmapsto (2\;3)\ . \tag{4.3.9}$$

The corresponding three sheeted cover may be depicted schematically as



(4.3.10)

with the parallelogram representing the base torus $\mathbb{T}^2$. The sheets 2 and 3 are ramified over the branch point corresponding to γ, while the sheets 1 and 2 are ramified over the other branch point corresponding to $[\alpha,\beta]\,\gamma$ (since $\Phi_1 : [\alpha,\beta]\,\gamma \mapsto (1\ 2)$).

The stabilizer subgroup of $\Gamma = \pi_1(\mathbb{T}^2_{\underline{w}}) = <\alpha,\beta,\gamma>$ can be presented by[4]

$$H_1 \quad = \; < g_1, \ldots, g_7 > \; := $$

$$< \alpha,\, \beta^2,\, \gamma,\, \beta\,\alpha^2\,\beta^{-1},\, \beta\,\gamma\,\alpha^{-1}\,\beta^{-1},\, \beta\,\alpha\,\gamma\,\beta^{-1},\, \beta\,\alpha\,\beta\,\alpha^{-1}\,\beta^{-1} > \quad (4.3.11)$$

The elements $g_1, \ldots, g_7$ generate the the group H_1 freely. One can then determine the generators of $\ker(\hat{\imath}_*)$ as

$$\begin{aligned}
\hat{g}_1 &= g_3 = \underline{\gamma}\,,\\
\hat{g}_2 &= g_6\,g_5 = \beta\,\alpha\,\underline{\gamma^2}\,\alpha^{-1}\,\beta^{-1}\,,\\
\hat{g}_3 &= g_4\,g_2\,g_1^{-1}\,g_2^{-1}\,g_5 = \beta\,\alpha\,\underline{[\alpha,\beta]\,\gamma}\,\alpha^{-1}\,\beta^{-1}\,,\\
\hat{g}_4 &= g_1\,g_4^{-1}\,g_7^{-1}\,g_6\,g_7\,g_3 = \underline{\big([\alpha,\beta]\,\gamma\big)^2}\,,
\end{aligned} \qquad (4.3.12)$$

where we have underlined the curves on the base that they are conjugate to. Finally, it is possible to write down the generators

$$\hat{\alpha}_1 = g_7\,, \quad \hat{\beta}_1 = g_1\,g_4^{-1}\,, \quad \hat{\alpha}_2 = g_1 \quad \text{and} \quad \hat{\beta}_2 = g_2 \qquad (4.3.13)$$

[4]In practice it is easier to determine the monodromy homomorphism corresponding to a given presentation of a finite index subgroup.

such that[5]

$$H_1/N_{H_1}(\hat{g}_1,\hat{g}_2,\hat{g}_3,\hat{g}_4) \;=\; <\hat{\alpha}_1,\hat{\alpha}_2,\hat{\beta}_1,\hat{\beta}_2 \,\big|\, [\hat{\alpha}_1,\hat{\beta}_1]\,[\hat{\alpha}_2,\hat{\beta}_2]=1> \;=\; \pi_1\big(\hat{\Sigma}\big) \quad (4.3.14)$$

We can now count the powers of α,β appearing in (4.3.13) to determine the matrix representation $\mathsf{M}=\mathsf{M}_1$ of f_* in (4.1.50) with

$$\mathsf{M}_1 = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix} \ . \qquad (4.3.15)$$

The remaining 15 admissible finite index subgroups are similarly treated. All instances provide a matrix representation M which satisfies the Hopf condition and which leads to the normal form (4.1.49) after reduction using the symplectic group $Sp(4,\mathbb{Z})$. However, the map from the set of admissible finite index subgroups to the set of normal forms (4.1.49) obeying the Hopf condition is *not* unique, and there is a large degree of arbitrariness in this procedure. The reason is that the partial reduction leading to (4.1.49) involves only $Sp(4,\mathbb{Z})$ transformations, but not modular transformations of the base. It may happen that an admissible finite index subgroup H_a is invariant under an $SL(2,\mathbb{Z})$ transformation of the base (*e.g.*, $\alpha\leftrightarrow\beta$), in which case one may get matrices M leading to period matrices which are not related by a modular transformation on the cover $\hat{\Sigma}$. Thus it is only onto the set of fully reduced Poincaré normal forms of M, which incorporates a sum over all such $SL(2,\mathbb{Z})$ transformations of the base $\mathbb{T}^2$, that this reduction map is unique. However, the reduced moduli space for the Poincaré normal form is very complicated and depends sensitively on number theoretic properties of the degree N (Appendix A).

4.3.3 Equivariance of the DVV Correlator

The construction of Section 4.3.2 above determines the dependence of the 2×2 period matrix τ_H on a given admissible monodromy homomorphism, or equivalently a given admissible finite index subgroup $H<\Gamma$, with $\tau_H=\tau_{r,m,s,t}$ in (4.1.51). At this stage we are faced with the problem of finding the dependence (either explicit or implicit) of

[5] One can check that the elements (4.3.13) are independent representatives of the generators of the quotient modulo $\hat{g}_1$, $\hat{g}_2$ and $\hat{g}_4$, except for $\hat{g}_3=[\hat{\alpha}_1,\hat{\beta}_1]\,[\hat{\alpha}_2,\hat{\beta}_2]$.

the Prym modulus $\Pi = r\,\tau^{\#}$ on the branch point location $z \in \mathbb{T}^2$. The construction of Prym differentials in Section 4.2.3 does not carry through to the higher degree branched covers, because for any genus two cover $f : \hat{\Sigma} \to \mathbb{T}^2$ of degree $N \geq 3$ there are no non-trivial automorphisms $\iota : \hat{\Sigma} \to \hat{\Sigma}$ such that $f \circ \iota = f$ [88]. As any genus two Riemann surface is a hyperelliptic curve, the cover $\hat{\Sigma}$ does have a canonical hyperelliptic involution $\iota_{\hat{\Sigma}}$ and its hyperelliptic divisor which is the effective divisor of degree six consisting of the fixed points of $\iota_{\hat{\Sigma}}$. Then there is a unique involution $\iota^2_{\mathbb{T}} : \mathbb{T}^2 \to \mathbb{T}^2$ of the base such that $f \circ \iota_{\hat{\Sigma}} = \iota^2_{\mathbb{T}} \circ f$ [88]. However, given that the above construction is not invariant under $SL(2,\mathbb{Z})$ transformations of the base, it is not clear how to exploit the hyperelliptic representation of $\hat{\Sigma}$, and the corresponding Schottky relations, to determine the branch point dependence as before. This is further reflected in the fact that the standard constructions of cut abelian differentials (such as (4.2.42)) for cyclic orbifolds [21] become ambiguous for nonabelian monodromy. We are not aware of any constructions of Prym differentials or Prym moduli for higher degree genus two covers $f : \hat{\Sigma} \to \mathbb{T}^2$ in terms of branch point loci.

On general grounds it follows that the complex structure on the covering surface $\hat{\Sigma}$ is uniquely determined by the holomorphic map $f : \hat{\Sigma} \to \mathbb{T}^2$ in terms of the moduli $\tau^\bullet$ and z, but not necessarily in an explicit parametrization. We can use results of [89] to ascertain that the desired explicit branch point dependence *does* exist and can be used to give some insight into the modular behaviour of the DVV correlator. One of the advantages of the formalism of Section 4.3.2 over that of Section 4.3.1 above is that one can study equivariance properties in the genus two modular group $Sp(4,\mathbb{Z})$, rather than in the more complicated mapping class group $\mathcal{M}(2,1)$. For fixed monodromy given by an admissible finite index subgroup $H < \Gamma$, there is a holomorphic map

$$\tau_H \ : \ \mathcal{T}(0,1) \times \mathbb{U} \ \longrightarrow \ \mathbb{U}^2 \ , \qquad (\tau^\bullet, z) \ \longmapsto \ \tau_H(\tau^\bullet, z) \tag{4.3.16}$$

which is determined generically in [89] via a sewing construction on twice-punctured tori in terms of Jacobi-Erdélyi theta functions, Weierstrass functions and Eisenstein series on the base torus $\mathbb{T}^2$. The primary difference in our specific case is that the modulus $\tau^{\#}$ has a square root cut singularity at each of the branch points $w_1 = z$ and $w_2 = 0$, rather than the logarithmic cut singularity which arises in [89].

Consider the monomorphism $SL(2,\mathbb{Z}) \hookrightarrow Sp(4,\mathbb{Z})$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{4.3.17}$$

This lift of $SL(2,\mathbb{Z})$ acts in the expected way on the domain of the map (4.3.16) as

$$(\tau^\bullet, z) \longmapsto \left(\tfrac{a\,\tau^\bullet+b}{c\,\tau^\bullet+d}\,,\,\tfrac{z}{c\,\tau^\bullet+d}\right) . \tag{4.3.18}$$

For each choice of branch for $\tau^\#$, the map τ_H is equivariant with respect to this action of $SL(2,\mathbb{Z}) < Sp(4,\mathbb{Z})$ [89] and there is a commutative diagram

$$\begin{array}{ccc} \mathcal{T}(0,1)\times\mathbb{U} & \xrightarrow{\ \tau_H\ } & \mathbb{U}^2 \\ {\scriptstyle SL(2,\mathbb{Z})}\downarrow & & \downarrow{\scriptstyle SL(2,\mathbb{Z})} \\ \mathcal{T}(0,1)\times\mathbb{U} & \xrightarrow[\ \tau_H\ ]{} & \mathbb{U}^2 \end{array} . \tag{4.3.19}$$

This property determines the equivariance of the DVV correlator (4.1.42), represented by the genus two DLCQ free energy (4.1.48) at a fixed value of the degree N. Since under (4.3.18) the flat area form on the torus transforms as $\mathrm{d}^2z \mapsto \mathrm{d}^2z/|c\,\tau^\bullet + d|^2$, and since the local twist field correlation functions $\langle\sigma_{a_1b_1}(z)\,\sigma_{a_2b_2}(0)\rangle^{S_N}$ have total scaling dimension 6, the scaling properties of the measure $\mu(z)$ under (4.3.18) can be explicitly determined.

Given the remarkable agreement of the $N=2$ free energy with the twist field two-point function in the $\mathbb{Z}_2$ orbifold, it is natural to extrapolate this correspondence and to take the fixed N DLCQ free energy integrand in (4.1.48) as the *definition* of the local twist field correlation function $\langle\sigma_{a_1b_1}(z)\,\sigma_{a_2b_2}(0)\rangle^{S_N}$ on the $\mathbb{R}^{24}\wr S_N$ permutation orbifold, according to the covering surface principle of Section 4.1.4. However, the explicit form of the mapping (4.3.16) displayed in [89, Proposition 6.2] is far too complicated for an explicit determination of the required Jacobian $|\mathrm{d}\tau^\#/\mathrm{d}z|^2$ (and furthermore one needs an $Sp(4,\mathbb{Z})$ transformation relating their period matrix to ours). Moreover, it is difficult to arrive at explicit formulas which are illuminating, as the products (4.1.44) of theta functions (4.1.54) are rather involved for $N \geq 3$.

4.4 Fermionic Orbifolds

In this final section we will study fermionic extensions of the permutation orbifolds considered thus far, in particular those orbifold sigma models arising in discrete light-cone quantization of superstrings and heterotic strings in ten spacetime dimensions. We will describe the modifications of the covering surface principle and twist field operators of Section 4.1 required in these cases. The genus two DLCQ free energy amplitudes in these instances are derived in (3.2.7), (3.3.4) and (3.4). Given the success of the bosonic $\mathbb{Z}_2$ orbifold model of Section 4.2, we will use the appropriately modified versions of the generic covering space principle of Section 4.1.4 to compute local one-loop correlation functions of (spin) twist field operators in supersymmetric and heterotic $\mathbb{Z}_2$ orbifolds. To the best of our knowledge these correlation functions have not been previously computed. The analysis of this section thus provides a powerful application of DLCQ string theory to producing new explicit expressions for correlation functions in orbifold superconformal field theories on the one hand, and for the forms of the leading cubic string interactions in the associated superstring field theories in ten dimensions on the other hand. Throughout we work in the Neveu-Schwarz-Ramond formalism.

4.4.1 Spin Twist Fields

Consider the superconformal sigma model on the torus with target space $\mathbb{R}^8$ defined by the action

$$I(X,\psi) = \frac{1}{4\pi\,\alpha'} \int_{\mathbb{T}}^{2} \mathrm{d}^2 z \, \frac{1}{2\,\mathrm{i}\,\tau_2} \, \big(\partial X_i(z)\,\overline{\partial} X_i(z) + \psi_i(z)\,\overline{\partial}\psi_i(z) + \overline{\psi}_i(z)\,\partial\overline{\psi}_i(z)\big) \tag{4.4.1}$$

where the real bosonic fields X_i, $i = 1,\dots,8$ transform in the eight-dimensional vector representation $\mathbf{8}_v$ of the R-symmetry group $SO(8)$, while the components $\psi_i, \overline{\psi}_i$, $i = 1,\dots,8$ of the 16-component Majorana-Weyl spinor field ψ transform in the spinor $\mathbf{8}_s$ and conjugate spinor $\mathbf{8}_c$ representations of $SO(8)$, respectively. The spinor fields are sections of the twisted spin line bundle $S_{\mathbb{T}}^2 \otimes L_{\boldsymbol{\delta}}$ over the torus, where $L_{\boldsymbol{\delta}}$ is a real line bundle over $\mathbb{T}^2$ with flat connection determined by one of the four spin structures $\boldsymbol{\delta} = \binom{\delta_\alpha}{\delta_\beta} \in H^1(\mathbb{T}^2, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}^2/2\mathbb{Z}^2$ and $[S_{\mathbb{T}}^2] \in \mathrm{Pic}^0(\mathbb{T}^2)$ is chosen to correspond to the theta divisor in the given homology basis (α, β). The $\mathcal{N} = 8$ worldsheet supersymmetry of the sigma-model is generated by the fermionic supercurrents

$$\mathsf{G}^\ell(z) = -\tfrac{1}{2}\, \gamma^i_{\ell'\ell}\, \big(\psi_{\ell'}(z)\, \partial X_i(z) + \overline{\psi}_{\ell'}(z)\, \overline{\partial} X_i(z)\big) \tag{4.4.2}$$

where γ^i are the $Spin(8)$ Dirac matrices.

In the corresponding permutation orbifold, the monodromy conditions on the bosonic fields X in a given twisted sector (P, Q) are as in (4.1.1), while the fermion monodromy is given by

$$\psi^a(z+1) \ = \ (-1)^{\delta_\alpha}\, \psi^{P(a)}(z) \qquad \text{and} \quad \psi^a(z+\tau) \ = \ (-1)^{\delta_\beta}\, \psi^{Q(a)}(z) \ , \tag{4.4.3}$$

where for simplicity we have omitted a potential extra sign depending on the reference spin structure $[S_{\mathbb{T}}^2]$. This symmetry is compatible with $\mathcal{N} = 8$ worldsheet superconformal invariance [90], and it means that on the fermionic fields the twist group G is extended to $G \times (\mathbb{Z}_2)^N$. The consistency condition $P\,Q = Q\,P$ implies [32] that the spin structure phases in (4.4.3) are independent of the coordinate label a in the permutation orbifold, and hence that only the diagonal subgroup of $(\mathbb{Z}_2)^N$ acts nontrivially on the fermions. The asymmetry between the twistings of bosons and fermions implies that the modular invariant sum over monodromy homomorphisms breaks spacetime supersymmetry of the orbifold sigma model.

Generally, the sum over $(\mathbb{Z}_2)^N$ monodromy in the fermionic sector is weighted by a consistent set of GSO phases $\zeta[\boldsymbol{\delta}; \Phi]$, generically dependent upon the twisted sector $\Phi : \pi_1(\Sigma) \to G$, which are constrained by modular covariance requirements. In the untwisted sector $\Phi(-) = e$, the phase corresponding to a spin structure $\boldsymbol{\delta} = \binom{\boldsymbol{\delta}_\alpha}{\boldsymbol{\delta}_\beta} \in \mathbb{Z}^{2g}/2\mathbb{Z}^{2g}$ is the mod 2 index of the Dirac operator on Σ twisted by the flat line bundle $L_{\boldsymbol{\delta}} \to \Sigma$ given by [64]

$$\zeta[\boldsymbol{\delta}; e] \ = \ (-1)^{\dim H^0(\Sigma, S_\Sigma \otimes L_{\boldsymbol{\delta}})} \ = \ (-1)^{\boldsymbol{\delta}_\alpha \cdot \boldsymbol{\delta}_\beta} \ , \tag{4.4.4}$$

where $\dim H^0(\Sigma, S_\Sigma \otimes L_{\boldsymbol{\delta}})$ is the number of linearly independent holomorphic sections of the spin bundle $S_\Sigma \otimes L_{\boldsymbol{\delta}}$. Schematically then, the modification of the formula (4.1.3) for the partition function of the supersymmetric permutation orbifold is given by

$$Z^{G\times(\mathbb{Z}_2)^N}(\tau)=\frac{1}{2^N\,|G|}\sum_{\Phi:\pi_1(\Sigma)\to G}\;\sum_{\boldsymbol{\delta}\in H^1(\Sigma,\mathbb{Z}/2\mathbb{Z})}\zeta[\boldsymbol{\delta};\Phi]\,\Big(\prod_{\xi\in\mathcal{O}(\Phi)}Z_{\boldsymbol{\delta}}\big(\tau^\xi\big)\Big) \tag{4.4.5}$$

where $Z_{\boldsymbol{\delta}}(\tau^\xi)$ is the partition function of the parent superconformal field theory computed with the global fermionic monodromy determined by the spin structure $\boldsymbol{\delta}$.

For example, the partition function of the supersymmetric $\mathbb{R}^8\wr(S_N\times(\mathbb{Z}_2)^N)$ permutation orbifold on $\Sigma=\mathbb{T}^2$ can be determined by first calculating the contribution from a given spin structure (say the Ramond-Ramond sector) to the path integral over the complex fermionic fields, and then summing over the modular orbits using either of the two GSO projections of Type II string theory. Then the parent partition function appearing in the formula (4.1.3) is given by [32]

$$Z(\tau)=\tfrac{1}{2}\,\Big|\mathfrak{z}\big(\begin{smallmatrix}0\\0\end{smallmatrix}\big)(\tau)^4-\mathfrak{z}\big(\begin{smallmatrix}0\\1\end{smallmatrix}\big)(\tau)^4-\mathfrak{z}\big(\begin{smallmatrix}1\\0\end{smallmatrix}\big)(\tau)^4\pm\mathfrak{z}\big(\begin{smallmatrix}1\\1\end{smallmatrix}\big)(\tau)^4\Big|^2\ , \tag{4.4.6}$$

where the $+/-$ sign corresponds to the Type IIA/B string amplitude and

$$\begin{aligned}\mathfrak{z}(\boldsymbol{\delta})(\tau)\;&=\;\left(\frac{4\pi^2\,\alpha'}{\tau_2}\right)^4\ \mathrm{e}^{\,\frac{\pi\,\mathrm{i}}{12}\,(2\delta_\alpha^2-1)\,\tau}\ \ \mathrm{e}^{\,\pi\,\mathrm{i}\,\delta_\alpha\,\delta_\beta/4}\\ &\quad\times\prod_{n=1}^\infty\big(1-(-1)^{\delta_\beta}\ \mathrm{e}^{\,\pi\,\mathrm{i}\,\tau\,(2n-1+\delta_\alpha)}\big)\ \big(1-(-1)^{\delta_\beta}\ \mathrm{e}^{\,\pi\,\mathrm{i}\,\tau\,(2n-1-\delta_\alpha)}\big)\end{aligned} \tag{4.4.7}$$

The corresponding grand canonical partition function (4.1.12) matches the Type II DLCQ free energy at finite temperature, with (4.4.6) producing the action of the (restricted) Hecke operator on the partition function of the first quantized Green-Schwarz superstring [5, 6, **?**]. A completely analogous correspondence holds for the thermal partition function of Type IIB DLCQ superstrings on the maximally supersymmetric plane wave background in ten dimensions [91].

The operators which create local monodromy in the superconformal sigma model with respect to the action of S_N are products $\sigma_P(z)\,\mathcal{S}_P(z)$ of bosonic and fermionic twist fields. Let us work in the sector of trivial global $\mathbb{Z}_2$ monodromy for the spinor fields, *i.e.*, with the Ramond-Ramond spin structure $\boldsymbol{\delta}=\big(\begin{smallmatrix}0\\0\end{smallmatrix}\big)$. The other sectors are treated similarly as in [14]. In a $\mathbb{Z}_n$-twisted sector corresponding to a cyclic permutation $P=(n)$, the vacuum state then carries an irreducible representation of the Clifford algebra for $Spin(8)$. By using an $SO(8)$ triality isomorphism, the representation space can be taken to be the direct sum $\mathbf{8}_v \oplus \mathbf{8}_c$. The corresponding components of the 16-dimensional ground state vector are created respectively by the primary spin fields $\mathcal{S}^i_{(n)}(z)$ and $\widetilde{\mathcal{S}}^i_{(n)}(z)$, $i = 1, \ldots, 8$. They each have conformal dimension [23]

$$\Delta^{\rm RR}_{(n)} = \frac{n}{6} + \frac{1}{3n} \ . \tag{4.4.8}$$

To describe the supersymmetric version of the DVV interaction vertex [3], we need another kind of spin twist field to ensure that the operators generating the basic joining and splitting of superstrings yield an irrelevant deformation of the superconformal sigma model. The bosonic twist field $\sigma_{ab}(z)$ transposing the fields X^a and X^b has conformal dimension $\frac{1}{2}$ when $d = 8$ (see (4.1.32)), as does the fermionic twist field $\mathcal{S}_{ab}(z)$ interchanging ψ^a and ψ^b. To increase the scaling dimension by $\frac{1}{2}$ in a supersymmetric fashion, we use the supersymmetric descendent of the primary twist field operators $\sigma(z)\,\widetilde{\mathcal{S}}(z)$ given by

$$\big[\mathsf{Q}^\ell \,,\, \sigma(z)\,\widetilde{\mathcal{S}}^{\,\ell'}(z)\big] + \big[\sigma(z)\,\widetilde{\mathcal{S}}^{\,\ell}(z) \,,\, \mathsf{Q}^{\ell'}\big] \ = \ \varrho^i(z)\,\mathcal{S}^i(z)\;\delta^{\ell\ell'} \ =: \ \Lambda(z)\;\delta^{\ell\ell'} \tag{4.4.9}$$

where

$$\mathsf{Q}^\ell = \oint \frac{{\rm d}z}{2\pi\,{\rm i}}\ \mathsf{G}^\ell(z) \tag{4.4.10}$$

are the $\mathcal{N} = 8$ supercharges and the contour integral is taken around the origin $z = 0$. The descendent bosonic twist fields $\varrho^i_{[P]}(z)$ create the first excited states in the twisted sector $[P]$. Since the combination $\psi^a - \psi^b$ has Ramond boundary conditions under transposition in S_N, the corresponding spin field carries a representation of the Clifford algebra. The twist field $\mathcal{S}^i_{ab}(z)$ transforms as a vector of $SO(8)$, and it coincides with the standard spin field of the supersymmetric $\mathbb{R}^8 \wr \mathbb{Z}_2$ permutation orbifold which can be constructed explicitly via bosonization of the fermion fields ψ_i [92, 93].

The fermionic DVV vertex operator is now defined by

$$V_{\rm ferm} = -\frac{\lambda\,N}{{\rm vol}(\mathbb{T}^2)}\int_{\mathbb{T}}^2 {\rm d}\mu(z)\ \sum_{1\le a<b\le N} \Lambda_{ab}(z) \ . \tag{4.4.11}$$

The descendent twist field $\Lambda_{ab}(z)$ is a primary field of conformal weight $\frac{3}{2}$. The interaction vertex (4.4.11) is spacetime supersymmetric, $SO(8)$ invariant and describes elementary string interactions [3].

The computation of the local twist field correlations functions $\langle \Lambda_{a_1 b_1}(z)\,\Lambda_{a_2 b_2}(0)\rangle^{S_N\times(\mathbb{Z}_2)^N}$ requires a modification of the covering surface principle of Section 4.1.4. This is because one should no longer simply close the punctures on the covering space $\hat\Sigma$ corresponding to the branch points to get the identity state at those points. Rather, one must insert the operator that creates a Ramond vacuum at the insertion points in order to give the fermions the correct local monodromy. Thus in the supersymmetric orbifold theory one uses the same covering spaces $\hat\Sigma$ as in the case of the bosonic orbifold, but instead of computing the partition function on $\hat\Sigma$ one computes a correlation function of spin fields on $\hat\Sigma$. A similar statement is also true in the NS–NS sector. In the mixed R–NS and NS–R sectors, there are no combinations of ψ^a which possess zero modes, so that these sectors have trivial local spin monodromy and the prescription instead follows that of Section 4.1.4. Schematically, the modification of a generic, normalized bosonic twist field correlation function (4.1.36) is given by

$$
\Big\langle \prod_{i=1}^{k} \Lambda_{[P_i]}(w_i) \Big\rangle^{G\times(\mathbb{Z}_2)^N} \tag{4.4.12}
$$

$$
= \frac{1}{2^N\,|G|} \sum_{\Phi:\pi_1(\Sigma_{\underline{w}})\to G\times(\mathbb{Z}_2)^N} \frac{1}{Z^{G\times(\mathbb{Z}_2)^N}(\tau)} \prod_{\xi\in\mathcal{O}(\Phi)} \Big\langle \prod_{i=1}^{\hat k} \hat{\mathbb{S}}_{[P_i]}(\hat w_i) \Big\rangle\big(\tau^{\xi,\underline{w}}\big)~,
$$

where the global $(\mathbb{Z}_2)^N$ monodromy acts trivially in the bosonic sector and diagonally in the fermionic sector as in (4.4.5). A similar prescription for $\mathcal{N}=4$ supersymmetric orbifold sigma models is used in [11]. When $\Sigma=\mathbb{T}^2$, this will be provided by the corresponding DLCQ free energy through the required modification of the GSO projection at finite temperature which breaks supersymmetry by making spacetime fermions antiperiodic around the thermal cycle (3.3.2).

Similar considerations also apply to the heterotic sigma model on the torus with target space $\mathbb{R}^8$, which is defined by the action

$$
I(X,\psi,\chi) = \frac{1}{4\pi\,\alpha'} \int_{\mathbb{T}}^{2} \mathrm{d}^2 z~\frac{1}{2\,\mathrm{i}\,\tau_2}~\big(\partial X_i(z)\,\overline{\partial} X_i(z) + \psi_i(z)\,\overline{\partial}\psi_i(z) + \chi_A(z)\,\partial\chi_A(z)\big) \tag{4.4.13}
$$

where the Majorana-Weyl fermion fields χ_A, $A=1,\dots,32$ are $SO(8)$ singlets. The holomorphic sector of this worldsheet field theory coincides with that of the supersymmetric sigma model (4.4.1), while after bosonization of χ_A the antiholomorphic sector coincides with the bosonic sigma model (4.1.16) in $d = 24$ with 16 of the bosons compactified on the Cartan torus of the heterotic gauge group $\mathcal{G} = SO(16) \times SO(16)$. The heterotic sigma model (4.4.13) is a superconformal field theory with chiral $(8,0)$ worldsheet supersymmetry.

The corresponding $(8,0)$ supersymmetric permutation orbifold [38, 41] is $(\mathbb{R}^8 \times \mathcal{G}) \wr (G \ltimes (\mathbb{Z}_2)^N)$. The twist subgroup $(\mathbb{Z}_2)^N$ acts on the holomorphic sector exactly as in the supersymmetric case. In the antiholomorphic sector, the gauge fermions χ_A are sections of flat real line bundles $L_{\boldsymbol{\delta}} \to \mathbb{T}^2$ like ψ_i, and so have global fermionic monodromy conditions as in (4.4.3). In contrast to the fields ψ_i, however, the spin structure phases for χ_A in the permutation orbifold generally depend on the coordinate label a. Perturbative string interactions are now generated by the heterotic version of the DVV vertex operator [38, 41]. For this, we must explicitly write worldsheet fields as products of holomorphic and antiholomorphic fields (which was implicitly understood in all previous formulae). As the holomorphic sector consists of the usual supersymmetric orbifold theory in eight dimensions, the holomorphic part of the vertex is constructed using the dimension $\frac{3}{2}$ spin twist operators $\Lambda_{ab}(z)$ defined in (4.4.9). On the other hand, the antiholomorphic sector consists of $d = 24$ bosons, and since the local monodromies about branch points are insensitive to the compactness of the 16 bosons on the Cartan torus, the antiholomorphic part of the vertex is built from the dimension $\frac{3}{2}$ bosonic twist fields $\overline{\sigma}_{ab}(z)$ of Section 4.1.4. It follows that the heterotic DVV vertex operator is defined by

$$V_{\text{het}} = -\frac{\lambda\, N}{\text{vol}(\mathbb{T}^2)} \int_{\mathbb{T}}^{2} \text{d}\mu(z) \sum_{1 \le a < b \le N} \big(\Lambda \otimes \overline{\sigma}\big)_{ab}(z) \ . \tag{4.4.14}$$

The computation of local twist field two-point functions proceeds by using a formula analogous to (4.4.12).

4.4.2 Supersymmetric DLCQ Strings

The genus two DLCQ free energy $F^{(2)}_{\text{ferm}}(\tau^\bullet, \kappa)$ for Type IIA superstrings at finite temperature is computed in (3.3.4). To write the result, we require some preliminary definitions. The ten even reduced, genus two integer characteristics $\binom{\boldsymbol{a}}{\boldsymbol{b}} = \binom{a_1\ b_1}{a_2\ b_2} \in H^1(\hat{\Sigma}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}^4/2\mathbb{Z}^4$ obey $\boldsymbol{a}\cdot\boldsymbol{b} \equiv 0 \bmod 2$ and are denoted by

$$\begin{aligned} \boldsymbol{\delta}_1 &= \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) \ , \quad \boldsymbol{\delta}_2 = \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right) \ , \quad \boldsymbol{\delta}_3 = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right) \ , \quad \boldsymbol{\delta}_4 = \left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right) \ , \\ \boldsymbol{\delta}_5 &= \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right) \ , \quad \boldsymbol{\delta}_6 = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \ , \quad \boldsymbol{\delta}_7 = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) \ , \quad \boldsymbol{\delta}_8 = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) \ , \\ \boldsymbol{\delta}_9 &= \left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right) \quad \text{and} \quad \boldsymbol{\delta}_0 = \left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right) \ . \end{aligned} \tag{4.4.15}$$

We use the shorthand notation $\vartheta_i := \Theta(\boldsymbol{\delta}_i)(\tau)^4$, where the genus two period matrix $\tau = \tau_{r,m,s,t}(\tau^\bullet, \tau^\#)$ is given by (4.1.51). On the last four characteristics in (4.4.15) we define genus two functions $\Xi_6(\boldsymbol{\delta}_i)(\tau)$ of modular weight six by the formulae

$$\begin{aligned} \Xi_6(\boldsymbol{\delta}_7) &= \vartheta_2\,\vartheta_3\,\vartheta_5 + \vartheta_8\,\vartheta_9\,\vartheta_0 - \vartheta_1\,\vartheta_4\,\vartheta_6 \ , \\ \Xi_6(\boldsymbol{\delta}_8) &= \vartheta_7\,\vartheta_9\,\vartheta_0 - \vartheta_1\,\vartheta_4\,\vartheta_5 + \vartheta_2\,\vartheta_3\,\vartheta_6 \ , \\ \Xi_6(\boldsymbol{\delta}_9) &= \vartheta_7\,\vartheta_8\,\vartheta_0 - \vartheta_1\,\vartheta_2\,\vartheta_6 + \vartheta_3\,\vartheta_4\,\vartheta_5 \ , \\ \Xi_6(\boldsymbol{\delta}_0) &= \vartheta_7\,\vartheta_8\,\vartheta_9 + \vartheta_3\,\vartheta_4\,\vartheta_6 - \vartheta_1\,\vartheta_2\,\vartheta_5 \ . \end{aligned} \tag{4.4.16}$$

Then one has

$$\begin{aligned} F^{(2)}_{\rm ferm}\big(\tau^\bullet\,,\,\kappa\big) &= -\frac{g_s^2}{4}\,\left|\frac{\tau^\bullet}{64\pi^2\,\alpha'}\right|^4\,\sum_{N=2}^\infty\,\frac{\kappa^N}{N}\,\sum_{\substack{r\,m=N\\ m\ \text{odd}}}\,\frac{1}{m^4}\,\sum_{\substack{s,t\in\mathbb{Z}/r\,\mathbb{Z}\\ t\neq 0}}\,\int_\triangle\,\frac{{\rm d}^2\tau^\#}{\big(\tau_2^\#\big)^4}\,\big|\Psi_{10}(\tau)\big|^{-2} \\ &\quad\times\Big|\Xi_6(\boldsymbol{\delta}_7)(\tau)\,\Theta(\boldsymbol{\delta}_7)(\tau)^4 + \Xi_6(\boldsymbol{\delta}_8)(\tau)\,\Theta(\boldsymbol{\delta}_8)(\tau)^4 \\ &\qquad + \Xi_6(\boldsymbol{\delta}_9)(\tau)\,\Theta(\boldsymbol{\delta}_9)(\tau)^4 + \Xi_6(\boldsymbol{\delta}_0)(\tau)\,\Theta(\boldsymbol{\delta}_0)(\tau)^4\Big|^2 \ . \end{aligned} \tag{4.4.17}$$

Note that the fermionic contribution to (4.4.17) consists of a sum of four terms in the Weierstrass-Poincaré reduction. We may identify these terms as resulting from the modular invariant sum over genus one spin structures, as in (4.4.6). The free energy (4.4.17) should now be equated to the translationally invariant correlator $\langle\,{}^\circ_\circ\,V_{\rm ferm}\,V_{\rm ferm}\,{}^\circ_\circ\,\rangle^{S_N\times(\mathbb{Z}_2)^N}$. As in the bosonic case, one is then faced with the problem of equating the two continuous parametrizations of the partially discretized genus two moduli space, one in terms of the elliptic Prym modulus $\Pi = r\,\tau^{\#}$ and the other in terms of the branch point location $z \in \mathbb{T}^2$. This can again be done explicitly for the degree two contribution to (4.4.17), corresponding to double covers of the torus $\mathbb{T}^2$, and used to compute local spin twist field correlation functions explicitly in each twisted sector of the $\mathbb{R}^8 \wr (\mathbb{Z}_2)^3$ permutation orbifold.

The $N = 2$ contribution to (4.4.17) is given by

$$\mathcal{F}_2^{\rm ferm}\big(\tau^\bullet\big) = -\frac{g_s^2}{4}\,\left|\frac{\tau^\bullet}{64\pi^2\,\alpha'}\right|^4\,\int_\triangle \frac{{\rm d}^2\tau^{\#}}{\big(\tau_2^{\#}\big)^4}\,\left|\frac{\mathcal{C}\big(\tau_0(\tau^\bullet,\tau^{\#})\big)}{\Psi_{10}\big(\tau_0(\tau^\bullet,\tau^{\#})\big)}\right|^2 \tag{4.4.18}$$

where

$$\mathcal{C} = \Xi_6(\boldsymbol{\delta}_7)\,\vartheta_7 + \Xi_6(\boldsymbol{\delta}_8)\,\vartheta_8 + \Xi_6(\boldsymbol{\delta}_9)\,\vartheta_9 + \Xi_6(\boldsymbol{\delta}_0)\,\vartheta_0\ . \tag{4.4.19}$$

We will begin by simplifying the elliptic function (4.4.19) using the decomposition (4.1.54) for $N = 2$. We use the notation of Section 4.2.2 throughout. To simplify the formulae somewhat, we momentarily omit the overall factor of $1/2\,\sqrt{-{\rm i}\,\tau^{\#}}$ in (4.2.11) and reinstate it at the end of the calculation. For reference, let us tabulate the ten reduced even genus two theta constants according to the spin structures (4.4.15) as

$$\begin{array}{c|c|c|c|c}
\boldsymbol{\delta}_1 & \boldsymbol{\delta}_2 & \boldsymbol{\delta}_3 & \boldsymbol{\delta}_4 & \boldsymbol{\delta}_5 \\
\theta_3^\bullet\,\theta_3^{\#} + \theta_4^\bullet\,\theta_4^{\#} & \theta_3^\bullet\,\theta_3^{\#} - \theta_4^\bullet\,\theta_4^{\#} & \theta_4^\bullet\,\theta_3^{\#} + \theta_3^\bullet\,\theta_4^{\#} & \theta_4^\bullet\,\theta_3^{\#} - \theta_3^\bullet\,\theta_4^{\#} & 2\,\tilde\theta_3^\bullet\,\tilde\theta_3^{\#} \\
\hline
\boldsymbol{\delta}_6 & \boldsymbol{\delta}_7 & \boldsymbol{\delta}_8 & \boldsymbol{\delta}_9 & \boldsymbol{\delta}_0 \\
2\,\tilde\theta_3^\bullet\,\tilde\theta_3^{\#} & \theta_2^{\#}\,\theta_2^\bullet & \theta_2^{\#}\,\theta_2^\bullet & 2\,\tilde\theta_1^\bullet\,\tilde\theta_1^{\#} & -2\,{\rm i}\,\tilde\theta_1^{\#}\,\tilde\theta_1^\bullet
\end{array}\ . \tag{4.4.20}$$

Since one has the equalities $\vartheta_5 = \vartheta_6$, $\vartheta_7 = \vartheta_8$ and $\vartheta_9 = \vartheta_0$ for the given reduction, we immediately find that $\Xi_6(\boldsymbol{\delta}_7) = \Xi_6(\boldsymbol{\delta}_8)$ and $\Xi_6(\boldsymbol{\delta}_9) = \Xi_6(\boldsymbol{\delta}_0)$.

After some elementary manipulations we can bring (4.4.19) into the form

$$\begin{aligned}
\mathcal{C} \;=\;& 2^8\ \theta_2^{\bullet\,4}\,\theta_2^{\#\,4}\,\theta_3^\bullet\,\theta_4^\bullet\,\theta_3^{\#}\,\theta_4^{\#}\,\big(\theta_3^{\bullet\,2} - \theta_4^{\bullet\,2}\big)\,\big(\theta_3^{\#\,2} - \theta_4^{\#\,2}\big)\,\tilde\theta_3^{\bullet\,4}\,\tilde\theta_3^{\#\,4} \\
&\qquad \times \Big[\theta_3^{\bullet\,2}\,\theta_4^{\bullet\,2}\,\big(\theta_3^{\#\,2} - \theta_4^{\#\,2}\big)^2 + \theta_3^{\#\,2}\,\theta_4^{\#\,2}\,\big(\theta_3^{\bullet\,2} - \theta_4^{\bullet\,2}\big)^2\Big] \\
&+ 2^{10}\ \theta_2^{\bullet\,8}\,\theta_2^{\#\,8}\,\tilde\theta_1^{\bullet\,8}\,\tilde\theta_1^{\#\,8} + 2^{11}\ \theta_2^{\bullet\,4}\,\theta_3^{\bullet\,2}\,\theta_4^{\bullet\,2}\,\theta_2^{\#\,4}\,\theta_3^{\#\,2}\,\theta_4^{\#\,2}\,\tilde\theta_1^{\bullet\,4}\,\tilde\theta_3^{\bullet\,4}\,\tilde\theta_1^{\#\,4}\,\tilde\theta_3^{\#\,4} \\
&- 2^9\ \tilde\theta_1^{\bullet\,4}\,\tilde\theta_3^{\bullet\,4}\,\tilde\theta_1^{\#\,4}\,\tilde\theta_3^{\#\,4}\,\theta_2^{\bullet\,4}\,\theta_2^{\#\,4}\,\big(\theta_3^{\bullet\,4} - \theta_4^{\bullet\,4}\big)\,\big(\theta_3^{\#\,4} - \theta_4^{\#\,4}\big)\ .
\end{aligned} \tag{4.4.21}$$

We can now proceed as in Section 4.2.2 by doubling the modulus of the theta functions. In addition to the identities displayed in (4.2.20), we will also require the doubling identities

$$\begin{aligned} \theta_1(z|\tau)\,\theta_2(z|\tau) &= \theta_1(2z|2\tau)\,\theta_4(0|2\tau)\;, \\ \theta_3(z|\tau)\,\theta_4(z|\tau) &= \theta_4(2z|2\tau)\,\theta_4(0|2\tau) \end{aligned} \tag{4.4.22}$$

with $z=\frac14$. We may then take into account that the theta functions with argument $z=\frac14$ satisfy $\tilde\theta_3^\bullet=\tilde\theta_4^\bullet$ and $\tilde\theta_1^\bullet=-\tilde\theta_2^\bullet$, and analogously for $\tilde\theta_i^\#$. The calculation is neither difficult nor illuminating, and the result is

$$\mathcal{C}=\frac{2}{\left(\tau^\#\right)^8}\;\bar\theta_2^{\bullet\;8}\,\bar\theta_3^{\bullet\;4}\,\bar\theta_4^{\bullet\;4}\,\bar\theta_2^{\#\;8}\,\bar\theta_3^{\#\;4}\,\bar\theta_4^{\#\;4}\;, \tag{4.4.23}$$

where we have inserted back the factor $\left(1/2\,\sqrt{-\,\mathrm{i}\,\tau^\#}\,\right)^{16}$ and the bar stands for doubled modulus as in Section 4.2.2.

Let us now perform a modular S transformation (4.2.23) on the modulus of both types of theta functions in (4.4.23). Then the final result for the numerator of the integrand in (4.4.35) reads

$$\left|\mathcal{C}\big(\tau_0(\tau^\bullet,\tau^\#)\big)\right|^2=2^{34}\left|\left(\tau^\bullet\right)^{16}\eta\left(2\tau^\#\right)^{24}\eta\left(\tau^\bullet\right)^{24}\theta_4\left(2\tau^\#\right)^8\theta_4\left(\tau^\bullet\right)^8\right|\;. \tag{4.4.24}$$

Substituting (4.4.24) along with (4.2.24) into (4.4.35), and using the abstruse identity (4.2.21), we arrive at the final form for the supersymmetric two-loop DLCQ free energy given by

$$\begin{aligned} \mathcal{F}_2^{\rm ferm}\left(\tau^\bullet\right) &= -\,\frac{16\,g_s^2}{\left(\pi^2\,\alpha'\right)^4}\;\left|\theta_4\left(\tau^\bullet\right)\right|^8 \\ &\quad\times\int_\triangle\frac{\mathrm{d}^2\tau^\#}{\left(\tau_2^\#\right)^4}\;\left|\frac{\theta_4\left(2\tau^\#\right)^2}{\theta_3\left(\tau^\bullet\right)^4\theta_4\left(2\tau^\#\right)^4-\theta_4\left(\tau^\bullet\right)^4\theta_3\left(2\tau^\#\right)^4}\right|^4\;. \end{aligned} \tag{4.4.25}$$

Analogously to the bosonic case of Section 4.2.5, this integral should be matched to the worldsheet averaged two-point correlation function of spin twist field operators $\Lambda(z)=\Lambda_{12}(z)$ in the $\mathbb{R}^8\wr(\mathbb{Z}_2)^3$ permutation orbifold given by

$$\mathcal{F}_2^{\rm ferm}\left(\tau^\bullet\right)=\frac{4\lambda^2}{\tau_2^\bullet\,\mu(0)}\int_{\mathbb{T}}^2\mathrm{d}\mu(z)\;\left\langle\Lambda(z)\,\Lambda(0)\right\rangle^{(\mathbb{Z}_2)^3}\;. \tag{4.4.26}$$

We recall that, by modular invariance at genus two, the branch point integration in (4.4.26) projects all contributions to the correlation function onto the trivial twist sector $(\varepsilon,\delta)=(0,0)$, so that the local integrand that we can read off from (4.4.26) is $4\cdot\frac{1}{2^3}\,\langle\Lambda(z)\,\Lambda(0)\rangle_{0,0}^{(\mathbb{Z}_2)^3}$. We substitute (4.2.69) with $d=8$ and (4.2.70), and recall that the Prym modulus is given by $\Pi=\Pi_{0,0}=2\tau^{\#}$. The crucial observation is that the bosonic contribution to (4.4.25) involving the difference of theta functions is identical to that of the purely bosonic case (4.2.25), due to the universal dimension independent contribution of the modular form $\Psi_{10}(\tau)$ to the bosonic genus two partition function (4.1.43). We can therefore use the same calculation of the Jacobian $|\mathrm{d}\tau^{\#}/\mathrm{d}z|^2$ that was carried out in Section 4.2.5, wherein it was shown that

$$\begin{aligned}&\left|\frac{\mathrm{d}\tau^{\#}}{\mathrm{d}z}\right|^2\,\Big|\theta_3\big(\tau^\bullet\big)^4\,\theta_4\big(2\tau^{\#}\big)^4-\theta_4\big(\tau^\bullet\big)^4\,\theta_3\big(2\tau^{\#}\big)^4\Big|^{-4} \\ &\qquad = \frac{1}{2^{21}}\left|\frac{\eta(\Pi)^4\,\eta\big(\tau^\bullet\big)^{-1}}{\theta_1\big(z\,|\,\tau^\bullet\big)}\right|^6\,\left|\frac{\theta\big[{a\atop b}\big]\big(\frac{z}{2}\,|\,\tau^\bullet\big)\,\theta\big[{a\atop b}\big]\big(0\,|\,\tau^\bullet\big)}{\theta\big[{a\atop b}\big]\big(0\,|\,\Pi\big)^2}\right|^{24}\end{aligned} \tag{4.4.27}$$

for an arbitrary fixed characteristic $(a,b)\neq(1,1)$. Using the identity (4.2.54) and recalling the definition of the prime form (4.2.52), after a little algebra we can use (4.4.25)–(4.4.27) to compute

$$\begin{aligned}\big\langle\Lambda(z)\,\Lambda(0)\big\rangle_{0,0}^{(\mathbb{Z}_2)^3} &= \hat{\mathfrak{z}}\big(\tau^\bullet\big)^8\,\big|E(z)\big|^{-6}\,\big|64\tau^\bullet\,\eta(\Pi)^3\,\theta_4(\Pi)\big|^8 \\ &\quad\times\left|\frac{\theta\big[{a\atop b}\big]\big(\frac{z}{2}\,|\,\tau^\bullet\big)^3\,\theta\big[{a\atop b}\big]\big(0\,|\,\tau^\bullet\big)^3}{\theta\big[{a\atop b}\big]\big(0\,|\,\Pi\big)^6}\right|^8\end{aligned} \tag{4.4.28}$$

where

$$\hat{\mathfrak{z}}(\tau)=\sqrt{\frac{4\pi^2\,\alpha'}{\tau_2}}\,\frac{1}{\big|\eta(\tau)\big|^2}\,\left|\frac{\theta_4(\tau)}{\eta(\tau)}\right| \tag{4.4.29}$$

is the one-loop, first quantized partition function of the Green-Schwarz superstring in $\mathbb{R}$ evaluated with the genus one spin structure $\big[{0\atop 1}\big]$.

We can generate from (4.4.28) the contribution of a generic twisted sector $(\varepsilon,\delta)\in(\mathbb{Z}/2\mathbb{Z})^2$ to the spin twist field correlation function by using a crossing transformation $z\mapsto z+\delta+\varepsilon\,\tau^\bullet$ and the corresponding twisted Prym modulus (4.2.46), along with the transformation formula for Jacobi elliptic functions given by

$$\theta\binom{a}{b}\big(z+\delta+\varepsilon\,\tau^\bullet\,\big|\,\tau^\bullet\big) = \exp\big(-\tfrac{\pi\,\mathrm{i}}{4}\,\varepsilon^2\,\tau^\bullet-\pi\,\mathrm{i}\,\varepsilon\,z-\tfrac{\pi\,\mathrm{i}}{2}\,(b+\delta)\,\varepsilon\big)\ \ \theta\binom{a+\varepsilon}{b+\delta}\big(z\,\big|\,\tau^\bullet\big) \tag{4.4.30}$$

which is valid for arbitrary $a,b\in\mathbb{Q}$ and $\varepsilon,\delta\in\mathbb{Q}$. In fact, the z-dependence of the correlation function (4.4.28) is identical to that of Section 4.2.4 (up to an overall power), and hence the twisted sector two-point function is an appropriate supersymmetric completion of the bosonic correlation function (4.2.50) (with $R=\infty$ and $d=8$). The final result is

$$\big\langle\Lambda(z)\,\Lambda(0)\big\rangle_{\varepsilon,\delta}^{(\mathbb{Z}_2)^3} = \mathfrak{z}\big(\tau^\bullet\big)^8\,\big|\hat c\binom{\varepsilon}{\delta}\big|^{-16}\ , \tag{4.4.31}$$

where

$$\hat c\binom{\varepsilon}{\delta} = \frac{c\binom{\varepsilon}{\delta}^3}{8\,\sqrt{\tau^\bullet\,\eta(\Pi_{\varepsilon,\delta})^3\,\theta_4(\Pi_{\varepsilon,\delta})}} \tag{4.4.32}$$

and the twisted bosonic determinant $c\binom{\varepsilon}{\delta}$ is given by (4.2.51). The cubic power in the supersymmetric twisted determinant (4.4.32) reflects the fact that the effective twist group of the supersymmetric permutation orbifold is $(\mathbb{Z}_2)^3$.

4.4.3 Heterotic DLCQ Strings

Finally, we come to the thermodynamic, genus two DLCQ free energy $F_{\rm het}^{(2)}(\tau^\bullet,\kappa)$ for heterotic strings with heterotic gauge group $\hat{\mathcal{G}}=Spin(32)/\mathbb{Z}_2$ or $\hat{\mathcal{G}}=E_8\times E_8$. The holomorphic sector consists of the usual chiral superstring contribution at genus two. In the antiholomorphic sector, the non-compact bosons produce the usual antichiral bosonic contribution, while the compactified bosonic fields produce an instanton sum over the root lattice of $\hat{\mathcal{G}}$. The latter contribution yields a theta function of the root lattice which is the unique genus two modular form of weight eight given by

$$\Psi_8(\tau) = \sum_{i=0}^{9}\Theta(\boldsymbol{\delta}_i)(\tau)^{16}\ . \tag{4.4.33}$$

In the notation of Section 4.4.2 above, one then has (3.4.4)

$$F_{\rm het}^{(2)}\big(\tau^\bullet\,,\,\kappa\big) = \frac{g_s^2}{8}\,\left|\frac{\tau^\bullet}{2048\pi^4\,(\alpha'\,)^2}\right|^4\,\sum_{N=2}^{\infty}\frac{\kappa^N}{N}\,\sum_{\substack{r\,m=N\\ m\ \mathrm{odd}}}\frac{1}{m^4}\,\sum_{\substack{s,t\in\mathbb{Z}/r\,\mathbb{Z}\\ t\neq0}}\,\int_\triangle\frac{\mathrm{d}^2\tau^\#}{\big(\tau_2^\#\big)^4}\,\frac{\overline{\Psi_8(\tau)}}{\big|\Psi_{10}(\tau)\big|^2}$$

$$\times \Big(\Xi_6(\boldsymbol{\delta}_7)(\tau)\, \Theta(\boldsymbol{\delta}_7)(\tau)^4 + \Xi_6(\boldsymbol{\delta}_8)(\tau)\, \Theta(\boldsymbol{\delta}_8)(\tau)^4$$

$$+ \Xi_6(\boldsymbol{\delta}_9)(\tau)\, \Theta(\boldsymbol{\delta}_9)(\tau)^4 + \Xi_6(\boldsymbol{\delta}_0)(\tau)\, \Theta(\boldsymbol{\delta}_0)(\tau)^4 \Big) \ . \tag{4.4.34}$$

Again we deal explicitly only with the contribution of double covers to the formula (4.4.34), which is given by

$$\mathcal{F}_2^{\rm het}\big(\tau^\bullet\big) = \frac{g_s^2}{8}\, \left|\frac{\tau^\bullet}{2048\pi^4\,(\alpha'\,)^2}\right|^4\, \int_\triangle\, \frac{{\rm d}^2\tau^\#}{\big(\tau_2^\#\big)^4}\, \frac{\overline{\Psi_8\big(\tau_0(\tau^\bullet,\tau^\#)\big)}\ \mathcal{C}\big(\tau_0(\tau^\bullet,\tau^\#)\big)}{\big|\Psi_{10}\big(\tau_0(\tau^\bullet,\tau^\#)\big)\big|^2} \tag{4.4.35}$$

where from (4.4.24) one has

$$\mathcal{C}\big(\tau_0(\tau^\bullet,\tau^\#)\big) = 2^{17}\, \big(\tau^\bullet\big)^8\, \eta\big(2\tau^\#\big)^{12}\, \eta\big(\tau^\bullet\big)^{12}\, \theta_4\big(2\tau^\#\big)^4\, \theta_4\big(\tau^\bullet\big)^4 \ . \tag{4.4.36}$$

To simplify the combination of elliptic functions arising in the genus two modular form (4.4.33), we follow the same steps as in the bosonic and supersymmetric calculations. Namely, we expand the terms in the sum over even genus two spin structures in (4.4.33) using the table (4.4.20), transform it to a form that is suitable for doubling the moduli of the Jacobi theta functions, write the doubling identities, and then make an elliptic S transformation. The final result is again conveniently written in terms of theta functions of moduli $2\tau^\#$ and $\tau^\bullet$ as

$$\Psi_8\big(\tau_0(\tau^\bullet,\tau^\#)\big) = 2^{10}\, \big(\tau^\bullet\big)^8\, \theta_4\big(\tau^\bullet\big)^{16}\, \theta_4\big(2\tau^\#\big)^{16}\ P_{\hat{\mathcal{G}}}\Big(\tfrac{\theta_3(\tau^\bullet)^4}{\theta_4(\tau^\bullet)^4}\,,\ \tfrac{\theta_3(2\tau^\#)^4}{\theta_4(2\tau^\#)^4}\Big)\ , \tag{4.4.37}$$

where $P_{\hat{\mathcal{G}}}(x,y)$ is the symmetric polynomial defined by

$$\begin{aligned} P_{\hat{\mathcal{G}}}(x,y) \ &= \ 256\, \big(x^4\, y^4 + 1\big) - 512\, \big(x^4\, y^3 + x^3\, y^4 + x + y\big) + 1984\, \big(x^3\, y^3 + x\, y\big) \\ &\quad + 288\, \big(x^4\, x^2 + x^2\, y^4 + x^2 + y^2\big) - 2016\, \big(x^3\, y^2 + x^2\, y^3 + x^2\, y + x\, y^2\big) \\ &\quad + x^4 + y^4 + 604\, \big(x^3\, y + x\, y^3\big) + 3654\, x^2\, y^2 \ . \end{aligned} \tag{4.4.38}$$

Substituting (4.4.36) and (4.4.37), along with (4.2.24) and the abstruse identity (4.2.21), we find that the heterotic DLCQ free energy is given by

$$\mathcal{F}_2^{\rm het}\big(\tau^\bullet\big) = \frac{g_s^2}{64}\, \left|\frac{\theta_4\big(\tau^\bullet\big)}{4\pi^2\,\alpha'}\right|^8\, \left(\frac{\theta_4\big(-\overline{\tau^\bullet}\,\big)}{\eta\big(-\overline{\tau^\bullet}\,\big)}\right)^{12}$$

$$\times \int_\triangle\, \frac{{\rm d}^2\tau^\#}{\big(\tau_2^\#\big)^4}\, \left|\frac{\theta_4\big(2\tau^\#\big)^2}{\theta_3\big(\tau^\bullet\big)^4\,\theta_4\big(2\tau^\#\big)^4 - \theta_4\big(\tau^\bullet\big)^4\,\theta_3\big(2\tau^\#\big)^4}\right|^4$$

$$\times \left(\frac{\theta_4\big(-2\,\overline{\tau^{\#}}\,\big)}{\eta\big(-2\,\overline{\tau^{\#}}\,\big)}\right)^{12} P_{\hat{\mathcal{G}}}\Big(\tfrac{\theta_3(-\overline{\tau^{\bullet}}\,)^4}{\theta_4(-\overline{\tau^{\bullet}}\,)^4}\,,\,\tfrac{\theta_3(-2\,\overline{\tau^{\#}}\,)^4}{\theta_4(-2\,\overline{\tau^{\#}}\,)^4}\Big) \tag{4.4.39}$$

where we have used the complex conjugation properties $\overline{\theta_i(\tau)^4} = \theta_i(-\overline{\tau}\,)^4$ and $\overline{\eta(\tau)^{12}} = \eta(-\overline{\tau}\,)^{12}$. We equate (4.4.39) to the integrated two-point correlation function in the heterotic $(\mathbb{R}^8 \times \mathcal{G}) \wr (\mathbb{Z}_2 \ltimes (\mathbb{Z}_2)^2)$ permutation orbifold given by

$$\mathcal{F}_2^{\rm het}(\tau^\bullet) = \frac{4\lambda^2}{\tau_2^\bullet\,\mu(0)} \int_{\mathbb{T}}^2 {\rm d}\mu(z)\ \big\langle (\,\Lambda\otimes\overline{\sigma}\,)(z)\,(\,\Lambda\otimes\overline{\sigma}\,)(0)\big\rangle^{\mathbb{Z}_2\ltimes(\mathbb{Z}_2)^2}\ . \tag{4.4.40}$$

Using the identities (4.2.52), (4.2.54) and (4.4.27) we then arrive at the two-point function of twist fields in the untwisted sector given by

$$\begin{aligned}
&\big\langle (\,\Lambda\otimes\overline{\sigma}\,)(z)\,(\,\Lambda\otimes\overline{\sigma}\,)(0)\big\rangle_{0,0}^{\mathbb{Z}_2\ltimes(\mathbb{Z}_2)^2} \\
&\quad = 32\left(\frac{\hat{\mathfrak{z}}(\tau^\bullet)}{\sqrt{4\pi^2\,\alpha'}}\right)^8 \left(\frac{\theta_4\big(-\overline{\tau^\bullet}\,\big)}{\eta\big(-\overline{\tau^\bullet}\,\big)}\right)^{12} \big|E(z)\big|^{-6}\,\big|8\tau^\bullet\,\eta(\Pi)^3\,\theta_4(\Pi)\big|^8 \qquad (4.4.41)\\
&\qquad \times \left(\frac{\theta_4\big(-\overline{\Pi}\,\big)}{\eta\big(-\overline{\Pi}\,\big)}\right)^{12} P_{\hat{\mathcal{G}}}\Big(\tfrac{\theta_3(-\overline{\tau^\bullet}\,)^4}{\theta_4(-\overline{\tau^\bullet}\,)^4}\,,\,\tfrac{\theta_3(-\overline{\Pi}\,)^4}{\theta_4(-\overline{\Pi}\,)^4}\Big) \left|\frac{\theta\big(\begin{smallmatrix}a\\b\end{smallmatrix}\big)\big(\tfrac{z}{2}\,\big|\,\tau^\bullet\big)^3\,\theta\big(\begin{smallmatrix}a\\b\end{smallmatrix}\big)\big(0\,\big|\,\tau^\bullet\big)^3}{\theta\big(\begin{smallmatrix}a\\b\end{smallmatrix}\big)\big(0\,\big|\,\Pi\big)^6}\right|^8
\end{aligned}$$

with $(a,b) \neq (1,1)$, where $\hat{\mathfrak{z}}(\tau)$ is the supersymmetric partition function (4.4.29).

The structure of the formula (4.4.41) can be understood as follows. Generally, the separating degeneration limit $\tau_{12} \to 0$ of the genus two modular form (4.4.33) factorizes into the unique elliptic modular form of weight eight under $SL(2,\mathbb{Z})$ as

$$\Psi_8(\tau) = \big(\theta_2(\tau_{11})^{16}+\theta_3(\tau_{11})^{16}+\theta_4(\tau_{11})^{16}\big)\,\big(\theta_2(\tau_{22})^{16}+\theta_3(\tau_{22})^{16}+\theta_4(\tau_{22})^{16}\big)+O\big(\tau_{12}^2\big)\ . \tag{4.4.42}$$

For the covering surface $\hat{\Sigma}$, in the homology basis wherein the period matrix is given by (4.2.35) this degeneration limit corresponds to $\Pi \to \tau^\bullet$, or equivalently $z \to 0$. Since the $x \to y$ limit of the symmetric polynomial (4.4.38) factorizes as

$$P_{\hat{\mathcal{G}}}(x,x) = 64\,\big((x-1)^4 + x^4 + 1\big)^2\ , \tag{4.4.43}$$

we see that the $z \to 0$ limit of the two-point function (4.4.41) factors into the one-loop heterotic string partition function on $\mathbb{R}^8$ evaluated with the spin structure $\big(\begin{smallmatrix}0\\1\end{smallmatrix}\big)$ which is given by

$$\hat{\mathfrak{z}}_{\rm het}(\tau) = \left(\frac{4\pi^2\,\alpha'}{\tau_2}\right)^4 \frac{1}{\big|\eta(\tau)\big|^{16}} \left(\frac{\theta_4(\tau)}{\eta(\tau)}\right)^4$$

$$\times \left(\frac{\theta_2\big(-\overline{\tau}\,\big)^{16} + \theta_3\big(-\overline{\tau}\,\big)^{16} + \theta_4\big(-\overline{\tau}\,\big)^{16}}{2\,\eta\big(-\overline{\tau}\,\big)^{16}} \right) \ . \tag{4.4.44}$$

However, in contrast to the bosonic and supersymmetric twist field correlation functions, for distinct branch points the two-point function (4.4.41) does not neatly factor out a component corresponding to the untwisted fluctuation determinant of the heterotic orbifold sigma model. The reason generally is that the effective twist group is now a semi-direct product $S_N \ltimes (\mathbb{Z}_2)^N$ acting on the gauge fermions χ^a. This means that the discrete $(\mathbb{Z}_2)^N$ gauge symmetry acts in the gauge sector together with the monodromy conditions of the permutation orbifold, and a disentanglement of the twisted and untwisted determinants arising from integration over the fermion fields χ in terms of branch point data as previously is not possible.

For example, by applying a crossing transformation to (4.4.41) as before one arrives at the twisted sector two-point functions

$$\begin{aligned} \big\langle(\,\Lambda\otimes\overline{\sigma}\,)(z)\,(\,\Lambda\otimes\overline{\sigma}\,)(0)\big\rangle_{\varepsilon,\delta}^{\mathbb{Z}_2\ltimes(\mathbb{Z}_2)^2} &= 32\,\left(\frac{\hat{\mathfrak{z}}(\tau^\bullet)}{\sqrt{4\pi^2\,\alpha'}}\right)^8\,\big|\hat{c}\big(\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\big)\big|^{-16}\,\left(\frac{\theta_4\big(-\overline{\tau^\bullet}\,\big)}{2\eta\big(-\overline{\tau^\bullet}\,\big)}\right)^{12} \\ &\quad\times\left(\frac{\theta_4\big(-\overline{\Pi_{\varepsilon,\delta}}\,\big)}{2\eta\big(-\overline{\Pi_{\varepsilon,\delta}}\,\big)}\right)^{12}\,P_{\hat{\mathcal{G}}}\Big(\tfrac{\theta_3(-\overline{\tau^\bullet})^4}{\theta_4(-\overline{\tau^\bullet})^4}\,,\,\tfrac{\theta_3(-\overline{\Pi_{\varepsilon,\delta}})^4}{\theta_4(-\overline{\Pi_{\varepsilon,\delta}})^4}\Big) \end{aligned} \tag{4.4.45}$$

with the supersymmetric twisted determinant $\hat{c}\big(\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\big)$ given by (4.4.32). The extra gauge symmetry is implemented by $O(N)$ vector reflections of χ^a and holonomies of the corresponding flat real line bundles $L_{\boldsymbol{\delta}} \to \mathbb{T}^2$. The latter phases correspond to $\mathbb{Z}_2$-valued Wilson lines which break the spacetime heterotic gauge group $\hat{\mathcal{G}}$ to $\mathcal{G} = SO(16)\times SO(16)$. They yield the extra GSO projection required to match to the spectrum of the free $E_8 \times E_8$ heterotic string [38, 41, 43] and to light-cone heterotic string field theory.

Appendix A: Moduli Space for the Poincaré Normal Form

In this apendix we will sketch the computation of the two-loop free energy from the fully reduced Poincaré normal form (3.1.27). This is done for the sake of completeness and because it provides some interesting alternative characterizations of the genus two Hurwitz moduli space which may be of independent interest. As we will see, the free energy in this case cannot be made as explicit as in the main text, but the same reduction features do carry through nonetheless.

A.1 Reduced Moduli

The genus two Poincaré normal form is given by

$$\mathsf{P} = r \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & s & t & 0 \end{pmatrix} . \tag{A-1.1}$$

The matrix T which appears in the Frobenius normal form (3.1.26) can be absorbed into the period matrix as in (3.2.14) but the symplectic unimodular matrix

$$\mathsf{S} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \tag{A-1.2}$$

which acts on the base torus $\mathbb{T}^2_{\mathrm{i}\nu}$ as a modular transformation, remains [49, 51]. This is one of the reasons why the full reduction is undesirable, as both the moduli space and the GSO projection depend explicitly on the four integers a, b, c, d which are functions of the parameters r, s and t. We have to keep S explicitly in all of our calculations, and then sum over all the corresponding $SL(2, \mathbb{Z})$ modular transformations of the base. On the other hand, the full reduction to (A-1.1) leads to a somewhat simpler decomposition of genus two theta-constants into elliptic theta-functions [52].

Given the homology matrix (3.1.26), we can rewrite the constraint equation (3.2.2) using (A-1.1) and (A-1.2) to get

$$\mathsf{H}^\top(\mathbb{1}_2,\Omega) = (1,\,\mathrm{i}\,\nu)\,\mathsf{S}\,\mathsf{P}\,\mathsf{T} = (1,\,\mathrm{i}\,\nu)\begin{pmatrix} r\,a & r\,b\,s & r\,b\,t & 0 \\ r\,c & r\,d\,s & r\,d\,t & 0 \end{pmatrix}\mathsf{T}\,. \tag{A-1.3}$$

In order to factorize the genus two theta-constants in terms of elliptic functions as in Section 3.2.5, the period matrix must have rational-valued off-diagonal elements. This will happen if we modify (A-1.3) by multiplying P with the intersection form $-\mathsf{J}_2 = (\mathsf{J}_2)^{-1}$ to obtain

$$\mathsf{H}^\top(\mathbb{1}_2,\Omega) = (1,\,\mathrm{i}\,\nu)\begin{pmatrix} r\,b\,t & 0 & -r\,a & -r\,b\,s \\ r\,d\,t & 0 & -r\,c & -r\,d\,s \end{pmatrix}\mathsf{J}_2\,\mathsf{T}\,. \tag{A-1.4}$$

The matrix $\mathsf{J}_2\,\mathsf{T} \in Sp(4,\mathbb{Z})$ is invertible. The inverse $(\mathsf{J}_2\,\mathsf{T})^{-1}$ acts on the left-hand side of (A-1.4) as a modular transformation on the period matrix Ω and on the pullback matrix H as in (3.2.14).

We can now solve the constraint equation for the period matrix by first computing

$$\mathsf{H} = (1,\,\mathrm{i}\,\nu)\begin{pmatrix} r\,b\,t & 0 \\ r\,d\,t & 0 \end{pmatrix} = r\,t\,(b+d\,\mathrm{i}\,\nu,0) \tag{A-1.5}$$

to get

$$\mathsf{H}\,\Omega = r\,t\,(b+d\,\mathrm{i}\,\nu,0)\begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12} & \Omega_{22} \end{pmatrix} = (1,\,\mathrm{i}\,\nu)\begin{pmatrix} -r\,a & -r\,b\,s \\ -r\,c & -r\,d\,s \end{pmatrix}\,. \tag{A-1.6}$$

After a $\mathbb{Z}_2$ reflection, we thereby find

$$\Omega = \begin{pmatrix} \frac{\tau_\nu}{\mu\,s} & \frac{1}{\mu} \\ \frac{1}{\mu} & \tau \end{pmatrix} \tag{A-1.7}$$

where the integer μ is defined through $t = \mu\,s$ and is related to the degree N of the cover by $N = r^2\,\mu\,s$. As before $\tau := \Omega_{22} \in \mathcal{H}_1$ parametrizes an auxilliary torus $\mathbb{T}^2_\tau$, while

$$\tau_\nu = \frac{a + c\,(\mathrm{i}\,\nu)}{b + d\,(\mathrm{i}\,\nu)} \tag{A-1.8}$$

labels a torus $\mathbb{T}^2_{\tau_\nu}$ in the same elliptic modular orbit as the base $\mathbb{T}^2_{{\rm i}\,\nu}$. This is in contrast to the partial reduction carried out in the main text, in which the discretely parametrized tori were unramified covers Σ_1 over the base.

A.2 Residual Modular Group

The Poincaré normal form is obtained through a change of homology basis of Σ_2. The residual modular group $\mathcal{G}$ is the subgroup of $Sp(4,\mathbb{Z})$ which preserves the form (A-1.4). It consists of integral matrices of the form

$$\begin{pmatrix} 1 & -\mu\,\alpha & \alpha & \beta \\ 0 & 1-\mu\,\gamma & \gamma & \delta \\ 0 & 0 & 1 & 0 \\ 0 & -\mu^2\,\alpha & \mu\,\alpha & 1+\mu\,\beta \end{pmatrix} \tag{A-2.1}$$

which obey the $Sp(4,\mathbb{Z})$ condition

$$\gamma-\beta=\mu\,(\alpha\,\delta-\beta\,\gamma)\ . \tag{A-2.2}$$

The extended fundamental domain $\mathcal{F}_2' = \mathcal{H}_2/\mathcal{G}$ is then constructed as the quotient of the Siegal upper half-plane by the residual modular group.

By using the $Sp(4,\mathbb{Z})$ transformation rule (3.1.24), one finds that under the action of the residual modular group the period matrix elements transform as

$$\begin{aligned} \Omega_{22} &\longmapsto \frac{\Omega_{22}\,(1-\mu\,\gamma)+\delta}{1+\mu\,\beta-\mu^2\,\alpha\,\Omega_{22}}\ , \\ \Omega_{12} &\longmapsto \frac{\Omega_{12}-\mu\,\alpha\,\Omega_{22}+\beta}{1+\mu\,\beta-\mu^2\,\alpha\,\Omega_{22}}\ , \\ \Omega_{11} &\longmapsto \Omega_{11}+\frac{\alpha\,(\mu\,\Omega_{12}-1)^2}{1+\mu\,\beta-\mu^2\,\alpha\,\Omega_{22}}\ . \end{aligned} \tag{A-2.3}$$

The Möbius transformations of $\tau=\Omega_{22}$ in the first line of (A-2.3) form a congruence subgroup of the elliptic modular group $SL(2,\mathbb{Z})$ defined by

$$\Gamma_{(\mu)} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \ \middle|\ a,d\equiv 1 \bmod \mu\ ,\ c\equiv 0 \bmod \mu^2\ ,\ b\in\mathbb{Z} \right\}\ . \tag{A-2.4}$$

Once the elliptic fundamental domain for $\tau \in \mathcal{H}_1$ is determined, the full genus two fundamental domain $\mathcal{F}_2'$ will follow from the other transformation rules in (A-2.3).

A.3 Moduli Space

We will now construct a fundamental modular domain in the upper complex half-plane $\mathcal{H}_1$ for the action of the congruence subgroup $\Gamma_{(\mu)} \subset SL(2,\mathbb{Z})$. Let

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad , \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{A-3.1}$$

be the standard generators of $SL(2,\mathbb{Z})$. Consider the fundamental domain Δ for the action of $SL(2,\mathbb{Z})$ given by (3.1.40), which is a triangle with one vertex at infinity. The three edges separate Δ from the Möbius images $S \bullet \Delta$, $T \bullet \Delta$ and $T^{-1} \bullet \Delta$. A *Schreier transversal* $\mathfrak{C}_{(\mu)}$ for $\Gamma_{(\mu)}$ in $SL(2,\mathbb{Z})$ with respect to $\{S,T\}$ is a set of right coset representatives $SL(2,\mathbb{Z}) = \bigcup_g \Gamma_{(\mu)} g$ (i.e. $\Gamma_{(\mu)} g \cap \mathfrak{C}_{(\mu)}$ has precisely one element for each $g \in SL(2,\mathbb{Z})$) expressed as words in the generating set $\{S,T\}$ such that each prefix (or initial segment) of an element of $\mathfrak{C}_{(\mu)}$ is also in $\mathfrak{C}_{(\mu)}$. Then the region

$$\mathfrak{C}_{(\mu)} \bullet \Delta = \bigcup_{C \in \mathfrak{C}_{(\mu)}} C \bullet \Delta \tag{A-3.2}$$

is a polygonal fundamental domain for the action of $\Gamma_{(\mu)}$ on $\mathcal{H}_1$. For example, if $S\,T\,S \in \mathfrak{C}_{(\mu)}$, then also $S\,T, S, \mathbb{1}_2 \in \mathfrak{C}_{(\mu)}$. The triangles $(S\,T\,S) \bullet \Delta$ and $(S\,T) \bullet \Delta$ share a common edge, as do $(S\,T) \bullet \Delta$ and $S \bullet \Delta$, and so on.

Since the subgroup $\Gamma_{(\mu)} \subset SL(2,\mathbb{Z})$ has finite index, there are finite Schreier transversals. The group $\Gamma_{(\mu)}$ is the preimage of the subgroup

$$\phi\big(\Gamma_{(\mu)}\big) = \tilde{\Gamma}_{(\mu)} := \left\{ \begin{pmatrix} \mu\,a+1 & b \\ 0 & \mu\,d+1 \end{pmatrix} \ \middle| \ a+d \equiv 0 \bmod \mu \ , \ b \in \mathbb{Z}/\mu^2\,\mathbb{Z} \right\} \tag{A-3.3}$$

of the finite group $SL(2,\mathbb{Z}/\mu^2\,\mathbb{Z})$ under the surjective homomorphism

$$\phi \ : \ SL(2,\mathbb{Z}) \ \longrightarrow \ SL(2,\mathbb{Z}/\mu^2\,\mathbb{Z}) \tag{A-3.4}$$

given by reduction modulo μ^2. The index of $\Gamma_{(\mu)}$ in $SL(2,\mathbb{Z})$ may thereby be computed from

$$\big[SL(2,\mathbb{Z}) : \Gamma_{(\mu)}\big] \ = \ \big[\mathrm{im}(\phi) : \tilde{\Gamma}_{(\mu)} \cap \mathrm{im}(\phi)\big]$$

$$= \big[SL(2,\mathbb{Z}/\mu^2\,\mathbb{Z}) : \tilde{\Gamma}_{(\mu)}\big] \;=\; \frac{\big|SL(2,\mathbb{Z}/\mu^2\,\mathbb{Z})\big|}{\big|\tilde{\Gamma}_{(\mu)}\big|}\;. \qquad \text{(A-3.5)}$$

We now need to work out the orders of the two finite groups $SL(2,\mathbb{Z}/\mu^2\,\mathbb{Z})$ and $\tilde{\Gamma}_{(\mu)}$. The order of $\tilde{\Gamma}_{(\mu)}$ can be easily determined by inspection of its definition (A-3.3) to be $|\tilde{\Gamma}_{(\mu)}| = \mu^3$. The order of $SL(2,\mathbb{Z}/\mu^2\,\mathbb{Z})$ is calculated as follows.

The index of $\Gamma_{(\mu)}$ turns out to depend crucially on the prime factorization of the integer μ. Suppose that $\mu = p_1^{k(1)} \cdots p_t^{k(t)}$ with p_j, $j = 1,\dots,t$ distinct prime numbers and $k(j) > 0$. By the Chinese remainder theorem the corresponding finite group factorizes as

$$SL\big(2,\mathbb{Z}/\mu^2\,\mathbb{Z}\big) = SL\big(2,\mathbb{Z}/p_1^{2k(1)}\,\mathbb{Z}\big) \;\times\; \cdots \;\times\; SL\big(2,\mathbb{Z}/p_t^{2k(t)}\,\mathbb{Z}\big) \qquad \text{(A-3.6)}$$

and its order is given by

$$\big|SL(2,\mathbb{Z}/\mu^2\,\mathbb{Z})\big| = \prod_{j=1}^{t} \big|SL(2,\mathbb{Z}/p_j^{2k(j)}\,\mathbb{Z})\big|\;. \qquad \text{(A-3.7)}$$

It thus suffices to compute the order of $SL(2,\mathbb{Z}/p^{2k}\,\mathbb{Z})$ for p prime and $k > 0$. Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}/p^{2k}\,\mathbb{Z})$. Then $a\,d - b\,c \equiv 1 \mod p^{2k}$. To ensure that the matrix is non-singular, the pair (a,b) must take values in the set

$$\big(\mathbb{Z}/p^{2k}\,\mathbb{Z} \;\times\; \mathbb{Z}/p^{2k}\,\mathbb{Z}\big) \setminus \big(p\,\mathbb{Z}/p^{2k}\,\mathbb{Z} \;\times\; p\,\mathbb{Z}/p^{2k}\,\mathbb{Z}\big)\;. \qquad \text{(A-3.8)}$$

The number of elements in this set is $p^{4k-2}\,(p^2-1)$. The pair (c,d) must be chosen so that p does not divide the determinant. There are $p^{4k-1}\,(p-1)$ such pairs (c,d) for each (a,b). This ensures that the matrix is non-singular. Thus the number of invertible matrices is given by

$$\big|GL(2,\mathbb{Z}/p^{2k}\,\mathbb{Z})\big| = p^{4k-2}\,\big(p^2-1\big)\,p^{4k-1}\,\big(p-1\big)\;. \qquad \text{(A-3.9)}$$

The determinant is a group homomorphism $\det : GL(2,\mathbb{Z}/p^{2k}\,\mathbb{Z}) \to \mathbb{Z}/p^{2k}\,\mathbb{Z}$. It follows that the index of $SL(2,\mathbb{Z}/p^{2k}\,\mathbb{Z})$ in $GL(2,\mathbb{Z}/p^{2k}\mathbb{Z})$ is

$$\big[GL(2,\mathbb{Z}/p^{2k}\,\mathbb{Z}) : SL(2,\mathbb{Z}/p^{2k}\,\mathbb{Z})\big] = \frac{\big|GL(2,\mathbb{Z}/p^{2k}\,\mathbb{Z})\big|}{\big|SL(2,\mathbb{Z}/p^{2k}\,\mathbb{Z})\big|} = p^{2k-1}\,(p-1)\;, \qquad \text{(A-3.10)}$$

which is just the Euler φ-function of the field $\mathbb{Z}/p^{2k}\,\mathbb{Z}$.

By combining all of these results we find finally that the index of $\Gamma_{(\mu)}$ in $SL(2,\mathbb{Z})$ is given by the Euler product expansion

$$\big[SL(2,\mathbb{Z}):\Gamma_{(\mu)}\big]=\mu^3\prod_{\mathrm{primes}\,p|\mu}\left(1-\frac{1}{p^2}\right)~. \tag{A-3.11}$$

We can now build a Schreier tranversal inductively, starting from $\{\mathbb{1}_2\}$. Suppose that we have a set $\mathfrak{C}_k$ of k words, satisfying the suffix condition, which contains at most one representative of any right coset. If k is strictly less than the index (A-3.11), then we can examine the right cosets $\Gamma_{(\mu)}S\,C$, $\Gamma_{(\mu)}T\,C$ and $\Gamma_{(\mu)}T^{-1}\,C$ for each $C\in\mathfrak{C}_k$ until we find one which is different from $\Gamma_{(\mu)}C$ for $C\in\mathfrak{C}_k$. Then add $S\,C$, $T\,C$ or $T^{-1}\,C$ to the list of words to form a new list $\mathfrak{C}_{k+1}$. This process terminates precisely when k is equal to the index (A-3.11), and then $\mathfrak{C}_k=\mathfrak{C}_{(\mu)}$ is the desired Schreier transversal for $\Gamma_{(\mu)}$. For example, when $\mu=2$ the subgroup $\Gamma_{(2)}$ has index 6 and $\mathfrak{C}_{(2)}=\{\mathbb{1}_2,S,S\,T,S\,T^2,S\,T^3,S\,T^2\,S\}$ is a Schreier transversal for $\Gamma_{(2)}$. The corresponding elliptic fundamental domain (A-3.2) is depicted in Figure A-3.1.

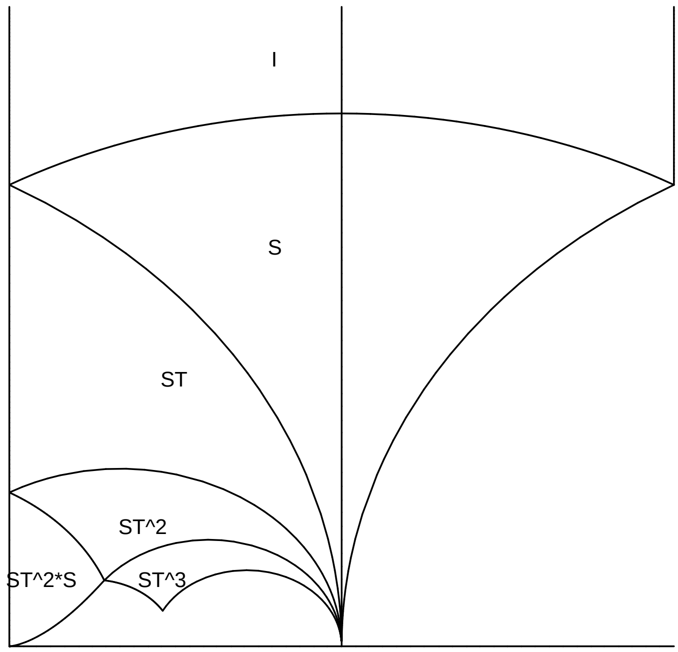


Figure A-3.1: The fundamental domain $\mathfrak{C}_{(2)}\bullet\Delta$ for the action of the congruence subgroup $\Gamma_{(2)}\subset SL(2,\mathbb{Z})$ on $\mathcal{H}_1$.

Using the modular transformations (A-2.3) along with the positivity constraint (3.2.24) on the period matrix, we find that the fundamental domain at genus two for the residual modular group preserving the Poincaré normal form is given by

$$\mathcal{F}_2'(\mu) = \big(\mathfrak{C}_{(\mu)} \bullet \Delta\big) \times \mathbb{C} \times \mathcal{H}_1 \tag{A-3.12}$$

with elements $(\Omega_{22}, \Omega_{12}, \Omega_{11})$. The integers μ, t, r, a, b, c and d are thus unrestricted except for the dependences of a, b, c and d on r, s and t. Because of this dependence and the complexity of the integration region $\tau \in \mathfrak{C}_{(\mu)} \bullet \Delta$, the free energy cannot be made as explicit as those computed in Sections 3.2.7, 3.3.4 and 3.4.

Appendix B: Explicit Form of Ξ_6

In this section we provide the explicit expressions for the modular covariant form $\Xi_6[\delta]$ on $\mathcal{H}_2$ defined in (3.3.14) for the ten even spin structures. Given an even characteristic δ_i, $i = 0, 1, \ldots, 9$, we denote $\vartheta_i(\Omega) := \Theta[\delta_i](\mathbf{0}|\Omega)^4$. By the mirror property [39], there are two equivalent expressions for $\Xi_6[\delta_i](\Omega)$ corresponding to the two triples of odd spin structures used to represent $\delta_i = \nu_{i_1} + \nu_{i_2} + \nu_{i_3} = \nu_{i_4} + \nu_{i_5} + \nu_{i_6}$ for each i. One then has

$$
\begin{aligned}
\Xi_6[\delta_1] &= -\vartheta_4\,\vartheta_5\,\vartheta_8 - \vartheta_2\,\vartheta_6\,\vartheta_9 - \vartheta_3\,\vartheta_7\,\vartheta_0 = -\,\vartheta_4\,\vartheta_7\,\vartheta_6 - \vartheta_3\,\vartheta_8\,\vartheta_9 - \vartheta_2\,\vartheta_5\,\vartheta_0 \ , \\
\Xi_6[\delta_2] &= \vartheta_3\,\vartheta_5\,\vartheta_7 + \vartheta_4\,\vartheta_8\,\vartheta_0 - \vartheta_1\,\vartheta_6\,\vartheta_9 = \vartheta_3\,\vartheta_6\,\vartheta_8 - \vartheta_1\,\vartheta_5\,\vartheta_0 + \vartheta_4\,\vartheta_7\,\vartheta_9 \ , \\
\Xi_6[\delta_3] &= \vartheta_2\,\vartheta_5\,\vartheta_7 - \vartheta_1\,\vartheta_8\,\vartheta_9 + \vartheta_4\,\vartheta_6\,\vartheta_0 = \vartheta_2\,\vartheta_6\,\vartheta_8 + \vartheta_5\,\vartheta_4\,\vartheta_9 - \vartheta_1\,\vartheta_7\,\vartheta_0 \ , \\
\Xi_6[\delta_4] &= -\vartheta_1\,\vartheta_5\,\vartheta_8 + \vartheta_3\,\vartheta_6\,\vartheta_0 + \vartheta_2\,\vartheta_7\,\vartheta_9 = -\,\vartheta_1\,\vartheta_6\,\vartheta_7 + \vartheta_2\,\vartheta_8\,\vartheta_0 + \vartheta_3\,\vartheta_5\,\vartheta_9 \ , \\
\Xi_6[\delta_5] &= \vartheta_2\,\vartheta_3\,\vartheta_7 - \vartheta_1\,\vartheta_4\,\vartheta_6 + \vartheta_6\,\vartheta_9\,\vartheta_0 = -\,\vartheta_1\,\vartheta_2\,\vartheta_0 + \vartheta_3\,\vartheta_4\,\vartheta_9 + \vartheta_6\,\vartheta_7\,\vartheta_8 \ , \\
\Xi_6[\delta_6] &= \vartheta_3\,\vartheta_4\,\vartheta_0 - \vartheta_1\,\vartheta_2\,\vartheta_9 + \vartheta_5\,\vartheta_7\,\vartheta_8 = -\,\vartheta_1\,\vartheta_4\,\vartheta_7 + \vartheta_5\,\vartheta_9\,\vartheta_0 + \vartheta_2\,\vartheta_3\,\vartheta_8 \ , \\
\Xi_6[\delta_7] &= \vartheta_2\,\vartheta_3\,\vartheta_5 + \vartheta_8\,\vartheta_9\,\vartheta_0 - \vartheta_1\,\vartheta_4\,\vartheta_6 = \vartheta_2\,\vartheta_4\,\vartheta_9 - \vartheta_1\,\vartheta_3\,\vartheta_0 + \vartheta_5\,\vartheta_6\,\vartheta_8 \ , \\
\Xi_6[\delta_8] &= \vartheta_7\,\vartheta_9\,\vartheta_0 - \vartheta_1\,\vartheta_4\,\vartheta_5 + \vartheta_2\,\vartheta_3\,\vartheta_6 = -\,\vartheta_1\,\vartheta_3\,\vartheta_9 + \vartheta_2\,\vartheta_4\,\vartheta_0 + \vartheta_5\,\vartheta_6\,\vartheta_7 \ , \\
\Xi_6[\delta_9] &= \vartheta_7\,\vartheta_8\,\vartheta_0 - \vartheta_1\,\vartheta_2\,\vartheta_6 + \vartheta_3\,\vartheta_4\,\vartheta_5 = \vartheta_5\,\vartheta_6\,\vartheta_0 - \vartheta_1\,\vartheta_3\,\vartheta_8 + \vartheta_2\,\vartheta_4\,\vartheta_7 \ , \\
\Xi_6[\delta_0] &= \vartheta_7\,\vartheta_8\,\vartheta_9 + \vartheta_3\,\vartheta_4\,\vartheta_6 - \vartheta_1\,\vartheta_2\,\vartheta_5 = \vartheta_5\,\vartheta_6\,\vartheta_9 + \vartheta_2\,\vartheta_4\,\vartheta_8 - \vartheta_1\,\vartheta_3\,\vartheta_7 \ . \qquad \text{(B-1.1)}
\end{aligned}
$$

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