Appendix A: Moduli Space for the Poincaré Normal Form

In this apendix we will sketch the computation of the two-loop free energy from the fully reduced Poincaré normal form (**??**). This is done for the sake of completeness and because it provides some interesting alternative characterizations of the genus two Hurwitz moduli space which may be of independent interest. As we will see, the free energy in this case cannot be made as explicit as in the main text, but the same reduction features do carry through nonetheless.

A.1 Reduced Moduli

The genus two Poincaré normal form is given by

$$\mathsf{P} = r \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & s & t & 0 \end{pmatrix} . \tag{A-1.1}$$

The matrix T which appears in the Frobenius normal form (**??**) can be absorbed into the period matrix as in (**??**) but the symplectic unimodular matrix

$$\mathsf{S} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \tag{A-1.2}$$

which acts on the base torus $\mathbb{T}^2_{\mathrm{i}\nu}$ as a modular transformation, remains [**?**, **?**]. This is one of the reasons why the full reduction is undesirable, as both the moduli space and the GSO projection depend explicitly on the four integers a, b, c, d which are functions of the parameters r, s and t. We have to keep S explicitly in all of our calculations, and then sum over all the corresponding $SL(2,\mathbb{Z})$ modular transformations of the base. On the other hand, the full reduction to (A-1.1) leads to a somewhat simpler decomposition of genus two theta-constants into elliptic theta-functions [?].

Given the homology matrix (??), we can rewrite the constraint equation (??) using (A-1.1) and (A-1.2) to get

$$\mathsf{H}^\top(\mathbb{1}_2, \Omega) = (1,\, \mathrm{i}\,\nu)\,\mathsf{S}\,\mathsf{P}\,\mathsf{T} = (1,\, \mathrm{i}\,\nu)\begin{pmatrix} r\,a & r\,b\,s & r\,b\,t & 0 \\ r\,c & r\,d\,s & r\,d\,t & 0 \end{pmatrix}\mathsf{T}\,. \tag{A-1.3}$$

In order to factorize the genus two theta-constants in terms of elliptic functions as in Section ??, the period matrix must have rational-valued off-diagonal elements. This will happen if we modify (A-1.3) by multiplying P with the intersection form $-\mathsf{J}_2 = (\mathsf{J}_2)^{-1}$ to obtain

$$\mathsf{H}^\top(\mathbb{1}_2, \Omega) = (1,\, \mathrm{i}\,\nu)\begin{pmatrix} r\,b\,t & 0 & -r\,a & -r\,b\,s \\ r\,d\,t & 0 & -r\,c & -r\,d\,s \end{pmatrix}\mathsf{J}_2\,\mathsf{T}\,. \tag{A-1.4}$$

The matrix $\mathsf{J}_2\,\mathsf{T} \in Sp(4,\mathbb{Z})$ is invertible. The inverse $(\mathsf{J}_2\,\mathsf{T})^{-1}$ acts on the left-hand side of (A-1.4) as a modular transformation on the period matrix Ω and on the pullback matrix H as in (??).

We can now solve the constraint equation for the period matrix by first computing

$$\mathsf{H} = (1,\, \mathrm{i}\,\nu)\begin{pmatrix} r\,b\,t & 0 \\ r\,d\,t & 0 \end{pmatrix} = r\,t\,(b + d\,\mathrm{i}\,\nu, 0) \tag{A-1.5}$$

to get

$$\mathsf{H}\,\Omega = r\,t\,(b + d\,\mathrm{i}\,\nu, 0)\begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12} & \Omega_{22} \end{pmatrix} = (1,\, \mathrm{i}\,\nu)\begin{pmatrix} -r\,a & -r\,b\,s \\ -r\,c & -r\,d\,s \end{pmatrix}\,. \tag{A-1.6}$$

After a $\mathbb{Z}_2$ reflection, we thereby find

$$\Omega = \begin{pmatrix} \frac{\tau_\nu}{\mu\,s} & \frac{1}{\mu} \\ \frac{1}{\mu} & \tau \end{pmatrix} \tag{A-1.7}$$

where the integer μ is defined through $t = \mu\,s$ and is related to the degree N of the cover by $N = r^2\,\mu\,s$. As before $\tau := \Omega_{22} \in \mathcal{H}_1$ parametrizes an auxilliary torus $\mathbb{T}^2_\tau$, while

$$\tau_\nu = \frac{a + c\,(\mathrm{i}\,\nu)}{b + d\,(\mathrm{i}\,\nu)} \tag{A-1.8}$$

labels a torus $\mathbb{T}^2_{\tau_\nu}$ in the same elliptic modular orbit as the base $\mathbb{T}^2_{\mathrm{i}\,\nu}$. This is in contrast to the partial reduction carried out in the main text, in which the discretely parametrized tori were unramified covers Σ_1 over the base.

A.2 Residual Modular Group

The Poincaré normal form is obtained through a change of homology basis of Σ_2. The residual modular group $\mathcal{G}$ is the subgroup of $Sp(4,\mathbb{Z})$ which preserves the form (A-1.4). It consists of integral matrices of the form

$$\begin{pmatrix} 1 & -\mu\,\alpha & \alpha & \beta \\ 0 & 1-\mu\,\gamma & \gamma & \delta \\ 0 & 0 & 1 & 0 \\ 0 & -\mu^2\,\alpha & \mu\,\alpha & 1+\mu\,\beta \end{pmatrix} \tag{A-2.1}$$

which obey the $Sp(4,\mathbb{Z})$ condition

$$\gamma-\beta=\mu\,(\alpha\,\delta-\beta\,\gamma)\ . \tag{A-2.2}$$

The extended fundamental domain $\mathcal{F}_2'=\mathcal{H}_2/\mathcal{G}$ is then constructed as the quotient of the Siegal upper half-plane by the residual modular group.

By using the $Sp(4,\mathbb{Z})$ transformation rule (**??**), one finds that under the action of the residual modular group the period matrix elements transform as

$$\begin{aligned} \Omega_{22} &\longmapsto \frac{\Omega_{22}\,(1-\mu\,\gamma)+\delta}{1+\mu\,\beta-\mu^2\,\alpha\,\Omega_{22}}\ , \\ \Omega_{12} &\longmapsto \frac{\Omega_{12}-\mu\,\alpha\,\Omega_{22}+\beta}{1+\mu\,\beta-\mu^2\,\alpha\,\Omega_{22}}\ , \\ \Omega_{11} &\longmapsto \Omega_{11}+\frac{\alpha\,(\mu\,\Omega_{12}-1)^2}{1+\mu\,\beta-\mu^2\,\alpha\,\Omega_{22}}\ . \end{aligned} \tag{A-2.3}$$

The Möbius transformations of $\tau=\Omega_{22}$ in the first line of (A-2.3) form a congruence subgroup of the elliptic modular group $SL(2,\mathbb{Z})$ defined by

$$\Gamma_{(\mu)}=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL(2,\mathbb{Z})\ \middle|\ a,d\equiv 1 \bmod \mu\ ,\ c\equiv 0 \bmod \mu^2\ ,\ b\in\mathbb{Z}\right\}\ . \tag{A-2.4}$$

Once the elliptic fundamental domain for $\tau \in \mathcal{H}_1$ is determined, the full genus two fundamental domain $\mathcal{F}_2'$ will follow from the other transformation rules in (A-2.3).

A.3 Moduli Space

We will now construct a fundamental modular domain in the upper complex half-plane $\mathcal{H}_1$ for the action of the congruence subgroup $\Gamma_{(\mu)} \subset SL(2,\mathbb{Z})$. Let

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ , \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{A-3.1}$$

be the standard generators of $SL(2,\mathbb{Z})$. Consider the fundamental domain Δ for the action of $SL(2,\mathbb{Z})$ given by (**??**), which is a triangle with one vertex at infinity. The three edges separate Δ from the Möbius images $S \bullet \Delta$, $T \bullet \Delta$ and $T^{-1} \bullet \Delta$. A *Schreier transversal* $\mathfrak{C}_{(\mu)}$ for $\Gamma_{(\mu)}$ in $SL(2,\mathbb{Z})$ with respect to $\{S,T\}$ is a set of right coset representatives $SL(2,\mathbb{Z}) = \bigcup_g \Gamma_{(\mu)} g$ (i.e. $\Gamma_{(\mu)} g \cap \mathfrak{C}_{(\mu)}$ has precisely one element for each $g \in SL(2,\mathbb{Z})$) expressed as words in the generating set $\{S,T\}$ such that each prefix (or initial segment) of an element of $\mathfrak{C}_{(\mu)}$ is also in $\mathfrak{C}_{(\mu)}$. Then the region

$$\mathfrak{C}_{(\mu)} \bullet \Delta = \bigcup_{C \in \mathfrak{C}_{(\mu)}} C \bullet \Delta \tag{A-3.2}$$

is a polygonal fundamental domain for the action of $\Gamma_{(\mu)}$ on $\mathcal{H}_1$. For example, if $S\,T\,S \in \mathfrak{C}_{(\mu)}$, then also $S\,T, S, \mathbb{1}_2 \in \mathfrak{C}_{(\mu)}$. The triangles $(S\,T\,S) \bullet \Delta$ and $(S\,T) \bullet \Delta$ share a common edge, as do $(S\,T) \bullet \Delta$ and $S \bullet \Delta$, and so on.

Since the subgroup $\Gamma_{(\mu)} \subset SL(2,\mathbb{Z})$ has finite index, there are finite Schreier transversals. The group $\Gamma_{(\mu)}$ is the preimage of the subgroup

$$\phi\big(\Gamma_{(\mu)}\big) = \tilde\Gamma_{(\mu)} := \left\{ \begin{pmatrix} \mu\,a+1 & b \\ 0 & \mu\,d+1 \end{pmatrix} \ \middle| \ a+d \equiv 0 \bmod \mu \ , \ b \in \mathbb{Z}/\mu^2\,\mathbb{Z} \right\} \tag{A-3.3}$$

of the finite group $SL(2,\mathbb{Z}/\mu^2\,\mathbb{Z})$ under the surjective homomorphism

$$\phi \ : \ SL(2,\mathbb{Z}) \ \longrightarrow \ SL(2,\mathbb{Z}/\mu^2\,\mathbb{Z}) \tag{A-3.4}$$

given by reduction modulo μ^2. The index of $\Gamma_{(\mu)}$ in $SL(2,\mathbb{Z})$ may thereby be computed from

$$\big[SL(2,\mathbb{Z}) : \Gamma_{(\mu)}\big] \ = \ \big[\mathrm{im}(\phi) : \tilde\Gamma_{(\mu)} \cap \mathrm{im}(\phi)\big]$$

$$= \left[SL(2,\mathbb{Z}/\mu^2\,\mathbb{Z}) : \tilde{\Gamma}_{(\mu)}\right] \; = \; \frac{\left|SL(2,\mathbb{Z}/\mu^2\,\mathbb{Z})\right|}{\left|\tilde{\Gamma}_{(\mu)}\right|} \; . \tag{A-3.5}$$

We now need to work out the orders of the two finite groups $SL(2,\mathbb{Z}/\mu^2\,\mathbb{Z})$ and $\tilde{\Gamma}_{(\mu)}$. The order of $\tilde{\Gamma}_{(\mu)}$ can be easily determined by inspection of its definition (A-3.3) to be $|\tilde{\Gamma}_{(\mu)}| = \mu^3$. The order of $SL(2,\mathbb{Z}/\mu^2\,\mathbb{Z})$ is calculated as follows.

The index of $\Gamma_{(\mu)}$ turns out to depend crucially on the prime factorization of the integer μ. Suppose that $\mu = p_1^{k(1)} \cdots p_t^{k(t)}$ with p_j, $j = 1,\dots,t$ distinct prime numbers and $k(j) > 0$. By the Chinese remainder theorem the corresponding finite group factorizes as

$$SL\big(2,\mathbb{Z}/\mu^2\,\mathbb{Z}\big) = SL\big(2,\mathbb{Z}/p_1^{2k(1)}\,\mathbb{Z}\big) \;\times\; \cdots \;\times\; SL\big(2,\mathbb{Z}/p_t^{2k(t)}\,\mathbb{Z}\big) \tag{A-3.6}$$

and its order is given by

$$\left|SL(2,\mathbb{Z}/\mu^2\,\mathbb{Z})\right| = \prod_{j=1}^{t} \left|SL(2,\mathbb{Z}/p_j^{2k(j)}\,\mathbb{Z})\right| \; . \tag{A-3.7}$$

It thus suffices to compute the order of $SL(2,\mathbb{Z}/p^{2k}\,\mathbb{Z})$ for p prime and $k > 0$. Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}/p^{2k}\,\mathbb{Z})$. Then $a\,d - b\,c \equiv 1 \mod p^{2k}$. To ensure that the matrix is non-singular, the pair (a,b) must take values in the set

$$\big(\mathbb{Z}/p^{2k}\,\mathbb{Z} \;\times\; \mathbb{Z}/p^{2k}\,\mathbb{Z}\big) \setminus \big(p\,\mathbb{Z}/p^{2k}\,\mathbb{Z} \;\times\; p\,\mathbb{Z}/p^{2k}\,\mathbb{Z}\big) \; . \tag{A-3.8}$$

The number of elements in this set is $p^{4k-2}\,(p^2-1)$. The pair (c,d) must be chosen so that p does not divide the determinant. There are $p^{4k-1}\,(p-1)$ such pairs (c,d) for each (a,b). This ensures that the matrix is non-singular. Thus the number of invertible matrices is given by

$$\left|GL(2,\mathbb{Z}/p^{2k}\,\mathbb{Z})\right| = p^{4k-2}\,\big(p^2-1\big)\,p^{4k-1}\,\big(p-1\big) \; . \tag{A-3.9}$$

The determinant is a group homomorphism $\det : GL(2,\mathbb{Z}/p^{2k}\,\mathbb{Z}) \to \mathbb{Z}/p^{2k}\,\mathbb{Z}$. It follows that the index of $SL(2,\mathbb{Z}/p^{2k}\,\mathbb{Z})$ in $GL(2,\mathbb{Z}/p^{2k}\mathbb{Z})$ is

$$\left[GL(2,\mathbb{Z}/p^{2k}\,\mathbb{Z}) : SL(2,\mathbb{Z}/p^{2k}\,\mathbb{Z})\right] = \frac{\left|GL(2,\mathbb{Z}/p^{2k}\,\mathbb{Z})\right|}{\left|SL(2,\mathbb{Z}/p^{2k}\,\mathbb{Z})\right|} = p^{2k-1}\,(p-1) \; , \tag{A-3.10}$$

which is just the Euler φ-function of the field $\mathbb{Z}/p^{2k}\,\mathbb{Z}$.

By combining all of these results we find finally that the index of $\Gamma_{(\mu)}$ in $SL(2,\mathbb{Z})$ is given by the Euler product expansion

$$\big[SL(2,\mathbb{Z}):\Gamma_{(\mu)}\big]=\mu^3\prod_{\mathrm{primes}\,p|\mu}\left(1-\frac{1}{p^2}\right)\ . \qquad \text{(A-3.11)}$$

We can now build a Schreier tranversal inductively, starting from $\{\mathbb{1}_2\}$. Suppose that we have a set $\mathfrak{C}_k$ of k words, satisfying the suffix condition, which contains at most one representative of any right coset. If k is strictly less than the index (A-3.11), then we can examine the right cosets $\Gamma_{(\mu)}S\,C$, $\Gamma_{(\mu)}T\,C$ and $\Gamma_{(\mu)}T^{-1}\,C$ for each $C\in\mathfrak{C}_k$ until we find one which is different from $\Gamma_{(\mu)}C$ for $C\in\mathfrak{C}_k$. Then add $S\,C$, $T\,C$ or $T^{-1}\,C$ to the list of words to form a new list $\mathfrak{C}_{k+1}$. This process terminates precisely when k is equal to the index (A-3.11), and then $\mathfrak{C}_k=\mathfrak{C}_{(\mu)}$ is the desired Schreier transversal for $\Gamma_{(\mu)}$. For example, when $\mu=2$ the subgroup $\Gamma_{(2)}$ has index 6 and $\mathfrak{C}_{(2)}=\{\mathbb{1}_2,S,S\,T,S\,T^2,S\,T^3,S\,T^2\,S\}$ is a Schreier transversal for $\Gamma_{(2)}$. The corresponding elliptic fundamental domain (A-3.2) is depicted in Figure A-3.1.

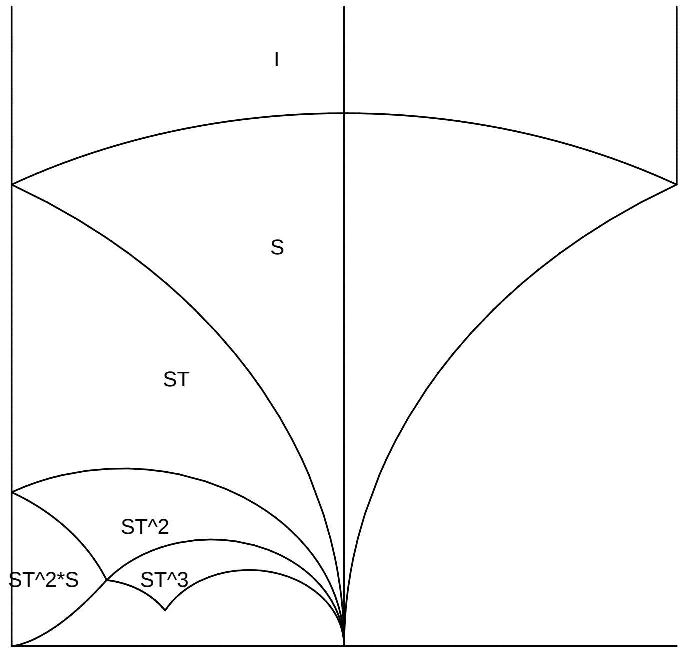

Figure A-3.1: The fundamental domain $\mathfrak{C}_{(2)}\bullet\Delta$ for the action of the congruence subgroup $\Gamma_{(2)}\subset SL(2,\mathbb{Z})$ on $\mathcal{H}_1$.

Using the modular transformations (A-2.3) along with the positivity constraint (??) on the period matrix, we find that the fundamental domain at genus two for the residual modular group preserving the Poincaré normal form is given by

$$\mathcal{F}_2'(\mu) = \big(\mathfrak{C}_{(\mu)} \bullet \Delta\big) \times \mathbb{C} \times \mathcal{H}_1 \tag{A-3.12}$$

with elements $(\Omega_{22}, \Omega_{12}, \Omega_{11})$. The integers μ, t, r, a, b, c and d are thus unrestricted except for the dependences of a, b, c and d on r, s and t. Because of this dependence and the complexity of the integration region $\tau \in \mathfrak{C}_{(\mu)} \bullet \Delta$, the free energy cannot be made as explicit as those computed in Sections **??**, **??** and **??**.

Appendix B: Explicit Form of Ξ_6

In this section we provide the explicit expressions for the modular covariant form $\Xi_6[\delta]$ on $\mathcal{H}_2$ defined in (**??**) for the ten even spin structures. Given an even characteristic δ_i, $i = 0, 1, \ldots, 9$, we denote $\vartheta_i(\Omega) := \Theta[\delta_i](\mathbf{0}|\Omega)^4$. By the mirror property [**?**], there are two equivalent expressions for $\Xi_6[\delta_i](\Omega)$ corresponding to the two triples of odd spin structures used to represent $\delta_i = \nu_{i_1} + \nu_{i_2} + \nu_{i_3} = \nu_{i_4} + \nu_{i_5} + \nu_{i_6}$ for each i. One then has

$$
\begin{aligned}
\Xi_6[\delta_1] &= -\vartheta_4\,\vartheta_5\,\vartheta_8 - \vartheta_2\,\vartheta_6\,\vartheta_9 - \vartheta_3\,\vartheta_7\,\vartheta_0 = -\vartheta_4\,\vartheta_7\,\vartheta_6 - \vartheta_3\,\vartheta_8\,\vartheta_9 - \vartheta_2\,\vartheta_5\,\vartheta_0 \ , \\
\Xi_6[\delta_2] &= \vartheta_3\,\vartheta_5\,\vartheta_7 + \vartheta_4\,\vartheta_8\,\vartheta_0 - \vartheta_1\,\vartheta_6\,\vartheta_9 = \vartheta_3\,\vartheta_6\,\vartheta_8 - \vartheta_1\,\vartheta_5\,\vartheta_0 + \vartheta_4\,\vartheta_7\,\vartheta_9 \ , \\
\Xi_6[\delta_3] &= \vartheta_2\,\vartheta_5\,\vartheta_7 - \vartheta_1\,\vartheta_8\,\vartheta_9 + \vartheta_4\,\vartheta_6\,\vartheta_0 = \vartheta_2\,\vartheta_6\,\vartheta_8 + \vartheta_5\,\vartheta_4\,\vartheta_9 - \vartheta_1\,\vartheta_7\,\vartheta_0 \ , \\
\Xi_6[\delta_4] &= -\vartheta_1\,\vartheta_5\,\vartheta_8 + \vartheta_3\,\vartheta_6\,\vartheta_0 + \vartheta_2\,\vartheta_7\,\vartheta_9 = -\vartheta_1\,\vartheta_6\,\vartheta_7 + \vartheta_2\,\vartheta_8\,\vartheta_0 + \vartheta_3\,\vartheta_5\,\vartheta_9 \ , \\
\Xi_6[\delta_5] &= \vartheta_2\,\vartheta_3\,\vartheta_7 - \vartheta_1\,\vartheta_4\,\vartheta_6 + \vartheta_6\,\vartheta_9\,\vartheta_0 = -\vartheta_1\,\vartheta_2\,\vartheta_0 + \vartheta_3\,\vartheta_4\,\vartheta_9 + \vartheta_6\,\vartheta_7\,\vartheta_8 \ , \\
\Xi_6[\delta_6] &= \vartheta_3\,\vartheta_4\,\vartheta_0 - \vartheta_1\,\vartheta_2\,\vartheta_9 + \vartheta_5\,\vartheta_7\,\vartheta_8 = -\vartheta_1\,\vartheta_4\,\vartheta_7 + \vartheta_5\,\vartheta_9\,\vartheta_0 + \vartheta_2\,\vartheta_3\,\vartheta_8 \ , \\
\Xi_6[\delta_7] &= \vartheta_2\,\vartheta_3\,\vartheta_5 + \vartheta_8\,\vartheta_9\,\vartheta_0 - \vartheta_1\,\vartheta_4\,\vartheta_6 = \vartheta_2\,\vartheta_4\,\vartheta_9 - \vartheta_1\,\vartheta_3\,\vartheta_0 + \vartheta_5\,\vartheta_6\,\vartheta_8 \ , \\
\Xi_6[\delta_8] &= \vartheta_7\,\vartheta_9\,\vartheta_0 - \vartheta_1\,\vartheta_4\,\vartheta_5 + \vartheta_2\,\vartheta_3\,\vartheta_6 = -\vartheta_1\,\vartheta_3\,\vartheta_9 + \vartheta_2\,\vartheta_4\,\vartheta_0 + \vartheta_5\,\vartheta_6\,\vartheta_7 \ , \\
\Xi_6[\delta_9] &= \vartheta_7\,\vartheta_8\,\vartheta_0 - \vartheta_1\,\vartheta_2\,\vartheta_6 + \vartheta_3\,\vartheta_4\,\vartheta_5 = \vartheta_5\,\vartheta_6\,\vartheta_0 - \vartheta_1\,\vartheta_3\,\vartheta_8 + \vartheta_2\,\vartheta_4\,\vartheta_7 \ , \\
\Xi_6[\delta_0] &= \vartheta_7\,\vartheta_8\,\vartheta_9 + \vartheta_3\,\vartheta_4\,\vartheta_6 - \vartheta_1\,\vartheta_2\,\vartheta_5 = \vartheta_5\,\vartheta_6\,\vartheta_9 + \vartheta_2\,\vartheta_4\,\vartheta_8 - \vartheta_1\,\vartheta_3\,\vartheta_7 \ . \qquad \text{(B-1.1)}
\end{aligned}
$$