Heriot-Watt University

Supersymmetric Monopole Dynamics

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March 2009

Submitted for the degree of
Doctor of Philosophy in Mathematics
on completion of research in the
Department of Mathematics,
School of Mathematical and Computer Sciences.

Abstract

We study the supersymmetric quantum mechanics of monopoles in bosonic, $N = 2$ and $N = 4$ supersymmetric Yang-Mills-Higgs theory, with particular emphasis on monopoles of charge–$(1, 1)$ in a theory with gauge group $SU(3)$ spontaneously broken to $U(1) \times U(1)$.

In the moduli space approximation, the quantum states of bosonic monopoles can be described by functions on the moduli space. For $N = 2$ supersymmetric monopoles, quantum states can be interpreted as either spinors or anti-holomorphic forms on the moduli space. The quantum states of the $N = 4$ supersymmetric monopole correspond to general differential forms on the moduli space. In each case, we review the moduli space approximation and derive general expressions for the supercharges as differential operators. In the geometrical language of forms on the moduli space, the Hamiltonian is proportional to the Laplacian acting on forms. We propose a general expression for the total angular momentum operator and verify its commutation relations with the supercharges.

We use the known metric structure of the moduli space of charge–$(1, 1)$ monopoles to show that there are no quantum bound states of such monopoles in the moduli space approximation. We exhibit scattering states and compute the corresponding differential cross sections. Using the general expressions for the supercharges we construct the short supermultiplet of supersymmetric monopoles, and study its decomposition under the proposed angular momentum operator.

Dedication

First of all I would like to thank my supervisor Bernd Schroers for all his guidance and support. In addition I am grateful to José Figueroa-O'Farrill for many helpful conversations. I also thank Heriot-Watt University and the School of Mathematical and Computer Sciences, as well as EPSRC, for giving me the opportunity to do this research.

On a more personal level, many people have helped to make my time in Edinburgh an unforgettable period in my life. These include my office mates, fellow PhD students and my flatmates throughout these years. I am happy to say that many of you have become very good friends.

Special thanks go out to my best friends who have always been there for me, to celebrate the good times, but also to help me through harder times: Desi, Sachi, Sander, Mark, and of course my jaarclub Scuderia.

Words cannot say how important you have been to me, and still are. Bébel. Anita. Julia. My life would not have been the same without you, and you will always have a place in my heart.

Finally, Thanks to my Mum and Dad, and my Sister. Thanks for loving me and supporting me unconditionally, and encouraging me to always follow my heart, to make my dreams come true.

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Updated February 2008, November 2008, February 2009

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Chapter 1

Introduction

One of the most intriguing aspects of Yang-Mills-Higgs theory is that it generically contains magnetic monopoles as classical, solitonic solutions, with properties which appear to be dual to those of the electrically charged quantum particles in a dual theory. The strongest formulation of this duality is the electromagnetic duality conjecture of Montonen and Olive [1] in the setting of supersymmetric Yang-Mills-Higgs theories, according to which the physics of massive, electrically charged particles (W-bosons) in the theory should be equivalent to the physics of magnetically charged particles (magnetic monopoles) in the dual theory. In this dual theory, the gauge group is replaced by its dual and the coupling constant is inverted. If the conjecture is correct, electromagnetic duality provides a means to investigate the physics of strongly coupled electric particles by studying the physics of magnetic monopoles at weak coupling, using perturbative or semiclassical techniques in the dual theory.

The properties of monopoles in supersymmetric Yang-Mills-Higgs theory have been studied extensively, often motivated by the electromagnetic duality conjectures, or their generalisations to S-duality. A crucial tool in these studies has been the moduli space approximation. This approximation was originally introduced by Manton in order to study the classical dynamics of several interacting monopoles [2]. He showed that the classical trajectories of monopoles correspond to geodesics on the moduli space. However, the metrics of moduli spaces prove exceedingly difficult to derive, and only for a few moduli spaces is this metric known. The moduli space of a single monopole is $\mathcal{M}_1 = \mathbb{R}^3 \times S^1$ with a flat metric. The moduli space of two monopoles has, roughly speaking, a factor corresponding to the centre of mass motion (which is equivalent to $\mathcal{M}_1$) and a factor corresponding to the relative motion of the two monopoles (which is curved, hence giving rise to non-trivial dynamics). For two identical $SU(2)$ monopoles (charge-2) this relative moduli space is the Atiyah-Hitchin manifold [3, 4], while for two distinct $SU(3)$ monopoles in a theory with maximal symmetry breaking (charge-$(1,1)$) it is the Taub-NUT manifold [5, 6]. Some useful facts about general monopole moduli spaces are known. One is that all monopole moduli spaces in theories with maximal symmetry breaking are hyperkähler manifolds. Another is that the multi-monopole moduli space decomposes into the product of single monopole moduli spaces in the limit of infinite separation between monopoles.

The quantisation of the effective theory in the moduli space approximation was first considered in the context of the non-supersymmetric, bosonic theory by supposing that quantum states are scalar functions on the moduli space and that the Hamiltonian is proportional to the Laplacian on the moduli space [4]. This model was then used [7, 8] to compute bound state energies and scattering cross sections for elastic and inelastic monopole-monopole scattering in Yang-Mills-Higgs theory with gauge group $SU(2)$ broken to $U(1)$.

Gauntlett [9] subsequently showed that the application of the moduli space approximation to $N = 2$ supersymmetric monopoles leads to a model where the quantum states can be described in terms of either anti-holomorphic forms or spinors on the moduli space. The equivalence of the two descriptions follows from the hyperkähler property of monopole moduli spaces. He explained how certain supercharges correspond to Dolbeault operators acting on forms, or the Dirac operator acting on spinors. The Hamiltonian is given by either the Laplacian acting on forms, or the square of the Dirac operator acting on spinors. In this thesis, we show that the remaining supercharges correspond to twisted Dolbeault operators, or twisted Dirac operators, and we demonstrate the equivalence of the two interpretations of the quantisation explicitly for the examples of the single monopole, and charge-$(1,1)$ monopoles.

In the second viewpoint, both fermionic and bosonic degrees of freedom of the original $N = 2$ supersymmetric field theory are encoded in spinors on the moduli space. This may seem puzzling at first, and this viewpoint is less convenient than the geometrical one, when it comes to interpreting the moduli space quantum mechanics in terms of the original fields. However, the spinorial viewpoint provides interesting links with the large literature on spectral properties of Dirac operators. We will see and exploit this explicitly in our case study where earlier work by Comtet and Horváthy [10] on the Dirac operator on the Taub-NUT manifold provides useful guidance.

The moduli space approximation to the quantum dynamics of $N = 4$ supersymmetric monopoles is closely related to, and a natural extension of, the effective theory for $N = 2$ supersymmetric monopoles. In the geometrical interpretation, wavefunctions are now arbitrary differential forms on the moduli space (as opposed to anti-holomorphic forms only); the supercharges correspond to the Dolbeault operators and their twisted counterparts, and the Hamiltonian is again the Laplacian [9, 11, 12]. This model played a crucial role in the genesis of the S-duality conjecture.

Until now, many investigations have focused on the calculation of lowest bound states, the so-called BPS states, which play a key role in testing the S-duality conjecture. Among the most important results in this area are the following. Taking the spin of particles and monopoles into account, S-duality can only possibly hold for $N = 4$ supersymmetric Yang-Mills-Higgs theories. Furthermore, Sen [13] showed, in the case of charge-2 monopoles, that it requires the existence of a unique, normalisable, harmonic form on the relative moduli space (which we now call a Sen-form), and he explicitly gave the formula for this form on the Atiyah-Hitchin manifold. Gauntlett and Lowe [5] later repeated his argument for charge-$(1, 1)$ monopoles and gave the corresponding Sen-form on the Taub-NUT manifold.

In this thesis we take a step further. We study the low energy dynamics of $N = 2$ supersymmetric monopoles in theories with maximally broken gauge symmetry using the moduli space approximation. We derive general expressions for all of the supercharges of the effective theory and show how they can be interpreted as natural differential operators in the quantum theory (either the (twisted) Dolbeault operators or the (twisted) Dirac operators). Furthermore, we propose a formula for the angular momentum operator as a differential operator in the quantum theory. Focussing on monopoles of charge-$(1, 1)$, we show that there are no bound states and study the scattering states. We find that the scattering cross section of two distinct monopoles in this theory is the same, at low energies, as that of two BPS monopoles in the $SU(2)$ theory with the symmetry breaking to $U(1)$. Using our expressions for the supercharges and the angular momentum operator, we discuss the multiplet structure of $N = 2$ supersymmetric monopoles.

We then extend these studies to $N = 4$ supersymmetric monopoles. Our goal is to illustrate how semiclassical techniques can be used to compute details of the quantum mechanics of magnetic monopole interactions. Hopefully, via electromagnetic duality, these calculations can be used to learn more about the physics of strongly interacting electric particles. A secondary purpose is to exhibit some of the interesting geometrical features of the low energy quantum dynamics of supersymmetric monopoles in the moduli space approximation.

Summary and Outline

This thesis consists of two parts. Part I deals with the general theory of monopoles and their low energy dynamics. In the second part, we work out two examples in detail: first of all the single, charge-1 monopole, and secondly the more interesting case of charge-$(1, 1)$ monopoles.

Our discussion of the low energy dynamics of monopoles starts off, in the following chapter, with a review of the moduli space approximation of bosonic monopoles. We briefly present the field theoretical model that gives rise to monopoles, fixing our conventions and notation. Next we review the moduli space approximation and the hyperkähler structure of the moduli space. We introduce a quaternionic description of zero-modes, which is closely related to the viewpoint that zero-modes correspond to solutions of a Dirac equation, the subjects of the subsequent sections. We finish the chapter by discussing the zero-modes of a single monopole of charge-1 and the hyperkähler structure of its moduli space.

In chapter 3 we start again with a presentation of the field theory, this time an $N = 2$ supersymmetric Yang-Mills-Higgs theory. The bosonic zero-modes of this theory are the same as those of the bosonic model in chapter 2, and we focus our discussion on the fermionic zero-modes. These are closely related to the bosonic zero-modes due to the supersymmetry of the theory, and the quaternionic description of zero-modes is useful to show how the hyperkähler structure acts. For the first time, now, we use the equivalence between spinors and anti-holomorphic forms, in this case to view the zero-modes (spinors) as anti-holomorphic forms on Euclidean 4-space. In section 3.3 we present the effective Lagrangian which governs the low energy dynamics of $N = 2$ supersymmetric monopoles. The next two sections discuss the quantisation of this effective theory, in terms of spinors on the moduli space and in terms of forms on the moduli space respectively. These two quantisation procedures are equivalent due to the hyperkähler structure of the moduli space. This is the second time we come across the equivalence between spinors and anti-holomorphic forms, now on the level of the moduli space. The effective Lagrangian of section 3.3 has $\mathcal{N} = 4$ real supersymmetries. We give general expressions for all of the corresponding supercharges and give their interpretation as (twisted) Dirac operators in the context of quantisation using spinors (section 3.4), and as (twisted) Dolbeault operators in the context of quantisation using anti-holomorphic forms on the moduli space (section 3.5).

Chapter 4 deals with $N = 4$ supersymmetric monopoles. The effective theory of the moduli space approximation now possesses $\mathcal{N} = 8$ real supersymmetries, and we quantise the effective theory in terms of forms on the moduli space. The doubled supersymmetry (compared to the previous chapter) leads to the fact that holomorphic and anti-holomorphic forms both appear as quantum states of the supersymmetric monopole, and play an equivalent role in the quantisation procedure. The set of supercharges is similarly doubled, and their geometrical interpretation is given by the set of (twisted) Dolbeault operators of the previous chapter, supplemented with their complex conjugates.

The final chapter of Part I, chapter 5, is devoted to a particularly important observable, namely the angular momentum operator. The naive guess that its components are simply the infinitesimal generators of the $SO(3)$ action on the moduli space cannot be correct. This action does not respect the complex structure used to define the quantum states (i.e. the anti-holomorphic and holomorphic forms). We propose a modification which does act on the spaces of anti-holomorphic forms and holomorphic forms independently, and we show that it has the required commutation relations with the supercharges.

In Part II of this thesis we illustrate the results of the previous chapters in two examples. Chapter 6 deals with the simplest example, namely the moduli space approximation of the dynamics of a single monopole. Then, in chapter 7, we consider the moduli space quantisation of two distinct monopoles in a (supersymmetric) Yang-Mills-Higgs theory with gauge group $SU(3)$ broken to $U(1) \times U(1)$. The relevant moduli space is an eight-dimensional hyperkähler manifold of the form $\mathbb{R}^3 \times (\mathbb{R} \times M_{TN})/\mathbb{Z}$, where M_{TN} is the complete self-dual Taub-NUT space with positive mass parameter [5, 6, 14]. The reason for choosing this example is that the scalar Laplace equation on this space can be solved exactly, and that the complex structures are known. As a result, we are able to exhibit many features of the bosonic and supersymmetric quantum mechanics explicitly: we show that there are no bound states, and give explicit formulae for differential cross sections of scattering states. Finally, using our expressions for the supercharges and the angular momentum operator, we discuss the multiplet structure of $N = 2$ and $N = 4$ supersymmetric monopoles.

Part I

Low Energy Monopole Dynamics

Chapter 2

Bosonic Monopoles

In this chapter, we review the construction of monopole moduli spaces. First of all, to get a feeling for the concept of moduli spaces, we construct the moduli space for a classical point particle moving along a 2-dimensional surface. Then we discuss the field theoretical model that gives rise to monopoles and the interpretation of monopoles as translationally invariant instantons in Euclidean $\mathbb{R}^4$. In the next sections we construct the moduli space of BPS monopoles and discuss its hyperkähler structure. We introduce a description of zero-modes in terms of quaternions, which is closely related to the interpretation of zero-modes as solutions to a Dirac equations. Finally, we briefly discuss the zero-modes of the charge-1 't Hooft-Polyakov monopole.

2.1 *Prelude:* The classical point particle

To introduce the idea of the moduli space approximation, we start off with the simplest example: a classical point particle moving along a 2-dimensional surface. We assume there is a potential energy function with a 1-dimensional space of minima.

For example, we may think of a particle moving along a curved surface, under influence of a vertical, uniform (Newtonian) gravitational field. In this section we will be interested in a surface with the shape of a Mexican hat. In this case, the minima of the potential energy function lie on a circle at the bottom of the hat's surface.

The Lagrangian we're interested in is

$$L(\vec{r}) \;=\; \frac{1}{2}m|\dot{\vec{r}}|^2 - V(r), \tag{2.1}$$

$$V(r) \;=\; \lambda(r^2 - a^2)^2 \qquad ; \quad \lambda > 0. \tag{2.2}$$

In polar coordinates the Lagrangian becomes

$$L(r,\theta) \;=\; \frac{1}{2}m(\dot{r}^2 + (r\dot{\theta})^2) - \lambda(r^2 - a^2)^2. \tag{2.3}$$

The equations of motion are given by the Euler-Lagrange equations:

$$\partial_t \frac{\partial L}{\partial \dot{r}} \;=\; \frac{\partial L}{\partial r} \qquad\qquad m\ddot{r} \;=\; mr\dot{\theta}^2 + 2\lambda(r^2 - a^2)r \tag{2.4a}$$

$$\partial_t \frac{\partial L}{\partial \dot{\theta}} \;=\; \frac{\partial L}{\partial \theta} \qquad\qquad m\partial_t\left(r^2\dot{\theta}\right) \;=\; 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} \;=\; 0 \tag{2.4b}$$

The conjugate momenta to r and θ are

$$p_r \;=\; \frac{\partial L}{\partial \dot{r}} \;=\; m\dot{r}, \qquad\qquad p_\theta \;=\; \frac{\partial L}{\partial \dot{\theta}} \;=\; mr^2\dot{\theta}. \tag{2.5}$$

The Hamiltonian, or energy, H is given by the following Legendre transformation of the Lagrangian L,

$$H \;=\; \dot{r}p_r + \dot{\theta}p_\theta - L \;=\; \frac{1}{2}m(\dot{r}^2 + (r\dot{\theta})^2) + V(r). \tag{2.6}$$

Suppose we do not know how to solve the equations of motion exactly. We will then try to find approximate solutions instead. We start by looking for solutions of lowest energy.

Solutions of lowest energy are static ($\dot{r} = \dot{\theta} = 0$), and satisfy $V(r) = 0$, which implies $r = a$. The space of these solutions of minimal energy, V_0, can be parameterised by the angular coordinate θ:

$$V_0 \;=\; \big\{(a,\theta) \in \mathbb{R}^2 \mid 0 \le \theta < 2\pi\big\} \;\cong\; S^1 \tag{2.7}$$

In this simple case, all elements of V_0 are physically distinct, and we call this set the *moduli space*, $M = V_0$. (Monopoles appear in the context of a gauge theory, and then the moduli space will be the set of gauge equivalence classes of solutions of minimal energy.)

When the energy of a particle is low enough, it will stay close to minima of the potential. In these cases we can approximate the exact behaviour of the particle with motion in the moduli space. In the present case we only have one moduli space parameter θ, so the moduli space approximation implies that we should look for solutions of the form

$$\vec{r}_0(t) = \vec{r}_0(\theta(t)) = (a, \theta(t)). \tag{2.8}$$

By inserting a solution of this form into the Lagrangian (2.3) we find the effective Lagrangian, which is now only a function of θ:

$$L_{\text{eff}}(\theta) = L(\vec{r}_0) = \frac{1}{2}ma^2\dot{\theta}^2. \tag{2.9}$$

The equation of motion for this Lagrangian is

$$ma^2\ddot{\theta} = 0. \tag{2.10}$$

and the solution has constant velocity in the moduli space, $\theta = v_\theta t$. This corresponds to a constant angular velocity of the particle moving along the surface:

$$\vec{r}_0(t) = (a, v_\theta t). \tag{2.11}$$

This is not an exact solution of the original equations of motion (unless $v_\theta = 0$): when we insert this solution into equation (2.4a), we find

$$av_\theta^2 = \ddot{r}, \tag{2.12}$$

and we see that for any non-zero velocity, the particle will radially accelerate away from the minima of the potential (which will in turn affect v_θ).

Tangent vectors to the moduli space are called zero-modes. The zero-mode corresponding to the coordinate θ on M is the tangent vector ∂_θ. Thinking of the moduli space as imbedded in the total space of solutions, we may also denote a zero-mode by $\partial_\theta(\vec{r}_0(\theta))$.

2.2 BPS monopoles

We now turn to the theory of magnetic monopoles, which appear in certain classes of Yang-Mills-Higgs field theories with spontaneously broken symmetries [15, 16]. We study Yang-Mills-Higgs models on $(3+1)$-dimensional space-time with a Lorentzian metric of signature $(+,-,-,-)$. The Lagrangian that we are interested in is the Georgi-Glashow Lagrangian, given by

$$L \;=\; \int \mathrm{d}^3x\, \mathcal{L} \;=\; \int \mathrm{d}^3x \left(-\frac{1}{4} F_{\mu\nu} \cdot F^{\mu\nu} + \frac{1}{2} D_\mu \Phi \cdot D^\mu \Phi \right). \tag{2.13}$$

The fields are in the adjoint representation of the gauge group $G = SU(n)$; i.e. they take values in the Lie-algebra $\mathfrak{g} = su(n)$. The dot-product is an invariant inner product on the Lie-algebra. The field strength F is given by

$$\begin{aligned} F_{\mu\nu} \;&=\; \partial_\mu A_\nu - \partial_\nu A_\mu - e\,[A_\mu, A_\nu] \\ &=\; \partial_\mu A_\nu - D_\nu A_\mu \;=\; D_\mu A_\nu - \partial_\nu A_\mu, \end{aligned} \tag{2.14}$$

where $-e$ is the coupling constant, and the covariant derivative of the Higgs field Φ is

$$D_\mu \Phi \;=\; (\partial_\mu - e\,\mathrm{ad}\, A_\mu)\, \Phi \;=\; \partial_\mu \Phi - e\,[A_\mu, \Phi]\,. \tag{2.15}$$

We impose boundary conditions for the fields at infinity to break the symmetry,

$$\lim_{r\to\infty} \Phi \cdot \Phi = a^2. \tag{2.16}$$

We will assume that the symmetry breaking is maximal; specifically, $SU(n)$ is broken to $U(1)^{n-1}$. The action of the model is given by

$$S \;=\; \int \mathrm{d}t L. \tag{2.17}$$

The equations of motions are

$$D_\mu D^\mu \Phi \;=\; 0, \tag{2.18a}$$

$$D_\mu F^{\mu\nu} \;=\; -e\,[D^\nu \Phi, \Phi]\,. \tag{2.18b}$$

The conjugate momenta to A^i and Φ are

$$E_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{A}^i} = F_{i0}, \tag{2.19}$$

$$\Pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = D_0 \Phi. \tag{2.20}$$

E_i is called the (non-Abelian) electric field; the magnetic field B_i is defined by

$$B_i = \frac{1}{2}\epsilon_{ijk} F^{jk}. \tag{2.21}$$

The conjugate momentum to A^0 vanishes. Therefore we must impose Gauss' Law (the equation of motion for A^0; equation (2.18b) for $\nu = 0$) as a constraint on the gauge fields.

The Hamiltonian H is defined by the Legendre transformation

$$H = \int \mathrm{d}^3x\, \mathcal{H} = \int \mathrm{d}^3x \left(E_i \dot{A}^i + \Pi \dot{\Phi} - \mathcal{L} \right). \tag{2.22}$$

The Lagrangian and the Hamiltonian can be written in terms of the kinetic and potential energy as

$$L = K - V, \qquad H = K + V, \tag{2.23}$$

where the kinetic and potential energy are given by

$$K = \int \mathrm{d}^3x \left(\frac{1}{2}|E_i|^2 + \frac{1}{2}|\Pi|^2 \right), \tag{2.24}$$

$$V = \int \mathrm{d}^3x \left(\frac{1}{2}|B_i|^2 + \frac{1}{2}|D_i\Phi|^2 \right). \tag{2.25}$$

Bogomol'nyi [17] first observed that the potential energy can be written as

$$V = \int \mathrm{d}^3x \left(\frac{1}{2} \left| B_i \mp D_i \Phi \right|^2 \pm \partial_i \left(\Phi \cdot B_i \right) \right). \tag{2.26}$$

Then

$$\left| \int \mathrm{d}^3x \partial_i \left(\Phi \cdot B_i \right) \right| = \left| \int_{S^2_\infty} \mathrm{d}S_i \left(\Phi \cdot B_i \right) \right| = \frac{4\pi a}{e} b(\vec{k}), \tag{2.27}$$

where $\vec{k} = (k_1, \ldots, k_{n-1}) \in \mathbb{Z}^{n-1}$ is the topological charge and $b(\vec{k})$ is a positive, real function of the topological charge which depends on the details of the vacuum expectation value of the Higgs field. Hence the potential energy satisfies the Bogomol'nyi bound

$$V \;\geq\; \frac{4\pi a}{e} b(\vec{k}). \tag{2.28}$$

BPS monopoles have minimal energy, which means that they are static and they saturate the Bogomol'nyi bound. The latter implies that they must satisfy the Bogomol'nyi equations

$$B_i \;=\; \pm D_i \Phi. \tag{2.29}$$

Here the upper sign corresponds to monopoles (positive topological charge, i.e. all integers $k_1, \ldots, k_{n-1}$ are ≥ 0), and the lower sign to anti-monopoles (negative topological charge, i.e. all integers $k_1, \ldots, k_{n-1}$ are ≤ 0). The Bogomol'nyi equations imply the equations of motion for static field configurations, which we will show below (at the end of the next section) in the temporal gauge, $A_0 = 0$.

2.3 Euclidean 4-space

The temporal gauge, $A_0 = 0$, is a convenient gauge to work in, and from now on we assume this gauge. We define

$$W_i \;=\; A_i, \qquad\qquad W_4 \;=\; \Phi, \tag{2.30}$$

so that we can think of $W_{\underline{i}}$ (underlined indices $\underline{i}$ run from 1 to 4) as a connection on Euclidean $\mathbb{R}^4$, if we introduce a fourth spatial dimension and assume all fields to be independent of this fourth dimension, $\partial_4 \equiv 0$. The covariant derivatives on this Euclidean $\mathbb{R}^4$ are defined by

$$D_{\underline{i}} \;=\; \partial_{\underline{i}} - e \,\mathrm{ad}\, W_{\underline{i}}. \tag{2.31}$$

Infinitesimal gauge transformations can be written as

$$\delta_\Lambda W_{\underline{i}} = -\frac{1}{e} D_{\underline{i}} \Lambda \tag{2.32}$$

for some gauge parameter Λ, which is again taken to be independent of the fourth dimension.

Gauss' Law, equation (2.18b) for $\nu = 0$, becomes

$$D_{\underline{i}} \dot{W}_{\underline{i}} \; = \; 0, \tag{2.33}$$

where a dot denotes a time derivative. In terms of the field strength $G_{\underline{ij}}$ corresponding to $W_{\underline{i}}$, the remaining equations of motion become

$$\ddot{W}_{\underline{j}} \; = \; D_{\underline{i}} G_{\underline{ij}}, \tag{2.34}$$

and the kinetic and potential energy can be written as

$$K \; = \; \int \mathrm{d}^3 x \; \frac{1}{2} \left| \dot{W}_{\underline{i}} \right|^2, \tag{2.35}$$

$$V \; = \; \int \mathrm{d}^3 x \; \frac{1}{4} |G_{\underline{ij}}|^2. \tag{2.36}$$

The Bogomol'nyi equations (2.29) become

$$G_{ij} \; = \; \varepsilon_{ijk} B_k \; = \; \pm \varepsilon_{ijk} D_k \Phi \; = \; \pm \varepsilon_{ijk4} G_{k4}. \tag{2.37}$$

Permuting the indices, this is equivalent to the (anti-)self-duality equations for G,

$$G_{\underline{ij}} \; = \; \pm \frac{1}{2} \varepsilon_{\underline{ijkl}} G_{\underline{kl}}. \tag{2.38}$$

Therefore we may think of monopoles as instantons in Euclidian 4-space that are independent of the fourth dimension.

Static field equations

It is now, in the temporal gauge $A_0 = 0$, fairly straightforward to show that the Bogomol'nyi equations imply the equations of motion for static field configurations. Using the Bianchi identity we have

$$D_i G_{i4} \; = \; D_i D_i \Phi \; = \; \pm D_i B_i \; = \; 0, \tag{2.39}$$

and, using equation (2.37),

$$\begin{aligned} D_i G_{ij} \; &= \; \pm \varepsilon_{ijk4} D_i G_{k4} \; = \; \pm \varepsilon_{ijk} D_i D_k \Phi \; = \\ &= \; \pm \frac{1}{2} \varepsilon_{ijk} \left[D_i, D_k \right] \Phi \; = \; \mp \frac{1}{2} \, e \, \varepsilon_{ijk} \left[F_{ik}, \Phi \right] \; = \\ &= \; \pm \, e \left[B_j, \Phi \right] \; = \; e \left[D_j \Phi, \Phi \right]. \end{aligned} \tag{2.40}$$

Together they can be summarised by

$$D_{\underline{i}} G_{\underline{ij}} \;=\; 0, \tag{2.41}$$

which is the same as equation (2.34) for static field configurations, $\dot{W}_{\underline{j}} = 0$.

2.4 The moduli space approximation

The moduli space is the space of physically distinct monopoles of minimal energy (within a particular topological class $\vec{k}$). Field configurations of monopoles related to each other by gauge transformations are not physically distinct, and the moduli space is therefore the space of gauge equivalence classes of BPS monopoles. The main reference for this section is the book on Topological Solitons by Manton and Sutcliffe [18].

We denote the set of finite energy field configurations $W_{\underline{i}}$ by $\mathcal{A}$, and the group of short-range gauge transformations by $\mathcal{G}$. Short-range gauge transformations are those gauge transformations that tend to the identity at infinity. The configuration space is obtained by identifying field configurations that are related via a short-range gauge transformation,

$$\mathcal{C} \;=\; \mathcal{A}/\mathcal{G}. \tag{2.42}$$

Long-range gauge transformations, with non-trivial action on the fields at infinity, are excluded from $\mathcal{G}$, because when we allow such gauge transformations to become time dependent, they have a physical effect on the monopoles (turning them into dyons with electric charge [19]), unlike time dependent short-range gauge transformations.

The set of field configurations corresponding to BPS monopoles of charge $\vec{k}$, $\mathcal{V}_{\vec{k}} \subset \mathcal{A}$, is the subspace of $\mathcal{A}$ of field configurations satisfying the Bogomol'nyi equations in the topological class $\vec{k}$. We now define the moduli space of charge-$\vec{k}$ monopoles, $\mathcal{M}_{\vec{k}}$, to be the subspace of $\mathcal{C}$ corresponding to static solutions of the Bogomol'nyi equations with topological charge $\vec{k}$,

$$\mathcal{M}_{\vec{k}} \;=\; \mathcal{V}_{\vec{k}}/\mathcal{G} \;\subset\; \mathcal{C}. \tag{2.43}$$

The idea of the moduli space approximation is to describe the low energy dynamics of monopoles by motion in the moduli space [2]. This is an adiabatic description and a good approximation provided the monopoles move slowly [20].

In order to compute the equation of motion governing the path in the moduli space, we lift this path to a path in $\mathcal{V}_{\vec{k}} \subset \mathcal{A}$ and insert it into the Lagrangian of the field theory, interpreting the parameter along the path as time. In practice this means picking a representative (static) field configuration in each equivalence class of solutions of the BPS equations along the path, thus leading to field configuration $W_{\underline{i}}(t, x^1, x^2, x^3)$ which depend on time and space coordinates. Time derivatives of such field configurations correspond to tangent vectors along the path, provided that they satisfy Gauss' Law, which ensures that the time evolution is orthogonal to gauge orbits. The potential energy on the moduli space is constant, saturating the Bogomol'nyi bound (2.28), and using expression (2.35) for the kinetic energy, the effective Lagrangian for charge-$\vec{k}$ monopoles is therefore

$$L_{\text{eff}} \;=\; \int \mathrm{d}^3x \; \frac{1}{2} \left| \dot{W}_{\underline{i}} \right|^2 - \frac{4\pi a}{e} b(\vec{k}). \tag{2.44}$$

Here Gauss' Law in the form (2.33) must be satisfied, which ensures that the effective Lagrangian is well defined, as we will show below.

Weinberg [21] first argued that the dimension of the moduli space $\mathcal{M}_{\vec{k}}$ is $4k$, where $k = |k_1 + \ldots + k_{n-1}|$, generalising an earlier index calculation by Callias [22] (see also section 2.7, and Taubes [23] and Atiyah and Hitchin [3]). Therefore, we can parameterise the moduli space with $4k$ parameters, or moduli, X^a. Using these coordinates, tangent vectors to $\mathcal{M}_{\vec{k}}$ can be decomposed as

$$\dot{W}_{\underline{i}} \;=\; \delta_a W_{\underline{i}} \, \dot{X}^a, \tag{2.45}$$

where the zero-modes $\delta_a W_{\underline{i}}$ form a basis of vector fields on $\mathcal{M}_{\vec{k}}$.

The metric on $\mathcal{M}_{\vec{k}}$ is obtained by restricting the metric on $\mathcal{A}$. The natural metric on $\mathcal{A}$ is given by

$$g(\dot{W}, \dot{V}) \;=\; \int \mathrm{d}^3x \; \dot{W}_{\underline{i}} \cdot \dot{V}_{\underline{i}}, \tag{2.46}$$

and the components of its restriction to the moduli space $\mathcal{M}_{\vec{k}}$, with respect to the basis of zero-modes $\delta_a W_{\underline{i}}$, are

$$g_{ab} \;=\; \int \mathrm{d}^3x \;\delta_a W_{\underline{i}} \cdot \delta_b W_{\underline{i}}. \tag{2.47}$$

From Gauss' Law (2.33), we derive the background gauge condition for the zero-modes,

$$D_{\underline{i}} \delta_a W_{\underline{i}} \;=\; 0. \tag{2.48}$$

which implies that the tangent vectors $\dot{W}_{\underline{i}}$ to the lifted path in $\mathcal{V}_{\vec{k}} \subset \mathcal{A}$ are orthogonal to gauge transformations:

$$g(\delta_a W, \delta_\varepsilon W) \;=\; \frac{1}{e}\int \mathrm{d}^3x \;\delta_a W_{\underline{i}} \cdot D_{\underline{i}}\varepsilon \;=\; -\frac{1}{e}\int \mathrm{d}^3x \;(D_{\underline{i}}\delta_a W_{\underline{i}}) \cdot \varepsilon \;=\; 0. \tag{2.49}$$

Therefore the effective Lagrangian (2.44) is well defined.

Since field configurations corresponding to points in $\mathcal{M}_{\vec{k}}$ satisfy the Bogomol'nyi equations, tangent vectors to $\mathcal{M}_{\vec{k}}$ must satisfy the linearised Bogomol'nyi equations,

$$D_{\underline{i}}\dot{W}_{\underline{j}} - D_{\underline{j}}\dot{W}_{\underline{i}} \;=\; \pm\varepsilon_{\underline{ijkl}} D_{\underline{k}}\dot{W}_{\underline{l}}. \tag{2.50}$$

Inserting the decomposition of tangent vectors with respect to the basis of zero-modes (2.45) into the effective Lagrangian (2.44), we find

$$\begin{aligned} L_{\text{eff}} \;&=\; \frac{1}{2} g(\dot{W}, \dot{W}) - \frac{4\pi a}{e} b(\vec{k}) \\ &=\; \frac{1}{2} g_{ab}\dot{X}^a \dot{X}^b - \frac{4\pi a}{e} b(\vec{k}). \end{aligned} \tag{2.51}$$

(This final step is analogous to the derivation of the effective Lagrangian (2.9) of the classical point particle in section 2.1, by inserting the parametrisation (2.8) into the Lagrangian (2.3).) Because the potential energy on $\mathcal{M}_{\vec{k}}$ is constant, the equations of motion for this Lagrangian are simply the geodesic equations for the moduli space. Therefore, classically, slowly moving monopoles follow geodesics on the moduli space [2]. The quantum mechanics of monopoles is described by wavefunctions on the moduli space, as we will discuss in the context of supersymmetric monopoles in the next two chapters.

2.5 The hyperkähler structure of $\mathcal{M}_{\vec{k}}$

In the case of maximal symmetry breaking, moduli spaces of monopoles are hyperkähler manifolds. A hyperkähler structure on a Hermitian manifold is generated by three parallel complex structures, $\mathcal{I}_i$, that obey the quaternion algebra,

$$\mathcal{I}_i \mathcal{I}_j \;=\; -\delta_{ij} + \varepsilon_{ijk} \mathcal{I}_k. \tag{2.52}$$

This implies that there is a whole two-sphere of complex structures, parameterised by $a\mathcal{I}_1 + b\mathcal{I}_2 + c\mathcal{I}_3$ such that $a^2 + b^2 + c^2 = 1$. A hyperkähler manifold is Kähler with respect to each of these complex structures. We will sometimes denote the complex structures by

$$\mathcal{I} \;=\; \mathcal{I}_3, \qquad\qquad \mathcal{J} \;=\; \mathcal{I}_1, \qquad\qquad \mathcal{K} \;=\; \mathcal{I}_2. \tag{2.53}$$

The standard reference for hyperkähler manifolds is the book by Besse [24].

The existence of a hyperkähler structure on the moduli space is deeply connected to the fact that the field theory allows for an $N = 4$ supersymmetric extension [25]. The hyperkähler structure on the moduli space derives from the hyperkähler structure on Euclidean $\mathbb{R}^4$. In the following we use the same symbols $\mathcal{I}_i$ for the action of the complex structures on $\mathbb{R}^4$, on the space of field configurations $\mathcal{A}$, and on the moduli space $\mathcal{M}_{\vec{k}}$.

Kähler structures

A Hermitian manifold is a Riemannian manifold with a metric g and a complex structure $\mathcal{I}$ that satisfy

$$g(\dot{W}, \dot{V}) \;=\; g(\mathcal{I}(\dot{W}), \mathcal{I}(\dot{V})). \tag{2.54}$$

On a Hermitian manifold, we define the Kähler form ω by

$$\omega(\dot{W}, \dot{V}) \;=\; g(\dot{W}, \mathcal{I}(\dot{V})). \tag{2.55}$$

If the Kähler form is closed, $\mathrm{d}\omega = 0$, the underlying manifold is called a Kähler manifold. A general reference on Kähler manifolds is the book by Moroianu [26].

For a basis of tangent vectors $\{\partial_a\}$, the components of the Kähler form are given by

$$\omega_{ab} \;=\; \omega(\partial_a, \partial_b) \;=\; g(\partial_a, \mathcal{I}(\partial_b)) \;=\; g(\partial_a, \partial_c \mathcal{I}^c{}_b) \;=\; g_{ac}\mathcal{I}^c{}_b. \tag{2.56}$$

Complex structures on $\mathcal{M}_{\vec{k}}$

When we interpret $W_{\underline{i}}$ as a connection on $\mathbb{R}^4$, the space of field configurations inherits complex structures from this $\mathbb{R}^4$, via [27, 9]

$$(\mathcal{I}_i(\dot{W}))_{\underline{j}} \;=\; (\mathcal{I}_i)_{\underline{jk}}\; \dot{W}_{\underline{k}}. \tag{2.57}$$

We will give an explicit matrix representation of the complex structures $\mathcal{I}_i$ in the next section. If $\dot{W}_{\underline{j}}$ is a tangent vector to $\mathcal{M}_{\vec{k}}$, then so is the linear combination $(\mathcal{I}_i)_{\underline{jk}}\dot{W}_{\underline{k}}$, and therefore the complex structures on $\mathcal{A}$ can be restricted to complex structures on the moduli space. The metric on $\mathcal{A}$ (2.46), and its restriction to $\mathcal{M}_{\vec{k}}$, are Hermitian with respect to these complex structures, i.e. they satisfy equation (2.54), as can be verified using equation (2.57).

We can define three Kähler forms on the space of field configurations by equations (2.55), using the three complex structures $\mathcal{I}_i$ inherited from $\mathbb{R}^4$. Using equation (2.57) we have

$$\omega_i(\dot{W}, \dot{V}) \;\equiv\; \omega_{\mathcal{I}_i}(\dot{W}, \dot{V}) \;\equiv\; g(\dot{W}, \mathcal{I}_i(\dot{V})) \;=\; \int \mathrm{d}^3x\; \dot{W}_{\underline{i}} \cdot (\mathcal{I}_i)_{\underline{ij}}\; \dot{V}_{\underline{j}}. \tag{2.58}$$

Since both the complex structures and the metric can be restricted to the moduli space, the Kähler forms can be restricted to $\mathcal{M}_{\vec{k}}$ as well. Decomposing vectors on $\mathcal{M}_{\vec{k}}$ as usual, via equation (2.45), the components of the Kähler forms are given by

$$(\omega_i)_{ab} \;=\; \int \mathrm{d}^3x\; \delta_a W_{\underline{i}} \cdot (\mathcal{I}_i)_{\underline{ij}}\; \delta_b W_{\underline{j}}, \tag{2.59}$$

which we now compare to the general expression (2.56),

$$(\omega_i)_{ab} \;=\; g_{ac}(\mathcal{I}_i)^c{}_b \;=\; \int \mathrm{d}^3x\; \delta_a W_{\underline{i}} \cdot \delta_c W_{\underline{i}} (\mathcal{I}_i)^c{}_b. \tag{2.60}$$

Since $\delta_b W_{\underline{i}}$ is orthogonal to gauge modes, so are the linear combinations $(\mathcal{I}_i)_{\underline{ij}}\ \delta_b W_{\underline{j}}$ and $\delta_a W_{\underline{i}}(\mathcal{I}_i)^a{}_b$. Therefore $\delta_a W_{\underline{i}}(\mathcal{I}_i)^a{}_b$ is completely defined by the integral (2.60), and we have

$$\delta_a W_{\underline{i}}(\mathcal{I}_i)^a{}_b \ = \ (\mathcal{I}_i)_{\underline{ij}}\ \delta_b W_{\underline{j}}. \tag{2.61}$$

It is now straightforward to check that the $(\mathcal{I}_i)^a{}_b$ obey the quaternion algebra,

$$\delta_a W_{\underline{i}}(\mathcal{I}_i)^a{}_b(\mathcal{I}_j)^b{}_c \ = \ \delta_a W_{\underline{i}}(-\delta_{ij}\mathbb{1}^a{}_c + \varepsilon_{ijk}(\mathcal{I}_k)^a{}_c) \tag{2.62}$$

The metric is parallel, and by explicit calculation it is possible to show that the Kähler forms are as well,

$$\nabla_a(\omega_i)_{bc} \ = \ 0. \tag{2.63}$$

This in turn implies that the complex structures must be parallel too, and therefore $\mathcal{M}_{\vec{k}}$ is hyperkähler. The hyperkähler structure of the moduli space can also be shown by interpreting the moduli space as a hyperkähler quotient [28, 3].

2.6 Quaternionic description

Some of the above statements can be understood most easily by combining the bosonic zero-modes into a quaternion as follows,

$$\mathsf{w}_a \ = \ \delta_a W_4 - \jmath\,\delta_a W_1 - \kappa\,\delta_a W_2 - \imath\,\delta_a W_3 \ = \ \bar{\mathsf{e}}_{\underline{i}}\,\delta_a W_{\underline{i}}. \tag{2.64}$$

The choice for minus signs in this definition will prove useful for the explicit calculations for the two examples in chapters 6 and 7. We have now identified $\mathbb{R}^4$ with the quaternions, via

$$\mathsf{e}_4 \ = \ 1 \qquad \mathsf{e}_1 \ = \ \jmath \qquad \mathsf{e}_2 \ = \ \kappa \qquad \mathsf{e}_3 \ = \ \imath. \tag{2.65}$$

They obey the quaternion algebra, $\imath^2 = \jmath^2 = \kappa^2 = \imath\jmath\kappa = -1$.

The action of the complex structures (2.61) corresponds to multiplication of the quaternions w_a, and its quaternionic conjugate $\overline{\mathsf{w}}_a = \delta_a W_4 + \jmath\,\delta_a W_1 + \kappa\,\delta_a W_2 + \imath\,\delta_a W_3$, with the unit quaternions. We may choose the definition of the three complex structures so that

$$\mathcal{I}_i(\mathsf{w}_a) \;=\; -\mathsf{w}_a \mathsf{e}_i, \qquad\qquad \mathcal{I}_i(\overline{\mathsf{w}}_a) \;=\; \mathsf{e}_i \overline{\mathsf{w}}_a. \tag{2.66}$$

With this definition, $\mathcal{I}_i(\mathsf{w}_a) = \mathsf{e}_{\underline{i}}(\mathcal{I}_i)_{\underline{ij}}\delta_a W_{\underline{j}} = -\mathsf{e}_{\underline{j}}\mathsf{e}_i\,\delta_a W_{\underline{j}}$, and the components of $\mathcal{I}_i$ are given by

$$(\mathcal{I}_i)_{\underline{ij}} \;=\; \left\langle \mathsf{e}_{\underline{i}}, \mathsf{e}_{\underline{k}}(\mathcal{I}_i)_{\underline{kj}} \right\rangle \;=\; \left\langle \mathsf{e}_{\underline{i}}, -\mathsf{e}_{\underline{j}}\mathsf{e}_i \right\rangle, \tag{2.67}$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{H} \cong \mathbb{R}^4$. Therefore we see that, in the ordered basis $\{e_1, e_2, e_3, e_4\} \leftrightarrow \{\jmath, \kappa, \imath, 1\}$, definition (2.66) corresponds to

$$(\mathcal{I}_1)_{\underline{ij}} \;=\; \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}_{\underline{ij}} \tag{2.68a}$$

$$(\mathcal{I}_2)_{\underline{ij}} \;=\; \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}_{\underline{ij}} \tag{2.68b}$$

$$(\mathcal{I}_3)_{\underline{ij}} \;=\; \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}_{\underline{ij}} \tag{2.68c}$$

We may summarise the action of the complex structures by

$$(\mathcal{I}_i)(e_j) \;=\; e_{\underline{k}}(\mathcal{I}_i)_{\underline{k}j} \;=\; \delta_{ij}e_4 + \varepsilon_{ijk}e_k, \tag{2.69a}$$

$$(\mathcal{I}_i)(e_4) \;=\; e_{\underline{k}}(\mathcal{I}_i)_{\underline{k}4} \;=\; -e_i. \tag{2.69b}$$

The components of the metric and the Kähler forms on the moduli space are given by the real and imaginary parts of

$$\int \mathrm{d}^3x\ \mathsf{w}_a\overline{\mathsf{w}}_b \;=\; g_{ab} + \mathsf{e}_i(\omega_i)_{ab} \tag{2.70}$$

respectively, which can be verified directly by inserting definition (2.64), and comparing with equation (2.59) using the expressions (2.68) for the complex structures. We see again that the metric and Kähler forms are invariant under the action of the complex structures:

$$\begin{aligned} \mathcal{I}_i\left(g_{ab}+\mathsf{e}_i(\omega_i)_{ab}\right) &= \mathcal{I}_i\left(\int \mathrm{d}^3x\ \mathsf{w}_a\overline{\mathsf{w}}_b\right) \\ &= \int \mathrm{d}^3x\left(-\mathsf{w}_a\mathsf{e}_i\right)\left(\mathsf{e}_i\overline{\mathsf{w}}_b\right) \\ &= \int \mathrm{d}^3x\left(\mathsf{w}_a\overline{\mathsf{w}}_b\right) \qquad = g_{ab}+\mathsf{e}_i(\omega_i)_{ab} \end{aligned} \tag{2.71}$$

The background gauge condition (2.48) and the linearised Bogomol'nyi equations (2.50) can also be written together in quaternionic form. We define the quaternionic differential operator

$$\mathsf{D} = \mathsf{e}_{\underline{j}}D_{\underline{j}}, \tag{2.72}$$

in terms of which the background gauge condition and the linearised Bogomol'nyi equations become the real and imaginary parts of

$$\mathsf{D}\mathsf{w}_a = 0 \tag{2.73}$$

respectively. Therefore, the action of the complex structures (2.66) leaves this set of equations invariant:

$$\mathcal{I}_i(\mathsf{D}\mathsf{w}_a) = \mathsf{D}(-\mathsf{w}_a\mathsf{e}_i) = -(\mathsf{D}\mathsf{w}_a)\mathsf{e}_i = 0. \tag{2.74}$$

2.7 Zero-modes as solutions to a Dirac-equation

Notice that equation (2.73) is very similar to a Dirac-equation. We may identify the unit quaternions with $SU(2)$ matrices,

$$\mathsf{e}_i = -i\sigma_i, \qquad \mathsf{e}_4 = \mathbb{1}_2, \tag{2.75}$$

where σ_i are the Pauli matrices. Under this identification, w_a becomes a $(2{\times}2)$-matrix, which is completely determined by either of its two columns:

$$\mathsf{w}_a \;=\; \delta_a\Phi + i\sigma_i\delta_a A_i \;=\; \begin{pmatrix} \delta_a\Phi + i\delta_a A_3 & i\delta_a A_1 + \delta_a A_2 \\ i\delta_a A_1 - \delta_a A_2 & \delta_a\Phi - i\delta_a A_3 \end{pmatrix} \;=\; \begin{pmatrix} \chi_1 & -\overline{\chi}_2 \\ \chi_2 & \overline{\chi}_1 \end{pmatrix}. \tag{2.76}$$

The operator D defined in equation (2.72) now becomes the Dirac operator

$$\mathsf{D} \;=\; D_4\mathbb{1}_2 - i\sigma_j D_j. \tag{2.77}$$

The fact that w_a is a zero-mode of D, as expressed by this equation, then implies that

$$\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \;=\; \chi, \qquad\qquad \begin{pmatrix} -\overline{\chi}_2 \\ \overline{\chi}_1 \end{pmatrix} \;=\; -i\sigma_2\,\overline{\chi}, \tag{2.78}$$

are zero-modes of D,

$$\mathsf{D}\chi \;=\; 0, \qquad\qquad -i\,\mathsf{D}\sigma_2\overline{\chi} \;=\; 0. \tag{2.79}$$

Conversely, if χ is a zero-mode of D, then so is w_a. Using the fact that $\overline{\sigma}_1 = \sigma_1$, $\overline{\sigma}_2 = -\sigma_2$ and $\overline{\sigma}_3 = \sigma_3$,

$$\begin{aligned} -i\,\mathsf{D}\,(\sigma_2\overline{\chi}) \;&=\; -i\,(-e\;\;\mathrm{ad}\,\Phi\mathbb{1}_2 - iD_j\sigma_j)\,\sigma_2\overline{\chi} \\ &=\; -i\,(-e\;\;\mathrm{ad}\,\Phi\sigma_2 + iD_1\sigma_2\sigma_1 - iD_2\sigma_2\sigma_2 + iD_3\sigma_2\sigma_3)\,\overline{\chi} \\ &=\; -i\,\sigma_2\,(-e\;\;\mathrm{ad}\,\Phi\mathbb{1}_2 + iD_1\sigma_1 - iD_2\sigma_2 + iD_3\sigma_3)\,\overline{\chi} \\ &=\; -i\,\sigma_2\overline{\mathsf{D}}\overline{\chi} \;=\; -i\,\sigma_2\overline{\mathsf{D}\chi} \\ &=\; 0. \end{aligned} \tag{2.80}$$

Together with the original assumption $\mathsf{D}\chi = 0$, this then implies $\mathsf{D}\mathsf{w}_a = 0$.

We conclude that the bosonic zero-modes w_a of the monopole, correspond to the (2-spinor) zero-modes of the Dirac operator D.

Dimension of the moduli space

We can deduce the dimension of the moduli-space from the dimension of the kernel of the Dirac operator, but we need to be careful. The Dirac operator D is a complex operator acting on complex 2-component spinors. Therefore its kernel is a complex vector space of complex dimension $\dim_{\mathbb{C}} \ker \mathsf{D}$. However, the matrices w_a and $i\mathsf{w}_a$ correspond to linearly independent zero-modes of the monopole, so that the dimension of the moduli-space is equal to the real dimension of kernel of D:

$$\dim \mathcal{M}_{\vec{k}} \;=\; \dim_{\mathbb{R}} \mathcal{M}_{\vec{k}} \;=\; \dim_{\mathbb{R}} \ker \mathsf{D} \;=\; 2 \dim_{\mathbb{C}} \ker \mathsf{D}. \tag{2.81}$$

The index of a complex operator is defined as

$$\operatorname{ind} \mathsf{D} \;=\; \dim_{\mathbb{C}} \ker \mathsf{D} - \dim_{\mathbb{C}} \ker \mathsf{D}^{\dagger}. \tag{2.82}$$

$\mathsf{D}^{\dagger}$ has no normalisable zero-modes, $\dim_{\mathbb{C}} \ker \mathsf{D}^{\dagger} = 0$, so that $\operatorname{ind} \mathsf{D} = \dim_{\mathbb{C}} \ker \mathsf{D}$, and the real dimension of the moduli space is twice the index of the Dirac operator D,

$$\dim \mathcal{M}_{\vec{k}} \;=\; 2 \operatorname{ind} \mathsf{D}. \tag{2.83}$$

Callias [22] has found that for the Dirac operator corresponding to $SU(2)$ monopoles $\operatorname{ind} \mathsf{D} = 2k$, so that

$$\dim \mathcal{M}_{k} \;=\; 4k. \tag{2.84}$$

As mentioned before in section 2.4, Weinberg [21] first argued that more generally

$$\dim \mathcal{M}_{\vec{k}} \;=\; 4k, \tag{2.85}$$

where $k = |k_1 + \ldots + k_{n-1}|$ (where again the integers $k_1, \ldots, k_{n-1}$ are either all positive or all negative).

2.8 Zero-modes of the 't Hooft-Polyakov monopole

For a single, charge-1 monopole, we know that there are 4 zero-modes. Three of these, $\delta_1 W_{\underline{j}}$, $\delta_2 W_{\underline{j}}$ and $\delta_3 W_{\underline{j}}$, correspond to translations in space. The fourth, $\delta_4 W_{\underline{j}} = \delta_\chi W_{\underline{j}}$, corresponds to gauge transformations $g(\chi)$ in the unbroken gauge group.

The naive guess for the zero-modes of translation would be $\delta_i W_{\underline{j}} = \partial_i W_{\underline{j}}$. However, as we have seen, the actual zero-modes are perpendicular to gauge modes, because the field configurations must satisfy Gauss' Law. Therefore, the zero-modes are given by a gauge transformation of the naive guess

$$\delta_i W_{\underline{j}} \;=\; \partial_i W_{\underline{j}} - \frac{1}{e} D_{\underline{j}} \Lambda_i, \tag{2.86}$$

and we need to choose Λ_i in a suitable way, to ensure that the zero-modes satisfy the background gauge condition. This can be done by choosing

$$\Lambda_i \;=\; eA_i. \tag{2.87}$$

The zero-modes of translation are then

$$\delta_i W_{\underline{j}} \;=\; \partial_i W_{\underline{j}} - D_{\underline{j}} W_i \;=\; G_{i\underline{j}}. \tag{2.88}$$

As stated above, the fourth zero-mode $\delta_\chi W_{\underline{j}}$ is given by an infinitesimal gauge transformation in the unbroken gauge group, with gauge parameter $\Lambda \sim \Phi$,

$$\delta_4 W_{\underline{j}} \;=\; -D_{\underline{j}}\Phi \;=\; G_{4\underline{j}}. \tag{2.89}$$

The four zero-modes together can therefore be written as

$$\delta_{\underline{i}} W_{\underline{j}} \;=\; G_{\underline{i}\underline{j}}. \tag{2.90}$$

Finally, since the Bogomol'nyi equations imply the equations of motion for static field configurations (2.41), we see that $\delta_{\underline{i}} W_{\underline{j}}$ satisfies the background gauge condition:

$$D_{\underline{j}}\delta_{\underline{i}} W_{\underline{j}} \;=\; D_{\underline{j}} G_{\underline{i}\underline{j}} \;=\; 0. \tag{2.91}$$

The translational zero-modes $\delta_i W_{\underline{j}}$ give rise to a factor of $\mathbb{R}^3$ in the moduli space of the 't Hooft-Polyakov monopole. In contrast, the gauge transformations $g(\chi) = e^{-\chi\frac{\Phi}{a}}$ have a periodic parameter: $g(\chi) = g(\chi + 2\pi)$. Therefore, the gauge transformations give rise to a factor of S^1 in the moduli space. The total moduli space is hence

$$\mathcal{M}_1 \;=\; \mathbb{R}^3 \times S^1. \tag{2.92}$$

Action of the complex structures

From the zero-mode $\delta_\chi W_{\underline{i}}$ corresponding to gauge transformations, we form the quaternion

$$\mathsf{w}_\chi \;=\; \delta_\chi \Phi - \mathsf{e}_i \delta_\chi A_i \;=\; \mathsf{e}_i (D_i \Phi). \tag{2.93}$$

Acting with the complex structures $\mathcal{I}_j$ we find, using equation (2.29),

$$
\begin{aligned}
\mathcal{I}_j(\mathsf{w}_\chi) &= -\mathsf{w}_\chi \mathsf{e}_j \\
&= -(D_i\Phi)(\mathsf{e}_i\mathsf{e}_j) \\
&= D_j\Phi - \varepsilon_{ijk}\mathsf{e}_k\, B_i \\
&= D_j\Phi - \mathsf{e}_k\, F_{jk} \\
&= \delta_j\Phi - \mathsf{e}_k\, \delta_j A_k \\
&= \mathsf{w}_j, \qquad (2.94a)
\end{aligned}
$$

where w_j is the quaternion corresponding to the translational zero-modes $\delta_j W_{\underline{i}}$. A similar calculation yields

$$
\mathcal{I}_i(\mathsf{w}_j) = -\delta_{ij}\mathsf{w}_\chi + \varepsilon_{ijk}\mathsf{w}_k. \qquad (2.94b)
$$

Chapter 3

$N = 2$ Supersymmetric Monopoles

We now turn to $N = 2$ supersymmetric monopoles. As in the bosonic case, we will first review BPS monopoles and the corresponding zero-modes. Then we will discuss the effective Lagrangian in the moduli space approximation of the supersymmetric model, and its quantisation. This can be done in terms of either spinors (section 3.4), or anti-holomorphic forms on the moduli space (section 3.5). We review the realisation of particular supersymmetry charges as a Dirac operator on spinors and a Dolbeault operator on anti-holomorphic forms. The standard reference for this discussion is the article by Gauntlett [9]. A recent review by Weinberg and Yi [29] discusses these topics in a wider context. The lecture notes on electromagnetic duality by Figueroa-O'Farrill [30] are a useful guide for many of the calculations.

To complete the discussion of the quantisation of the effective model, we construct the differential operators corresponding to the remaining supercharges, and interpret them as twisted Dirac operators acting on spinors (in section 3.4.4), and twisted Dolbeault operators acting on anti-holomorphic forms (in section 3.5.5). The identification of all the supercharges is essential for finding all the states in a supermultiplet, as we will illustrate in chapters 6 and 7.

3.1 $N = 2$ supersymmetric BPS monopoles

The Yang-Mills-Higgs Lagrangian (2.13) can be extended with $N = 2$ supersymmetry. The supersymmetric Lagrangian is given by [9]

$$L = \int \mathrm{d}^3x\, \mathcal{L} = \int \mathrm{d}^3x \Big(-\frac{1}{4} F^{\mu\nu} \cdot F_{\mu\nu} + \frac{1}{2} D_\mu S \cdot D^\mu S + \frac{1}{2} D_\mu P \cdot D^\mu P - \frac{e^2}{2} ||\, [P, S]\, ||^2$$

$$+ i\overline{\psi} \cdot \gamma^\mu D_\mu \psi \;+\; ie\overline{\psi} \cdot (\mathrm{ad}\, S - i\gamma_5\, \mathrm{ad}\, P)\psi \Big). \tag{3.1}$$

Here S is a scalar field, P a pseudo-scalar field, and ψ a Dirac spinor. The chiral operator γ_5 is defined by $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. The supersymmetries of the Lagrangian (3.1) are given by

$$\delta A_\mu = i\left(\overline{\alpha}\gamma_\mu\psi - \overline{\psi}\gamma_\mu\alpha\right), \tag{3.2a}$$

$$\delta P = \left(\overline{\psi}\gamma_5\alpha - \overline{\alpha}\gamma_5\psi\right), \tag{3.2b}$$

$$\delta S = i\left(\overline{\psi}\alpha - \overline{\alpha}\psi\right), \tag{3.2c}$$

$$\delta\psi = \left(\frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu} + ie\gamma_5\,[P, S] \;-\; \gamma^\mu D_\mu(S - i\gamma_5 P)\right)\,\alpha, \tag{3.2d}$$

where the parameter α is a Dirac spinor. A Dirac spinor is equivalent to two Majorana spinors, which explains the number of supersymmetries, $N = 2$. These supersymmetries are most easily exhibited by deriving this Lagrangian and its supersymmetries from an $N = 1$ supersymmetric Lagrangian in 6 dimensions by dimensional reduction [31, 32] (see appendix B.1 for details).

The rotational symmetry of the extra dimensions reduces to an $SO(2)$ chiral rotational symmetry in four dimensions [30]:

$$S + iP \;\mapsto e^{-i\mu}(S + iP),$$

$$\psi \;\mapsto e^{\mu\gamma_5/2}\psi,$$

$$A_\mu \;\mapsto A_\mu. \tag{3.3}$$

As in the bosonic case, the symmetry breaking is induced by choosing appropriate boundary conditions on the fields at infinity:

$$\lim_{r\to\infty}\left(||S||^2+||P||^2\right) \;=\; a^2. \tag{3.4}$$

The kinetic and potential energy for the $N=2$ supersymmetric model are

$$K \;=\; \int \mathrm{d}^3x\,\Big(-||\frac{1}{2}F_{0i}||^2+\frac{1}{2}||D_0S||^2+\frac{1}{2}||D_0P||^2+i\overline{\psi}\cdot\gamma^0D_0\psi\Big), \tag{3.5}$$

$$V \;=\; \int \mathrm{d}^3x\,\Big(\,\frac{1}{4}F^{ij}\cdot F_{ij}+\frac{1}{2}||D_iS||^2+\frac{1}{2}||D_iP||^2+\frac{e^2}{2}||\;[P,S]\;||^2$$

$$+\,i\overline{\psi}\cdot\gamma_iD_i\psi-ie\overline{\psi}\cdot[S-i\gamma_5P,\psi]\Big). \tag{3.6}$$

The BPS monopoles are defined, as before, to have minimal energy. This implies again that they are static.

To find the zero-modes we first use the $SO(2)$ chiral rotational symmetry, so that we may assume that only the scalar field S has a non-zero vacuum expectation value. (If we require the vacuum to be parity-invariant, the vacuum expectation value of pseudoscalar field P must be zero to begin with.) In this case $S=\Phi$ takes on the role of the Higgs field of the bosonic model, and it must satisfy the Bogomol'nyi equations (2.29). To minimise the potential energy (3.6), ψ must satisfy the following Dirac equation in the presence of a monopole background,

$$\gamma_0\gamma_iD_i\psi-e\gamma_0\,[S,\psi] \;=\; 0. \tag{3.7}$$

We define Euclidian gamma-matrices by

$$\overline{\gamma}_i \;=\; \gamma_0\gamma_i, \qquad\qquad \overline{\gamma}_4 \;=\; \gamma_0, \tag{3.8}$$

which satisfy $\{\overline{\gamma}_{\underline{i}},\overline{\gamma}_{\underline{j}}\}=2\delta_{\underline{ij}}$. In terms of these, the Dirac equation (3.7) becomes

$$\not{D}\psi \;\equiv\; \overline{\gamma}_{\underline{i}}D_{\underline{i}}\psi \;=\; 0, \tag{3.9}$$

where $D_{\underline{i}}$ is the covariant derivative in Euclidian space defined in equation (2.31).

3.2 Zero-modes

Since $\psi=0$ is a solution of the Dirac equation (3.9), the purely bosonic monopole solutions discussed in chapter 2 are solutions of the $N=2$ supersymmetric model as well. The bosonic zero-modes of this model are therefore exactly the same as those of the purely bosonic model. The fermionic zero-modes are the solutions of the Dirac equation (3.9).

A convenient representation for the Euclidian γ-matrices is given by

$$\overline{\gamma}_i = \begin{pmatrix} 0 & i\sigma_i \\ -i\sigma_i & 0 \end{pmatrix}, \qquad \overline{\gamma}_4 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}. \tag{3.10}$$

We now identify the unit quaternions with $SU(2)$ matrices, as in section 2.7,

$$\mathsf{e}_1 \sim -i\sigma_1, \qquad \mathsf{e}_2 \sim -i\sigma_2, \qquad \mathsf{e}_3 \sim -i\sigma_3, \qquad \mathsf{e}_4 \sim \mathbb{1}_2, \tag{3.11}$$

so that we may write the Dirac equation (3.9) as

$$\not{D}\psi \equiv \begin{pmatrix} 0 & \mathsf{D}^\dagger \\ \mathsf{D} & 0 \end{pmatrix} \psi = 0, \tag{3.12}$$

where D is defined in (2.73). Since $\mathsf{D}^\dagger$ has no normalisable zero-modes, the fermionic zero-modes can all be written in terms of the bosonic zero-modes as [9]

$$\psi = \psi_a \lambda^a, \tag{3.13a}$$

with

$$\psi_a = \overline{\gamma}_{\underline{i}} \delta_a W_{\underline{i}} \begin{pmatrix} 0 \\ \chi \end{pmatrix} = \begin{pmatrix} \bar{\mathsf{e}}_{\underline{i}} \delta_a W_{\underline{i}}\, \chi \\ 0 \end{pmatrix} = \begin{pmatrix} \mathsf{w}_a\, \chi \\ 0 \end{pmatrix}, \tag{3.13b}$$

where w_a is the bosonic zero-mode defined in equation (2.64), and χ is a constant, normalised, commuting two-component spinor. Since the fermionic zero-mode ψ is an anti-commuting spinor, and χ is a commuting spinor, λ^a must be a Grassmann number.

ψ defined in equation (3.13a) is a fermionic zero-mode for any constant, complex spinor χ and complex valued λ^a. Using equation (2.73), we have indeed

$$\not{D}\psi \equiv \begin{pmatrix} 0 & \mathsf{D}^\dagger \\ \mathsf{D} & 0 \end{pmatrix} \psi = \begin{pmatrix} (\mathsf{D}\mathsf{w}_a)\, \chi \\ 0 \end{pmatrix} \lambda^a = 0, \tag{3.14}$$

However, the complex structures can be used to identify many of the fermionic zero-modes obtained in this manner. We shall see that the vector space of fermionic zero-modes can be thought of as an $\mathbb{R}$-vector space with basis $\{\psi_a\}$, where the ψ_a are defined by equation (3.13b) using a single, fixed spinor χ.

3.2.1 Action of the complex structures

The complex structures act on the fermionic zero-modes (3.13b) via equation (2.61) or (2.66). Using hats to distinguish the action of the complex structures on fermionic zero-modes from the action of the complex structures on bosonic zero-modes, we have

$$\hat{\mathcal{I}}_i(\psi_a) \;=\; \begin{pmatrix} \mathcal{I}_i(\mathsf{w}_a)\chi \\ 0 \end{pmatrix} \;=\; \begin{pmatrix} -\mathsf{w}_a\mathsf{e}_i\chi \\ 0 \end{pmatrix}. \tag{3.15}$$

Choice of χ

The fermionic zero-modes $\psi_a = \begin{pmatrix} \mathsf{w}_a\,\chi \\ 0 \end{pmatrix}$ and $\psi'_a = \begin{pmatrix} \mathsf{w}_a\,\chi' \\ 0 \end{pmatrix}$ are related via the action of a complex structure. The spinors χ and χ' are related by a $U(2)$ transformation, $\chi' = U\chi$, so that

$$\psi'_a \;=\; \begin{pmatrix} \mathsf{w}_a\;\chi' \\ 0 \end{pmatrix} \;=\; \begin{pmatrix} \mathsf{w}_a\;U\;\chi \\ 0 \end{pmatrix} \;=\; \begin{pmatrix} \mathsf{w}'_a\;\chi \\ 0 \end{pmatrix}, \tag{3.16}$$

where $\mathsf{w}'_a = \mathsf{w}_a U$ is the bosonic zero-mode obtained from w_a using the action of the complex structure corresponding to U (which in general is a linear combination of the $\mathcal{I}_i$). For example, if $\chi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi' = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then $U = \mathsf{e}_2$, $\mathsf{w}'_a = -\mathcal{I}_2(\mathsf{w}_a)$ and $\psi'_a = -\hat{\mathcal{I}}_2(\psi_a)$. Therefore, we can obtain all fermionic zero-modes from the bosonic ones using equations (3.13) with a single, fixed χ.

Action of the complex structures

From equation (3.15) we see that there is a complex structure $\hat{\mathcal{I}}$ (which depends on the choice of χ) such that [9]

$$\hat{\mathcal{I}}(\psi_a) \;=\; i\psi_a. \tag{3.17}$$

For example, if we choose

$$\chi \;=\; \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{3.18}$$

then

$$\hat{\mathcal{I}} \;=\; \hat{\mathcal{I}}_3. \tag{3.19}$$

Having made a choice, and fixed χ, the remaining two complex structures that make up the hyperkähler structure, $\hat{\mathcal{J}}$ and $\hat{\mathcal{K}}$ act anti-linearly:

$$\hat{\mathcal{J}}(i\psi_a) \;=\; \hat{\mathcal{J}}\hat{\mathcal{I}}(\psi_a) \;=\; -\hat{\mathcal{I}}\hat{\mathcal{J}}(\psi_a) \;=\; -i\hat{\mathcal{J}}(\psi_a), \tag{3.20}$$

and similarly for $\hat{\mathcal{K}}$.

3.2.2 The vector space of fermionic zero-modes

Equation (3.17) shows that the $4k$ fermionic zero-modes ψ_a (defined by (3.13b) in terms of the $4k$ bosonic zero-modes using a single, fixed spinor χ) are not linearly independent over $\mathbb{C}$. Therefore, the $4k$-dimensional real vector space of bosonic zero-modes corresponds to a $2k$-dimensional complex vector space V of fermionic zero-modes, in agreement with Callias' index theorem [22]. If

$$\mathcal{B}_V^{\mathbb{C}} \;=\; \{\psi_1, \ldots, \psi_{2k}\} \tag{3.21}$$

is a basis of V over $\mathbb{C}$, then, by equation (3.17), a basis of V over $\mathbb{R}$ is given by $\{\psi_1, \ldots, \psi_{4k}\}$, where $\psi_{2k+\alpha} = \hat{\mathcal{I}}(\psi_\alpha)$:

$$\mathcal{B}_V^{\mathbb{R}} \;=\; \{\psi_1, \ldots, \psi_{2k}, \hat{\mathcal{I}}(\psi_1), \ldots, \hat{\mathcal{I}}(\psi_{2k})\}. \tag{3.22}$$

3.2.3 Zero-modes as anti-holomorphic forms on $\mathbb{R}^4$

The fermionic zero-modes ψ_a (3.13) are static spinors in (3+1)-dimensional space-time. Extending space-time to $\mathbb{R} \times \mathbb{R}^4$, as we did in section 2.3, we may also view them as spinors on $\mathbb{R}^4$ that are independent of the fourth dimension. These spinors can now be identified with anti-holomorphic forms [33]. For example, we can identify the fermionic zero-modes ψ_a with anti-holomorphic forms $\overline{\upsilon}_a$ by

$$\psi_a \;=\; \begin{pmatrix} \chi_1 \\ \chi_2 \\ 0 \\ 0 \end{pmatrix} \qquad \sim \qquad \overline{\upsilon}_a \;=\; \chi_1\overline{\alpha}_1 + \chi_2\overline{\alpha}_2, \tag{3.23}$$

where we use the basis $\overline{\alpha}_1 = \frac{1}{\sqrt{2}}(\mathrm{d}x^3 - i\mathrm{d}x^4)$ and $\overline{\alpha}_2 = \frac{1}{\sqrt{2}}(\mathrm{d}x^1 - i\mathrm{d}x^2)$ of anti-holomorphic forms (with respect to the complex structure $\mathcal{I}$) on $\mathbb{R}^4$. This identification agrees with equation (3.17).

The complex structures $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$ act naturally on the space of all differential forms, and hence on this basis. One finds (see also appendix A)

$$\begin{aligned} \mathcal{I}(\overline{\alpha}_1) &= i\overline{\alpha}_1, & \mathcal{J}(\overline{\alpha}_1) &= i\alpha_2, & \mathcal{K}(\overline{\alpha}_1) &= \alpha_2, \\ \mathcal{I}(\overline{\alpha}_2) &= i\overline{\alpha}_2, & \mathcal{J}(\overline{\alpha}_2) &= -i\alpha_1, & \mathcal{K}(\overline{\alpha}_2) &= -\alpha_1. \end{aligned} \tag{3.24}$$

These (linear) actions of the complex structures on forms are related to the (linear) action of $\hat{\mathcal{I}}$, and the (anti-linear) actions of $\hat{\mathcal{J}}$ and $\hat{\mathcal{K}}$ on spinors defined in (3.15) as follows. Using the identification (3.23), we can pull the maps $\hat{\mathcal{I}}$, $\hat{\mathcal{J}}$ and $\hat{\mathcal{K}}$ back to maps on anti-holomorphic forms. We will denote these pull-backs by the same letters $\hat{\mathcal{I}}$, $\hat{\mathcal{J}}$ and $\hat{\mathcal{K}}$. If we choose χ as in (3.18), and write the bosonic zero-mode corresponding to ψ_a in the form (2.76), we find that

$$\hat{\mathcal{J}}(\psi_a) = \hat{\mathcal{J}}\begin{pmatrix} \mathsf{w}_a\,\chi \\ 0 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \chi_1 & -\overline{\chi}_2 \\ \chi_2 & \overline{\chi}_1 \end{pmatrix} i\sigma_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -i\overline{\chi}_2 \\ i\overline{\chi}_1 \\ 0 \\ 0 \end{pmatrix}. \tag{3.25}$$

Under the identification (3.23) we therefore have

$$\hat{\mathcal{J}}(\psi_a) \qquad \sim \qquad \hat{\mathcal{J}}(\overline{v}_a) := -i\overline{\chi}_2\overline{\alpha}_1 + i\overline{\chi}_1\overline{\alpha}_2 = -\overline{\mathcal{J}(\overline{v}_a)}. \tag{3.26}$$

Summarising the action of the complex structures on anti-holomorphic forms we have

$$\hat{\mathcal{I}}(\overline{v}_a) = \mathcal{I}(\overline{v}_a) = i\,\overline{v}_a, \qquad \hat{\mathcal{I}} = \mathcal{I}, \tag{3.27a}$$

$$\hat{\mathcal{J}}(\overline{v}_a) = -\overline{\mathcal{J}}(\overline{v}_a) = -\overline{\mathcal{J}(\overline{v}_a)}, \qquad \hat{\mathcal{J}} = -\overline{\mathcal{J}}, \tag{3.27b}$$

$$\hat{\mathcal{K}}(\overline{v}_a) = -\overline{\mathcal{K}}(\overline{v}_a) = -\overline{\mathcal{K}(\overline{v}_a)}, \qquad \hat{\mathcal{K}} = -\overline{\mathcal{K}}, \tag{3.27c}$$

where $\overline{\mathcal{J}}$ and $\overline{\mathcal{K}}$ are anti-linear maps from the space of anti-holomorphic forms to itself. It is straightforward to verify that the complex structures acting on the fermionic zero-modes interpreted as forms obey the quaternion algebra.

3.3 The moduli space approximation

We now turn to the moduli space approximation. As before, we must parameterise the lowest energy states with moduli space parameters, and by inserting this parameterisation into the Lagrangian of the model, we can derive the effective Lagrangian for the moduli space approximation.

We view the space of fermionic zero-modes as a $4k$-dimensional real vector space, and we parameterise the fermionic zero-modes using equations (3.13) with real valued Grassmann variables λ^a. Inserting this parametrisation into the Lagrangian (3.1) and expanding to lowest non-trivial order, Gauntlett [9] has found (see also Weinberg and Yi [29]) that the effective Lagrangian for the moduli space approximation is given by

$$L_{\text{eff}} = \frac{1}{2} g(\dot{X}, \dot{X}) + \frac{i}{2} g(\lambda, D_t \lambda) - \frac{4\pi a}{e} b(\vec{k}), \tag{3.28}$$

where the covariant derivative $D_t = \dot{X}^a D_a$, and

$$(D_t \lambda)^a = \dot{\lambda}^a + \Gamma^a_{bc} \dot{X}^b \lambda^c. \tag{3.29}$$

We see that the $N = 2$ supersymmetry of the theory has added a fermionic term to the bosonic effective Lagrangian. The effective Lagrangian of the moduli space approximation of $N = 2$ supersymmetric monopoles (3.28) is a σ-model with target space $\mathcal{M}_{\vec{k}}$. In the moduli space approximation, half of the original supersymmetries are broken, and we discuss the remaining supersymmetries of the effective Lagrangian below in the context of the quantisation of the effective model.

On our way to a quantum mechanical description of the $N = 2$ supersymmetric monopoles at low energies we may now proceed in two different, but equivalent ways. If we continue to work with real coordinates X^a on the moduli space, we naturally end up with a quantum theory of spinors on the moduli space (section 3.4). Alternatively, if we choose to work with complex coordinates, the natural way to quantise the theory leads to anti-holomorphic forms on the moduli space (section 3.5). Since the moduli space is hyperkähler, and hence Ricci flat, these two descriptions are equivalent [27], as we shall demonstrate explicitly for the examples $\mathcal{M}_1$ and $\mathcal{M}_{1,1}$ in chapters 6 and 7.

3.4 Quantisation using spinors on the moduli space

The quickest route towards quantisation is to continue to work with real coordinates on the moduli space, but to start by introducing an orthonormal frame to parameterise the fermionic zero-modes (Friedan and Windey [34], Davis, Macfarlane, Popat and Van Holten [35, 36], and Gauntlett [27]). The effective Hamiltonian can be derived without such a frame, but the canonical momenta one finds in this case are not suitable for quantisation, and an orthonormal frame will have to be introduced eventually.

We define the orthonormal frame e by

$$g_{ab} \;=\; \delta_{AB}e^A{}_a e^B{}_b, \qquad\qquad e^A \;=\; e^A{}_a \mathrm{d}X^a. \tag{3.30}$$

We denote the inverse of $e^B{}_c$ by $e^c{}_B$, in the sense that $e^c{}_B e^B{}_d = \delta^c_d$ and $e^A{}_c e^c{}_B = \delta^A_B$, so that

$$\delta_{AB} \;=\; g_{ab}e^a{}_A e^b{}_B, \qquad\qquad \mathrm{d}X^a \;=\; e^a{}_A e^A. \tag{3.31}$$

Using the orthonormal frame, we define the fermionic variables

$$\lambda^A \;=\; e^A{}_a \lambda^a, \tag{3.32}$$

and in terms of the orthonormal frame, the covariant derivative of λ becomes

$$(D_t\lambda)^A \;=\; \dot{\lambda}^A + \omega_a{}^A{}_B \dot{X}^a \lambda^B, \tag{3.33}$$

where the spin connection ω is determined by a gauge transformation

$$\omega_a{}^A{}_B \;=\; e^A{}_b \Gamma^b_{ac} e^c{}_B + e^A{}_b \partial_a e^b{}_B. \tag{3.34}$$

The effective Lagrangian (3.28) becomes

$$L_{\text{eff}} \;=\; \frac{1}{2}g_{ab}\dot{X}^a\dot{X}^b + \frac{i}{2}\delta_{AB}\lambda^A (D_t\lambda)^B - \frac{4\pi a}{e} b(\vec{k}). \tag{3.35}$$

3.4.1 Effective Hamiltonian

The canonical momenta corresponding to the effective Lagrangian (3.35) are

$$p_a \;=\; \frac{\partial L_{\text{eff}}}{\partial \dot{X}^a} \;=\; g_{ab}\dot{X}^b + \frac{i}{2}\omega_{aAB}\lambda^A\lambda^B, \tag{3.36a}$$

$$\pi_A \;=\; \frac{\partial L_{\text{eff}}}{\partial \dot{\lambda}^A} \;=\; -\frac{i}{2}\delta_{AB}\lambda^B, \tag{3.36b}$$

where $\omega_{aAB} = \delta_{AC}\omega_a{}^C{}_B$. The effective Hamiltonian is given by

$$\begin{aligned} H_{\text{eff}} &= \dot{X}^a p_a + \dot{\lambda}^A \pi_A - L_{\text{eff}} \\ &= \dot{X}^a (g_{ab}\dot{X}^b + \frac{i}{2}\omega_{aAB}\lambda^A\lambda^B) - \frac{i}{2}\delta_{AB}\dot{\lambda}^A\lambda^B \\ &\quad - \frac{1}{2}g_{ab}\dot{X}^a\dot{X}^b - \frac{i}{2}\delta_{AB}\lambda^A(\dot{\lambda}^B + \omega_a{}^B{}_C\dot{X}^a\lambda^C) + \frac{4\pi a}{e}b(\vec{k}) \\ &= \frac{1}{2}g_{ab}\dot{X}^a\dot{X}^b + \frac{4\pi a}{e}b(\vec{k}) \\ &= H_0 + \frac{4\pi a}{e}b(\vec{k}). \end{aligned} \tag{3.37}$$

Here we have defined

$$H_0 = \frac{1}{2}g^{ab}\tilde{p}_a\tilde{p}_b, \tag{3.38}$$

and

$$\tilde{p}_a = p_a - \frac{i}{2}\omega_{aAB}\lambda^A\lambda^B = g_{ab}\dot{X}^b. \tag{3.39}$$

3.4.2 Dirac brackets

The canonical way of quantisation is to replace Poisson brackets by (anti-)commutator brackets. In this case, however, the expression for the fermionic momenta leads to constraints. We must therefore use Dirac brackets instead of Poisson brackets, so that brackets with the constraints vanish identically. This allows us to set the constraints equal to zero, and canonical quantisation is done by replacing Dirac brackets with (anti-)commutators. This also affects the discussion of the supersymmetry of the effective model later on. Normally a symmetry is generated by its corresponding charge through a Poisson bracket, but here too we shall have to employ Dirac brackets instead, due to the constraints.

Having used an orthonormal frame to define the fermionic variables λ^A, we find that the Dirac brackets of the bosonic and fermionic variables decouple. We can therefore quantise the effective model by changing the Dirac brackets into (anti-) commutators.

Dirac brackets are defined in terms of the canonical Poisson brackets by

$$\{f,g\}_{DB} \;=\; \{f,g\}_{PB} - \{f,\xi_u\}_{PB}\,(\Delta^{-1})^{uv}\,\{\xi_v,g\}_{PB}\,, \tag{3.40}$$

where Δ is given in terms of the constraints ξ by

$$\Delta_{rs} \;=\; \{\xi_r,\xi_s\}_{PB}\,. \tag{3.41}$$

It is now straightforward to show that Dirac brackets with a constraint function vanish:

$$\begin{aligned}\{f,\xi_r\}_{DB} \;&=\; \{f,\xi_r\}_{PB} - \{f,\xi_u\}_{PB}\,(\Delta^{-1})^{uv}\,\{\xi_v,\xi_r\}_{PB}\\ &=\; \{f,\xi_r\}_{PB} - \{f,\xi_u\}_{PB}\,(\Delta^{-1})^{uv}\Delta_{vr}\\ &=\; \{f,\xi_r\}_{PB} - \{f,\xi_r\}_{PB}\\ &=\; 0\end{aligned} \tag{3.42}$$

The canonical Poisson brackets are

$$-\{z^s,p_r\}_{PB} \;=\; \{p_r,z^s\}_{PB} \;=\; \delta_r^s \tag{3.43a}$$

$$\left\{\lambda^B,\pi_A\right\}_{PB} \;=\; \left\{\pi_A,\lambda^B\right\}_{PB} \;=\; \delta_A^B \tag{3.43b}$$

We will also need the following:

$$\{p_r,\delta_{AB}\}_{PB} \;=\; \frac{\partial p_r}{\partial p_a}\frac{\partial g_{AB}}{\partial z^a} \;=\; \partial_r\delta_{AB} \;=\; 0 \tag{3.44}$$

The constraint functions corresponding to the canonical momenta (3.36) are given by

$$\xi_A \;=\; \pi_A + \frac{i}{2}\delta_{AB}\lambda^B \tag{3.45}$$

and using the naive Poisson brackets we find

$$\begin{aligned}\Delta_{AB} \;&=\; \{\xi_A,\xi_B\}_{PB}\\ &=\; \left\{\pi_A,\frac{i}{2}\delta_{BD}\lambda^D\right\}_{PB} + \left\{\frac{i}{2}\delta_{AC}\lambda^C,\pi_B\right\}_{PB}\\ &=\; i\delta_{AB}\end{aligned} \tag{3.46}$$

The Dirac brackets are therefore

$$\{f,g\}_{DB} \;=\; \{f,g\}_{PB} + i\,\{f,\xi_A\}_{PB}\,\delta^{AB}\,\{\xi_B,g\}_{PB} \tag{3.47}$$

We now compute

$$\{z^r,\xi_A\}_{PB} \;=\; 0 \tag{3.48}$$

$$\{p_r,\xi_A\}_{PB} \;=\; \frac{i}{2}\,\{p_r,\delta_{AB}\}_{PB}\,\lambda^B \;=\; 0 \tag{3.49}$$

Therefore Dirac brackets with z^r or p_r are the same as the original Poisson brackets. Here we see the use of the orthonormal frame we have introduced: without it, the Poisson bracket between the bosonic momenta and the metric would in general be non-zero ($\{p_r, g_{ab}\}_{PB} = \partial_r g_{ab}$), and as a result the Poisson brackets between the momenta and the constraints would not vanish.

For the fermionic variables, however, we find

$$\left\{\lambda^A,\xi_B\right\}_{PB} \;=\; \delta^A_B \tag{3.50}$$

$$\{\pi_A,\xi_B\}_{PB} \;=\; \frac{i}{2}g_{BC}\left\{\pi_A,\lambda^C\right\}_{PB} \;=\; \frac{i}{2}\delta_{AB} \tag{3.51}$$

and therefore Dirac brackets between λ^A become

$$\left\{\lambda^B,\lambda^A\right\}_{DB} \;=\; \left\{\lambda^A,\lambda^B\right\}_{DB} \;=\; i\delta^A_C\delta^{CD}\delta^B_D \;=\; i\delta^{AB} \tag{3.52}$$

The only non-vanishing Dirac brackets are

$$\left\{p_a,X^b\right\}_{DB} \;=\; \delta^b_a, \qquad\qquad \left\{\lambda^A,\lambda^B\right\}_{DB} \;=\; i\delta^{AB}. \tag{3.53}$$

3.4.3 Quantisation

To quantise the theory we follow Friedan and Windey [34], Alvarez-Gaumé [37], and Gauntlett [9]. Dirac brackets of bosons are replaced with commutators, while Dirac brackets of fermions are replaced with anti-commutators.

$$\left\{p_a,X^b\right\}_{DB} \;=\; \delta^b_a \qquad\mapsto\qquad \left[\hat{p}_a,\hat{X}^b\right] \;=\; -i\delta^b_a \tag{3.54}$$

$$\left\{\lambda^A,\lambda^B\right\}_{DB} \;=\; i\delta^{AB} \qquad\mapsto\qquad \left\{\hat{\lambda}^A,\hat{\lambda}^B\right\} \;=\; \delta^{AB} \tag{3.55}$$

The anti-commutator of the fermions defines a Clifford bundle over $\mathcal{M}_{\vec{k}}$, and we identify the Hilbert space of states with the space of spinors on the moduli space. Such spinors are sections of a 2^{2k}-dimensional complex vector bundle over $\mathcal{M}_{\vec{k}}$ which is acted upon by the Dirac matrices generating the Clifford bundle.

The bosonic coordinates act by multiplication and the bosonic momenta are represented as derivatives,

$$p_a \;\mapsto\; -i\partial_a. \tag{3.56}$$

We have a natural map from the fermions to the Dirac matrices on the moduli space:

$$\lambda^A \;\mapsto\; \frac{i}{\sqrt{2}}\gamma^A, \qquad\qquad \{\gamma^A,\gamma^B\} \;=\; -2\delta^{AB}. \tag{3.57}$$

We see that $\tilde{p}_a$ acts as the covariant derivative on spinors,

$$\tilde{p}_a \;= p_a - \frac{i}{4}\omega_{aAB}\left[\lambda^A,\lambda^B\right] \quad\mapsto\quad -i\left(\partial_a - \frac{1}{8}\omega_{aAB}\left[\gamma^A,\gamma^B\right]\right) \;=\; -iD_a. \tag{3.58}$$

Finally, the quantisation of the effective Hamiltonian H_0 gives half the Laplacian,

$$H_0 \;=\; \frac{1}{2}g^{ab}\tilde{p}_a\tilde{p}_b \qquad\mapsto\qquad -\frac{1}{2}g^{ab}D_aD_b \;=\; \frac{1}{2}\Delta. \tag{3.59}$$

Here we have defined the Laplacian to be the positive definite operator, which corresponds to the usual definition for the Laplacian acting on forms, $\Delta = (\mathrm{d}+\mathrm{d}^\dagger)^2$.

3.4.4 Supersymmetry

The effective action corresponding to the effective Lagrangian (3.28) is invariant under $\mathcal{N}=4$ supersymmetry transformations [9]:

$$\delta_{\mathbb{1}}X^a \;=\; \varepsilon\lambda^a \qquad\qquad \delta_{\mathbb{1}}\lambda^a \;=\; i\varepsilon\dot{X}^a \tag{3.60a}$$

$$\delta_{\mathcal{I}_j}X^a \;=\; \varepsilon\,(\mathcal{I}_j)^a{}_b\lambda^b \qquad \delta_{\mathcal{I}_j}\lambda^a \;=\; \varepsilon\left[-i(\mathcal{I}_j)^a{}_b\dot{X}^b - \Gamma^a_{cd}(\mathcal{I}_j)^c{}_b\lambda^b\lambda^d\right] \tag{3.60b}$$

which have their origin in the unbroken supersymmetries of the field theory. The supersymmetries of the original field theory (3.2) have a Dirac spinor α as parameter, which has 8 real independent components. Of these, the $\mathcal{N}=4$ supersymmetries (3.60) remain in the effective model. The corresponding supercharges are

$$Q_{\mathbb{1}} \;=\; \tilde{p}_a\lambda^a, \qquad\qquad Q_{\mathcal{I}_i} \;=\; \tilde{p}_b(\mathcal{I}_i)^b{}_a\lambda^a. \tag{3.61}$$

The supercharges generate the supersymmetry transformations via Dirac brackets,

$$\{Q, X^a\}_{DB} = \delta X^a, \qquad \{Q, \lambda^a\}_{DB} = \delta\lambda^a, \tag{3.62}$$

and they obey the $\mathcal{N} = 4$ supersymmetry algebra:

$$\{Q_{\mathbb{1}}, Q_{\mathbb{1}}\}_{DB} = 2iH_0, \qquad \left\{Q_{\mathcal{I}_i}, Q_{\mathcal{I}_j}\right\}_{DB} = \delta_{ij}\, 2iH_0, \tag{3.63}$$

and all other brackets vanishing. This agrees with the fact that the supersymmetry transformations square to i times a time-derivative, and that time evolution is generated by the Hamiltonian.

When we quantise the supercharges using the quantisation procedures given above, $Q_{\mathbb{1}}$ becomes the Dirac operator for spinors on the moduli space

$$Q_{\mathbb{1}} = \tilde{p}_a \lambda^a \qquad \mapsto \qquad \frac{1}{\sqrt{2}}\gamma^a D_a = \frac{1}{\sqrt{2}} \not{D}, \tag{3.64}$$

while the remaining supercharges become twisted Dirac operators [38]

$$Q_{\mathcal{I}_j} = \tilde{p}_b (\mathcal{I}_j)^b{}_a \lambda^a \qquad \mapsto \qquad \frac{1}{\sqrt{2}} (\mathcal{I}_j)^b{}_a \gamma^a D_b =: \frac{1}{\sqrt{2}} \not{D}_{\mathcal{I}_j}. \tag{3.65}$$

From the fact that the Hamiltonian is given by the Dirac bracket of supercharges, we find again that the quantisation of H_0 gives half the Laplacian:

$$H_0 = -\frac{i}{2}\{Q_{\mathbb{1}}, Q_{\mathbb{1}}\}_{DB} \qquad \mapsto \qquad \frac{1}{2}\not{D}^2 = -\frac{1}{2} g^{ab} D_a D_b = \frac{1}{2}\Delta. \tag{3.66}$$

3.5 Quantisation using forms on the moduli space

3.5.1 Complex coordinates on the moduli space

We now take a few steps back to discuss the quantisation of the effective model on the moduli space in terms of anti-holomorphic forms on the moduli space. We choose $2k$ complex coordinates Z^α on the hyperkähler manifold $\mathcal{M}_{\vec{k}}$ (α runs from 1 to $\frac{1}{2}\dim \mathcal{M}_{\vec{k}} = 2k$) that diagonalise the complex structure $\mathcal{I} = \mathcal{I}_3$. The real and imaginary parts of Z^α form a basis of real coordinates X^a (the index a runs as usual from 1 to $\dim \mathcal{M}_{\vec{k}} = 4k$). We may choose this basis such that

$$\begin{aligned} Z^\alpha &= X^\alpha + iX^{\alpha+2k} & \qquad \partial_\alpha &= \frac{1}{2}\left(\frac{\partial}{\partial X^\alpha} - i\frac{\partial}{\partial X^{\alpha+2k}}\right) \\ \overline{Z}^{\overline{\alpha}} &= X^\alpha - iX^{\alpha+2k} & \qquad \partial_{\overline{\alpha}} &= \frac{1}{2}\left(\frac{\partial}{\partial X^\alpha} + i\frac{\partial}{\partial X^{\alpha+2k}}\right) \end{aligned} \tag{3.67}$$

which satisfy

$$\mathcal{I}(\partial_\alpha) \;=\; i\partial_\alpha \qquad\qquad \mathcal{I}\left(\frac{\partial}{\partial X^\alpha}\right) \;=\; \frac{\partial}{\partial X^{\alpha+2k}}$$

$$\mathcal{I}(\partial_{\overline{\alpha}}) \;=\; -i\partial_{\overline{\alpha}} \qquad\qquad \mathcal{I}\left(\frac{\partial}{\partial X^{\alpha+2k}}\right) \;=\; -\frac{\partial}{\partial X^\alpha} \tag{3.68}$$

The components of the metric in real and complex coordinates are defined by

$$g_{ab} \;= g(\partial_a, \partial_b), \qquad\qquad \mathsf{g}_{\alpha\overline{\beta}} \;= g(\partial_\alpha, \partial_{\overline{\beta}}). \tag{3.69}$$

The metric is Hermitian, $g(\dot{X}, \dot{Y}) = g(I(\dot{X}), I(\dot{Y}))$, so that

$$\mathsf{g}_{\alpha\beta} \;=\; g(\partial_\alpha, \partial_\beta) \;=\; g(i\partial_\alpha, i\partial_\beta) \;=\; -g(\partial_\alpha, \partial_\beta) \;=\; 0, \tag{3.70}$$

and similarly $\mathsf{g}_{\overline{\alpha}\overline{\beta}} \;=\; \overline{\mathsf{g}_{\alpha\beta}} \;=\; 0$. The first term in the effective Lagrangian can be written in complex coordinates as

$$\frac{1}{2}g(\dot{X}, \dot{X}) \;= \frac{1}{2}\left(\mathsf{g}_{\overline{\alpha}\beta}\dot{\overline{Z}}^{\overline{\alpha}}\dot{Z}^\beta + \mathsf{g}_{\alpha\overline{\beta}}\dot{Z}^\alpha\dot{\overline{Z}}^{\overline{\beta}}\right) \;=\; \mathsf{g}_{\alpha\overline{\beta}}\dot{Z}^\alpha\dot{\overline{Z}}^{\overline{\beta}} \;=\; \mathsf{g}_{\overline{\alpha}\beta}\dot{\overline{Z}}^{\overline{\alpha}}\dot{Z}^\beta. \tag{3.71}$$

Using real coordinates

$$\mathcal{I}(\dot{X}) \;= \dot{X}^\alpha\frac{\partial}{\partial X^{\alpha+2k}} - \dot{X}^{\alpha+2k}\frac{\partial}{\partial X^\alpha} \tag{3.72}$$

and the Hermiticity of the metric implies that

$$g_{\alpha\beta} \;= g_{\alpha+2k,\beta+2k}, \qquad\qquad g_{\alpha+2k,\beta} \;=\; -\,g_{\alpha,\beta+2k}. \tag{3.73}$$

The components of the metric in real and complex coordinates are related via

$$\begin{aligned}
\mathsf{g}_{\overline{\alpha}\beta} \;&= g(\partial_{\overline{\alpha}}, \partial_\beta) \\
&= \frac{1}{4}g\left(\frac{\partial}{\partial X^\alpha} + i\frac{\partial}{\partial X^{\alpha+2k}}, \frac{\partial}{\partial X^\beta} - i\frac{\partial}{\partial X^{\beta+2k}}\right) \\
&= \frac{1}{4}\left(g_{\alpha\beta} + g_{\alpha+2k,\beta+2k}\right) + \frac{i}{4}\left(g_{\alpha+2k,\beta} - g_{\alpha,\beta+2k}\right) \\
&= \frac{1}{2}\left(g_{\alpha\beta} + i\,g_{\alpha+2k,\beta}\right)
\end{aligned} \tag{3.74}$$

Equation (3.17) implies that in the basis $\mathcal{B}_V^{\mathbb{R}}$ (3.22) we have $\psi_{\alpha+2k} = \mathcal{I}(\psi_\alpha) = i\psi_\alpha$. Therefore, when we view the space of fermionic zero-modes as a $2k$-dimensional complex vector space with basis $\mathcal{B}_V^{\mathbb{C}}$ (3.21), the fermionic zero-modes are parameterised by

$$\begin{aligned} \psi &= \psi_a \lambda^a \\ &= \psi_\alpha \left(\lambda^\alpha + i\lambda^{\alpha+2k}\right), \end{aligned} \tag{3.75}$$

where as usual $a \in \{1, \ldots, 4k\}$ and $\alpha \in \{1, \ldots, 2k\}$. We now define

$$\zeta^\alpha = \lambda^\alpha + i\lambda^{\alpha+2k}, \qquad \zeta^{\overline{\alpha}} = \lambda^\alpha - i\lambda^{\alpha+2k}. \tag{3.76}$$

The fermionic zero-modes are then parameterised by

$$\psi = \psi_\alpha \zeta^\alpha. \tag{3.77}$$

For the second term of the effective Lagrangian we find

$$\begin{aligned} g_{ab}\lambda^a D_t \lambda^b &= 2\mathrm{Re}(\mathsf{g}_{\overline{\alpha}\beta}) \left(\lambda^\alpha D_t \lambda^\beta + \lambda^{\alpha+2k} D_t \lambda^{\beta+2k}\right) \\ &\quad + 2\mathrm{Im}(\mathsf{g}_{\overline{\alpha}\beta}) \left(\lambda^{\alpha+2k} D_t \lambda^\beta - \lambda^\alpha D_t \lambda^{\beta+2k}\right) \\ &= \left(\mathsf{g}_{\overline{\alpha}\beta} + \mathsf{g}_{\alpha\overline{\beta}}\right) \left(\lambda^\alpha D_t \lambda^\beta + \lambda^{\alpha+2k} D_t \lambda^{\beta+2k}\right) \\ &\quad - i\left(\mathsf{g}_{\overline{\alpha}\beta} - \mathsf{g}_{\alpha\overline{\beta}}\right) \left(\lambda^{\alpha+2k} D_t \lambda^\beta - \lambda^\alpha D_t \lambda^{\beta+2k}\right) \\ &= \mathsf{g}_{\overline{\alpha}\beta} \left(\left(\lambda^\alpha - i\lambda^{\alpha+2k}\right) D_t \lambda^\beta + i\left(\lambda^\alpha - i\lambda^{\alpha+2k}\right) D_t \lambda^{\beta+2k}\right) \\ &\quad + \mathsf{g}_{\alpha\overline{\beta}} \left(\left(\lambda^\alpha + i\lambda^{\alpha+2k}\right) D_t \lambda^\beta - i\left(\lambda^\alpha + i\lambda^{\alpha+2k}\right) D_t \lambda^{\beta+2k}\right) \\ &= \mathsf{g}_{\overline{\alpha}\beta} \left(\lambda^\alpha - i\lambda^{\alpha+2k}\right) D_t \left(\lambda^\beta + i\lambda^{\beta+2k}\right) \\ &\quad + \mathsf{g}_{\alpha\overline{\beta}} \left(\lambda^\alpha + i\lambda^{\alpha+2k}\right) D_t \left(\lambda^\beta - i\lambda^{\beta+2k}\right) \\ &= \mathsf{g}_{\overline{\alpha}\beta} \left(\lambda^\alpha - i\lambda^{\alpha+2k}\right) D_t \left(\lambda^\beta + i\lambda^{\beta+2k}\right) \\ &\quad - \mathsf{g}_{\alpha\overline{\beta}} D_t \left(\lambda^\alpha + i\lambda^{\alpha+2k}\right) \left(\lambda^\beta - i\lambda^{\beta+2k}\right) \\ &= 2\mathsf{g}_{\overline{\alpha}\beta} \zeta^{\overline{\alpha}} D_t \zeta^\beta \end{aligned} \tag{3.78}$$

so that the effective Lagrangian (3.28) in terms of these coordinates becomes

$$L_{\text{eff}} \;=\; \mathsf{g}_{\alpha\overline{\beta}}\dot{Z}^{\alpha}\dot{\overline{Z}}^{\overline{\beta}} + i\mathsf{g}_{\overline{\alpha}\beta}\zeta^{\overline{\alpha}}\left(\dot{\zeta}^{\beta} + \Gamma^{\beta}_{\gamma\delta}\dot{Z}^{\gamma}\zeta^{\delta}\right) - \frac{4\pi a}{e}b(\vec{k}). \tag{3.79}$$

We introduce an orthonormal frame θ for the fermionic variables, choosing it so that it respects holomorphicity [27].

$$\begin{aligned}
\mathsf{g}_{\alpha\overline{\beta}} \;&=\; \delta_{A\overline{B}}\theta^{A}{}_{\alpha}\theta^{\overline{B}}{}_{\overline{\beta}} \qquad & \theta^{A} \;&=\; \theta^{A}{}_{\alpha}\mathrm{d}Z^{\alpha} \qquad & \zeta^{A} \;&=\; \theta^{A}{}_{\alpha}\zeta^{\alpha} \\
& & \theta^{\overline{A}} \;&=\; \theta^{\overline{A}}{}_{\overline{\alpha}}\mathrm{d}\overline{Z}^{\overline{\alpha}} \qquad & \zeta^{\overline{A}} \;&=\; \theta^{\overline{A}}{}_{\overline{\alpha}}\zeta^{\overline{\alpha}}
\end{aligned} \tag{3.80}$$

Here

$$\delta_{A\overline{B}} \;=\; \begin{cases} 1 & \text{if } A = B, \\ 0 & \text{otherwise.} \end{cases} \tag{3.81}$$

The effective Lagrangian then becomes

$$L_{\text{eff}} \;=\; \mathsf{g}_{\alpha\overline{\beta}}\dot{Z}^{\alpha}\dot{\overline{Z}}^{\overline{\beta}} + i\delta_{\overline{A}B}\zeta^{\overline{A}}\left(\dot{\zeta}^{B} + \omega_{\overline{\alpha}}{}^{B}{}_{C}\dot{\overline{Z}}^{\overline{\alpha}}\zeta^{C} + \omega_{\alpha}{}^{B}{}_{C}\dot{Z}^{\alpha}\zeta^{C}\right) - \frac{4\pi a}{e}b(\vec{k}), \tag{3.82}$$

where the spin connection ω is again determined by a gauge transformation,

$$\omega_{\alpha}{}^{A}{}_{B} \;=\; \theta^{A}{}_{\beta}\Gamma^{\beta}_{\alpha\gamma}\theta^{\gamma}{}_{B} + \theta^{A}{}_{\beta}\partial_{\alpha}\theta^{\beta}{}_{B}, \qquad \omega_{\overline{\alpha}}{}^{A}{}_{B} \;=\; \theta^{A}{}_{\beta}\partial_{\overline{\alpha}}\theta^{\beta}{}_{B}. \tag{3.83}$$

3.5.2 Effective Hamiltonian

We compute the canonical momenta from the Lagrangian.

$$\begin{aligned}
P_{\alpha} \;&=\; \frac{\partial L_{\text{eff}}}{\partial \dot{Z}^{\alpha}} \;=\; \mathsf{g}_{\alpha\overline{\beta}}\dot{\overline{Z}}^{\overline{\beta}} + i\omega_{\alpha\overline{A}C}\zeta^{\overline{A}}\zeta^{C} \qquad & \Pi_{A} \;&=\; \frac{\partial L_{\text{eff}}}{\partial \dot{\zeta}^{A}} \;=\; -i\delta_{\overline{B}A}\zeta^{\overline{B}} \\
P_{\overline{\alpha}} \;&=\; \frac{\partial L_{\text{eff}}}{\partial \dot{\overline{Z}}^{\overline{\alpha}}} \;=\; \mathsf{g}_{\beta\overline{\alpha}}\dot{Z}^{\beta} + i\omega_{\overline{\alpha}\overline{A}C}\zeta^{\overline{A}}\zeta^{C} \qquad & \Pi_{\overline{A}} \;&=\; \frac{\partial L_{\text{eff}}}{\partial \dot{\zeta}^{\overline{A}}} \;=\; 0
\end{aligned} \tag{3.84}$$

The effective Hamiltonian is

$$
\begin{aligned}
H_{\text{eff}} &= \dot{Z}^\alpha P_\alpha + \dot{Z}^{\overline{\alpha}} \overline{P}_{\overline{\alpha}} + \dot{\zeta}^A \Pi_A - L_{\text{eff}} \\
&= \dot{Z}^\alpha (\mathsf{g}_{\alpha\overline{\beta}} \dot{\overline{Z}}^{\overline{\beta}} + i\omega_{\alpha\overline{A}C} \zeta^{\overline{A}} \zeta^C) + \dot{Z}^{\overline{\alpha}} (\mathsf{g}_{\beta\overline{\alpha}} \dot{Z}^\beta + i\omega_{\overline{\alpha}\overline{A}C} \zeta^{\overline{A}} \zeta^C) - i\delta_{\overline{B}A} \dot{\zeta}^A \zeta^{\overline{B}} \\
&\quad - \mathsf{g}_{\alpha\overline{\beta}} \dot{Z}^\alpha \dot{\overline{Z}}^{\overline{\beta}} - i\delta_{\overline{A}B} \zeta^{\overline{A}} \left(\dot{\zeta}^B + \omega_{\overline{\alpha}}{}^B{}_C \dot{\overline{Z}}^{\overline{\alpha}} \zeta^C + \omega_\alpha{}^B{}_C \dot{Z}^\alpha \zeta^C \right) + \frac{4\pi a}{e} b(\vec{k}) \\
&= \mathsf{g}_{\alpha\overline{\beta}} \dot{Z}^\alpha \dot{\overline{Z}}^{\overline{\beta}} + \frac{4\pi a}{e} b(\vec{k}) \\
&= H_0 + \frac{4\pi a}{e} b(\vec{k}), \qquad (3.85)
\end{aligned}
$$

where we have defined

$$
H_0 = \mathsf{g}^{\alpha\overline{\beta}} \tilde{P}_\alpha \tilde{P}_{\overline{\beta}}, \qquad (3.86)
$$

and

$$
\tilde{P}_\alpha = P_\alpha - i\omega_{\alpha\overline{A}C} \zeta^{\overline{A}} \zeta^C = \mathsf{g}_{\alpha\overline{\beta}} \dot{\overline{Z}}^{\overline{\beta}}, \qquad (3.87a)
$$

$$
\tilde{P}_{\overline{\alpha}} = P_{\overline{\alpha}} - i\omega_{\overline{\alpha}\overline{A}C} \zeta^{\overline{A}} \zeta^C = \mathsf{g}_{\overline{\alpha}\beta} \dot{Z}^\beta. \qquad (3.87b)
$$

3.5.3 Dirac brackets

Just like in section 3.4, where we used real coordinates, we must use Dirac brackets in order to quantise the theory, because of the constraints that arise from the expressions for the fermionic momenta. As before, having used an orthonormal frame for the fermionic variables, we find that the brackets of the bosonic and fermionic variables decouple.

The canonical Poisson brackets are

$$
\begin{aligned}
\left\{ P_\alpha, Z^\beta \right\}_{PB} &= \delta_\alpha^\beta & \qquad \left\{ \Pi_A, \zeta^B \right\}_{PB} &= \delta_A^B \\
\left\{ P_{\overline{\alpha}}, Z^{\overline{\beta}} \right\}_{PB} &= \delta_{\overline{\alpha}}^{\overline{\beta}} & \qquad \left\{ \Pi_{\overline{A}}, \zeta^{\overline{B}} \right\}_{PB} &= \delta_{\overline{A}}^{\overline{B}} \qquad (3.88)
\end{aligned}
$$

The constraints are now given by

$$\xi_A = \Pi_A + i\delta_{A\overline{B}}\zeta^{\overline{B}} \tag{3.89a}$$

$$\xi_{\overline{A}} = \Pi_{\overline{A}} \tag{3.89b}$$

We find that

$$\begin{aligned} \Delta_{\overline{A}A} = \Delta_{A\overline{A}} &= \{\xi_A, \xi_{\overline{A}}\}_{PB} \\ &= \left\{\Pi_A + i\delta_{A\overline{B}}\zeta^{\overline{B}}, \Pi_{\overline{A}}\right\}_{PB} \\ &= i\delta_{A\overline{A}} \end{aligned} \tag{3.90}$$

while

$$\Delta_{AB} = \Delta_{\overline{AB}} = 0 \tag{3.91}$$

Poisson brackets between Z or P and the constraints vanish, and therefore Dirac brackets of Z and P are the same as the original Poisson brackets. The only non-vanishing Dirac bracket involving the ζ is

$$\begin{aligned} \left\{\zeta^A, \zeta^{\overline{B}}\right\}_{DB} &= \left\{\zeta^A, \zeta^{\overline{B}}\right\}_{PB} + i\delta^{C\overline{D}}\left\{\zeta^A, \xi_C\right\}_{PB}\left\{\xi_{\overline{D}}, \zeta^{\overline{B}}\right\}_{PB} \\ &= i\delta^{A\overline{B}} \end{aligned} \tag{3.92}$$

The only non-vanishing Dirac brackets are hence

$$\left\{P_\alpha, Z^\beta\right\}_{DB} = \delta^\beta_\alpha \qquad \left\{P_{\overline{\alpha}}, \overline{Z}^{\overline{\beta}}\right\}_{DB} = \delta^{\overline{\beta}}_{\overline{\alpha}} \qquad \left\{\zeta^A, \zeta^{\overline{B}}\right\}_{DB} = i\delta^{A\overline{B}} \tag{3.93}$$

3.5.4 Quantisation

To quantise the theory we follow the usual procedure again. Dirac brackets of bosons are replaced with commutators, and Dirac brackets of fermions are replaced with anti-commutators.

$$\left\{P_\alpha, Z^\beta\right\}_{DB} = \delta^\beta_\alpha \quad \mapsto \quad \left[P_\alpha, Z^\beta\right] = -i\delta^\beta_\alpha \tag{3.94a}$$

$$\left\{P_{\overline{\alpha}}, \overline{Z}^{\overline{\beta}}\right\}_{DB} = \delta^{\overline{\beta}}_{\overline{\alpha}} \quad \mapsto \quad \left[P_{\overline{\alpha}}, \overline{Z}^{\overline{\beta}}\right] = -i\delta^{\overline{\beta}}_{\overline{\alpha}} \tag{3.94b}$$

$$\left\{\zeta^A, \zeta^{\overline{B}}\right\}_{DB} = i\delta^{A\overline{B}} \quad \mapsto \quad \left\{\zeta^A, \zeta^{\overline{B}}\right\} = \delta^{A\overline{B}} \tag{3.94c}$$

We interpret the Hilbert space of states as the space of square-integrable $(0,p)$-forms on $\mathcal{M}$ as follows. The bosonic coordinates act by multiplication and the bosonic momenta are represented as derivatives,

$$P_\alpha \;\mapsto\; -i\partial_\alpha \qquad\qquad P_{\overline{\alpha}} \;\mapsto\; -i\partial_{\overline{\alpha}} \tag{3.95}$$

while the quantisation of fermions is given by

$$\zeta^{\overline{A}} \;\mapsto\; \theta^{\overline{A}}\wedge \qquad\qquad \zeta^{A} \;\mapsto\; \iota(\theta^A) \tag{3.96}$$

where $\iota(\theta^A)(\theta^{\overline{B}}) = \delta^{A\overline{B}}$. The $\tilde{P}$ act as covariant derivatives,

$$\tilde{P}_\alpha \;=\; P_\alpha - i\omega_{\alpha\overline{A}C}\zeta^{\overline{A}}\zeta^C \qquad \mapsto \qquad -i\left(\partial_\alpha + \omega_{\alpha\overline{A}C}\theta^{\overline{A}}\wedge\iota(\theta^C)\right) \;=\; -i\nabla_\alpha, \tag{3.97}$$

$$\tilde{P}_{\overline{\alpha}} \;=\; P_{\overline{\alpha}} - i\omega_{\overline{\alpha}\overline{A}C}\zeta^{\overline{A}}\zeta^C \qquad \mapsto \qquad -i\left(\partial_{\overline{\alpha}} + \omega_{\overline{\alpha}\overline{A}C}\theta^{\overline{A}}\wedge\iota(\theta^C)\right) \;=\; -i\nabla_{\overline{\alpha}}, \tag{3.98}$$

and the quantisation of the effective Hamiltonian gives again half the Laplacian,

$$H_0 \;=\; \mathsf{g}^{\alpha\overline{\beta}}\tilde{P}_\alpha\tilde{P}_{\overline{\beta}} \qquad \mapsto \qquad -\mathsf{g}^{\alpha\overline{\beta}}\nabla_\alpha\nabla_{\overline{\beta}} \;=\; \frac{1}{2}\Delta. \tag{3.99}$$

We have seen above that the quantisation of the spinorial zero-modes of the original field theory can be interpreted in terms of anti-holomorphic forms on the moduli space. The quantisation also automatically allows for the possibility that multiple fermionic zero-modes are excited. Such excitations are represented by wedge products of anti-holomorphic forms, with the antisymmetric nature of the wedge product reflecting the fermionic nature of the spinor zero-modes.

3.5.5 Supersymmetry

The effective action is invariant under the $\mathcal{N}=4$ supersymmetry transformations [30]

$$\begin{aligned}
\delta_{\mathbb{1}} Z^{\alpha} &= \varepsilon\zeta^{\alpha} & \delta_{\mathbb{1}}\zeta^{\alpha} &= i\varepsilon\dot{Z}^{\alpha} \\
\delta_{\mathbb{1}} \overline{Z}^{\overline{\alpha}} &= \varepsilon\zeta^{\overline{\alpha}} & \delta_{\mathbb{1}}\zeta^{\overline{\alpha}} &= i\varepsilon\dot{\overline{Z}}^{\overline{\alpha}}
\end{aligned} \tag{3.100a}$$

$$\begin{aligned}
\delta_{\mathcal{I}} Z^{\alpha} &= i\varepsilon\zeta^{\alpha} & \delta_{\mathcal{I}}\zeta^{\alpha} &= \varepsilon\dot{Z}^{\alpha} \\
\delta_{\mathcal{I}} \overline{Z}^{\overline{\alpha}} &= -i\varepsilon\zeta^{\overline{\alpha}} & \delta_{\mathcal{I}}\zeta^{\overline{\alpha}} &= -\,\varepsilon\dot{\overline{Z}}^{\overline{\alpha}}
\end{aligned} \tag{3.100b}$$

$$\begin{aligned}
\delta_{\mathcal{J}} Z^{\alpha} &= \varepsilon\mathcal{J}^{\alpha}{}_{\overline{\beta}}\zeta^{\overline{\beta}} & \delta_{\mathcal{J}}\zeta^{\alpha} &= -i\varepsilon\mathcal{J}^{\alpha}{}_{\overline{\beta}}\dot{\overline{Z}}^{\overline{\beta}} - \varepsilon\Gamma^{\alpha}_{\beta\gamma}\mathcal{J}^{\beta}{}_{\overline{\delta}}\zeta^{\overline{\delta}}\zeta^{\gamma} \\
\delta_{\mathcal{J}} \overline{Z}^{\overline{\alpha}} &= \varepsilon\mathcal{J}^{\overline{\alpha}}{}_{\beta}\zeta^{\beta} & \delta_{\mathcal{J}}\zeta^{\overline{\alpha}} &= -i\varepsilon\mathcal{J}^{\overline{\alpha}}{}_{\beta}\dot{Z}^{\beta} - \varepsilon\Gamma^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}}\mathcal{J}^{\overline{\beta}}{}_{\delta}\zeta^{\delta}\zeta^{\overline{\gamma}}
\end{aligned} \tag{3.100c}$$

$$\begin{aligned}
\delta_{\mathcal{K}} Z^{\alpha} &= \varepsilon\mathcal{K}^{\alpha}{}_{\overline{\beta}}\zeta^{\overline{\beta}} & \delta_{\mathcal{K}}\zeta^{\alpha} &= -i\varepsilon\mathcal{K}^{\alpha}{}_{\overline{\beta}}\dot{\overline{Z}}^{\overline{\beta}} - \varepsilon\Gamma^{\alpha}_{\beta\gamma}\mathcal{K}^{\beta}{}_{\overline{\delta}}\zeta^{\overline{\delta}}\zeta^{\gamma} \\
\delta_{\mathcal{K}} \overline{Z}^{\overline{\alpha}} &= \varepsilon\mathcal{K}^{\overline{\alpha}}{}_{\beta}\zeta^{\beta} & \delta_{\mathcal{K}}\zeta^{\overline{\alpha}} &= -i\varepsilon\mathcal{K}^{\overline{\alpha}}{}_{\beta}\dot{Z}^{\beta} - \varepsilon\Gamma^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}}\mathcal{K}^{\overline{\beta}}{}_{\delta}\zeta^{\delta}\zeta^{\overline{\gamma}}
\end{aligned} \tag{3.100d}$$

The corresponding supercharges are

$$\begin{aligned}
\mathcal{Q}_{\mathbb{1}} &= \tilde{P}_{\alpha}\zeta^{\alpha} + \tilde{P}_{\overline{\alpha}}\zeta^{\overline{\alpha}}, & \mathcal{Q}_{\mathcal{J}} &= \tilde{P}_{\overline{\alpha}}\mathcal{J}^{\overline{\alpha}}{}_{\alpha}\zeta^{\alpha} + \tilde{P}_{\alpha}\mathcal{J}^{\alpha}{}_{\overline{\alpha}}\zeta^{\overline{\alpha}}, \\
\mathcal{Q}_{\mathcal{I}} &= i\tilde{P}_{\alpha}\zeta^{\alpha} - i\tilde{P}_{\overline{\alpha}}\zeta^{\overline{\alpha}}, & \mathcal{Q}_{\mathcal{K}} &= \tilde{P}_{\overline{\alpha}}\mathcal{K}^{\overline{\alpha}}{}_{\alpha}\zeta^{\alpha} + \tilde{P}_{\alpha}\mathcal{K}^{\alpha}{}_{\overline{\alpha}}\zeta^{\overline{\alpha}}.
\end{aligned} \tag{3.101}$$

They generate the supersymmetry transformations via Dirac brackets, and they obey the $\mathcal{N}=4$ supersymmetry algebra. The supersymmetry transformations square to i times a time-derivative, $\delta^2 = i\partial_t$, and the Hamiltonian is once more $H_0 = -\frac{i}{2}\left\{\mathcal{Q}_{\mathbb{1}}, \mathcal{Q}_{\mathbb{1}}\right\}_{DB} = -\frac{i}{2}\left\{\mathcal{Q}_{\mathcal{I}_i}, \mathcal{Q}_{\mathcal{I}_i}\right\}_{DB}$.

It is now convenient to define new linear combinations of the supercharges by

$$\tilde{\mathcal{Q}} = \frac{i}{2}(\mathcal{Q}_{\mathbb{1}} + i\mathcal{Q}_{\mathcal{I}}) = i\tilde{P}_{\overline{\alpha}}\zeta^{\overline{\alpha}}, \tag{3.102a}$$

$$\tilde{\mathcal{Q}}^* = -\frac{i}{2}(\mathcal{Q}_{\mathbb{1}} - i\mathcal{Q}_{\mathcal{I}}) = -i\tilde{P}_{\alpha}\zeta^{\alpha}, \tag{3.102b}$$

$$\tilde{\mathcal{Q}}_{\mathcal{J}} = \frac{i}{2}(\mathcal{Q}_{\mathcal{J}} - i\mathcal{Q}_{\mathcal{K}}) = i\tilde{P}_{\alpha}\mathcal{J}^{\alpha}{}_{\overline{\alpha}}\zeta^{\overline{\alpha}}, \tag{3.102c}$$

$$\tilde{\mathcal{Q}}^*_{\mathcal{J}} = -\frac{i}{2}(\mathcal{Q}_{\mathcal{J}} + i\mathcal{Q}_{\mathcal{K}}) = -i\tilde{P}_{\overline{\alpha}}\mathcal{J}^{\overline{\alpha}}{}_{\alpha}\zeta^{\alpha}. \tag{3.102d}$$

The only non-vanishing brackets of these supercharges are

$$\left\{\tilde{\mathcal{Q}}, \tilde{\mathcal{Q}}^*\right\}_{DB} = \left\{\tilde{\mathcal{Q}}_{\mathcal{J}}, \tilde{\mathcal{Q}}^*_{\mathcal{J}}\right\}_{DB} = iH_{\text{eff}}. \tag{3.103}$$

The supercharges $\tilde{\mathcal{Q}}$ and $\tilde{\mathcal{Q}}^*$ are quantised, using the quantisation procedures given above, as

$$\tilde{\mathcal{Q}} = i\tilde{P}_{\overline{\alpha}}\zeta^{\overline{\alpha}} \quad \mapsto \quad \theta^{\overline{\alpha}} \wedge \nabla_{\overline{\alpha}} = \overline{\partial}, \tag{3.104}$$

$$\tilde{\mathcal{Q}}^* = -i\tilde{P}_{\alpha}\zeta^{\alpha} \quad \mapsto \quad -\iota(\theta^{\alpha})\nabla_{\alpha} = \overline{\partial}^{\dagger}, \tag{3.105}$$

where $\overline{\partial}$ and $\overline{\partial}^{\dagger}$ are the Dolbeault operator and its adjoint operator respectively.

For the remaining supercharges, we find that they are quantised as [38]

$$\tilde{\mathcal{Q}}_{\mathcal{J}} = i\tilde{P}_{\alpha}\mathcal{J}^{\alpha}{}_{\overline{\alpha}}\zeta^{\overline{\alpha}} \quad \mapsto \quad \mathcal{J}(\theta^{\alpha}) \wedge \nabla_{\alpha} = \mathcal{J}\partial\mathcal{J}^{-1} = \overline{\partial}_{\mathcal{J}}, \tag{3.106}$$

$$\tilde{\mathcal{Q}}^*_{\mathcal{J}} = -i\tilde{P}_{\overline{\alpha}}\mathcal{J}^{\overline{\alpha}}{}_{\alpha}\zeta^{\alpha} \quad \mapsto \quad -\iota\left(\mathcal{J}(\theta^{\overline{\alpha}})\right) \wedge \nabla_{\overline{\alpha}} = \mathcal{J}\partial^{\dagger}\mathcal{J}^{-1} = \overline{\partial}^{\dagger}_{\mathcal{J}}, \tag{3.107}$$

where $\overline{\partial}_{\mathcal{J}}$ and $\overline{\partial}^{\dagger}_{\mathcal{J}}$ are the twisted Dolbeault operator and its adjoint respectively (see appendix A.2 for more details).

The quantisation of the Hamiltonian as the Dirac bracket of the supercharges gives once more half the Laplacian [12]:

$$H_0 = -i\left\{\tilde{\mathcal{Q}}, \tilde{\mathcal{Q}}^*\right\}_{DB} \quad \mapsto \quad \overline{\partial}\,\overline{\partial}^{\dagger} + \overline{\partial}^{\dagger}\overline{\partial} = \frac{1}{2}\Delta. \tag{3.108}$$

3.5.6 The action of the complex structures

The complex structure $\mathcal{I}$ acts on anti-holomorphic one-forms by multiplication with the complex number i, but the complex structures $\mathcal{J}$ and $\mathcal{K}$ map forms which are anti-holomorphic with respect to $\mathcal{I}$ to holomorphic forms. This should be contrasted with the maps $\hat{\mathcal{I}}$, $\hat{\mathcal{J}}$ and $\hat{\mathcal{K}}$ acting on the spinor zero-modes in the original field theory (3.15). We can implement these maps on the anti-holomorphic forms on the moduli space, using their relation to $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$ given in section 3.2.3, equations (3.27): $\hat{\mathcal{I}} = \mathcal{I}$, $\hat{\mathcal{J}} = -\overline{\mathcal{J}}$ and $\hat{\mathcal{K}} = -\overline{\mathcal{K}}$. This way we obtain again an (anti-linear) action of the quaternion algebra on the space of anti-holomorphic forms, this time on the moduli space.

Chapter 4

$N = 4$ Supersymmetric Monopoles

In this chapter we study $N = 4$ supersymmetric monopoles. As in the bosonic and $N = 2$ supersymmetric cases, we will first review BPS monopoles, the zero-modes of the theory and the moduli space approximation [9, 11]. Then we will discuss the quantisation of the effective action of the $N = 4$ supersymmetric model (section 4.4).

Following Weinberg and Yi [29], we identify the Hilbert space of states with the space of forms on the moduli space. However, we choose a slightly different quantisation prescription that identifies two independent (but equivalent) sets of fermionic zero-modes with holomorphic and anti-holomorphic forms, analogous to the quantisation procedure of section 3.5. Weinberg and Yi mix up the two sets of fermionic zero-modes in their quantisation prescription, obscuring their independence. As we will see in chapter 5, our quantisation prescription allows us to define a natural angular momentum operator as a differential operator acting on forms, that can be applied to both $N = 2$ and $N = 4$ supersymmetric monopoles.

As in the $N = 2$ supersymmetric model, we construct the differential operators corresponding to the supercharges, and interpret them as (twisted) Dolbeault operators and their adjoints. As mentioned before, the identification of all the supercharges is essential for finding all the states in a supermultiplet. We will illustrate this in the chapters 6 and 7.

4.1 $N = 4$ supersymmetric BPS monopoles

The extension of the Yang-Mills-Higgs Lagrangian (2.13) with $N = 4$ supersymmetry is given by

$$
\begin{aligned}
L \;=\; & \int \mathrm{d}^3x \Big(-\frac{1}{4} F^{\mu\nu} \cdot F_{\mu\nu} + \frac{1}{2} D_\mu S_\imath \cdot D^\mu S_\imath + \frac{1}{2} D_\mu P_\jmath \cdot D^\mu P_\jmath \\
& - \frac{e^2}{4} \left(||\, [S_\imath, S_\jmath] \,||^2 + 2||\, [S_\imath, P_\jmath] \,||^2 + ||\, [P_\imath, P_\jmath] \,||^2 \right) \\
& + \frac{i}{2} \overline{\psi}_r \cdot \gamma^\mu D_\mu \psi_r + \frac{e}{2} \overline{\psi}_r \cdot \left(\alpha^\imath_{rs} \text{ ad}\, S_\imath - i \beta^\jmath_{rs} \gamma^5 \text{ ad}\, P_\jmath \right) \psi_s \Big).
\end{aligned} \tag{4.1}
$$

Here $S_\imath$ are three scalar fields, $P_\jmath$ are three pseudo-scalar fields, and ψ_r are four Majorana spinors. The indices have the following ranges: $\imath, \jmath, \ldots \in \{1, 2, 3\}$ and $r, s, \ldots \in \{1, 2, 3, 4\}$. The chiral operator γ_5 is again defined by $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. $\alpha^\imath$ and $\beta^\jmath$ are 4×4 real anti-symmetric matrices, satisfying

$$
\begin{aligned}
[\alpha^\imath, \alpha^\jmath] &= -2\varepsilon^{\imath\jmath\kappa} \alpha^\kappa, & \{\alpha^\imath, \alpha^\jmath\} &= -2\delta^{ij} \mathbb{1}_4, \\
[\beta^\imath, \beta^\jmath] &= -2\varepsilon^{\imath\jmath\kappa} \beta^\kappa, & \{\beta^\imath, \beta^\jmath\} &= -2\delta^{ij} \mathbb{1}_4, \\
[\alpha^\imath, \beta^\jmath] &= 0.
\end{aligned} \tag{4.2}
$$

An explicit representation of these matrices is given by the following.

$$
\alpha^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \qquad
\beta^1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
$$

$$
\alpha^2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad
\beta^2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}
$$

$$\alpha^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad \beta^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{4.3}$$

The supersymmetries of the Lagrangian (4.1) are given by

$$\delta A_\mu = i\bar{\epsilon}_r \gamma_\mu \psi_r, \tag{4.4a}$$

$$\delta S_\imath = -\bar{\epsilon}_r \alpha^\imath_{rs} \psi_s, \tag{4.4b}$$

$$\delta P_\jmath = i\,\bar{\epsilon}_r \gamma_5 \beta^\jmath_{rs} \psi_s, \tag{4.4c}$$

$$\begin{aligned} \delta\psi_r = \Big(&\frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}\delta_{rs} - i\gamma^\mu D_\mu S_\imath \alpha^\imath_{rs} + \gamma^\mu\gamma_5 D_\mu P_\jmath \beta^\jmath_{rs} \\ &- \frac{e}{2}\varepsilon_{\imath\jmath\kappa}\left[S_\imath, S_\jmath\right]\alpha^\kappa_{rs} - ie\gamma_5\left[S_\imath, P_\jmath\right]\alpha^\imath_{rt}\beta^\jmath_{ts} - \frac{e}{2}\varepsilon_{\imath\jmath\kappa}\left[P_\imath, P_\jmath\right]\beta^\kappa_{rs}\Big)\epsilon_s, \end{aligned} \tag{4.4d}$$

where ϵ_r are four Majorana spinor paramaters. These supersymmetries are most easily exhibited by deriving the Lagrangian and its supersymmetries from an $N = 1$ supersymmetric Lagrangian in 10 dimensions by dimensional reduction [31, 32] (see appendix B.2 for details). Under the dimensional reduction, the rotational symmetry of the extra dimensions gives rise to an $SU(4)$ internal symmetry of the Lagrangian (4.1) in four dimensions. As in the bosonic and $N = 2$ supersymmetric cases, the symmetry breaking is induced by choosing appropriate boundary conditions on the fields at infinity:

$$\lim_{r\to\infty}\left(S_\imath \cdot S_\imath + P_\jmath \cdot P_\jmath\right) = a^2. \tag{4.5}$$

The kinetic and potential energy for the $N = 4$ supersymmetric model are

$$K = \int d^3x \left(-||\frac{1}{2}F_{0i}||^2 + \frac{1}{2}||D_0 S_\imath||^2 + \frac{1}{2}||D_0 P_\jmath||^2 + \frac{i}{2}\overline{\psi}_r \cdot \gamma^0 D_0 \psi_r \right), \tag{4.6}$$

$$\begin{aligned} V = \int d^3x \Big(&\frac{1}{4}F^{ij}\cdot F_{ij} + \frac{1}{2}||D_i S_\jmath||^2 + \frac{1}{2}||D_i P_\jmath||^2 \\ &- \frac{e^2}{4}\left(||\left[S_\imath, S_\jmath\right]||^2 + 2||\left[S_\imath, P_\jmath\right]||^2 + ||\left[P_\imath, P_\jmath\right]||^2\right) \\ &+ \frac{i}{2}\overline{\psi}_r \cdot \gamma_i D_i \psi_r - \frac{e}{2}\overline{\psi}_r \cdot \left(\alpha^\imath_{rs}\,\mathrm{ad}\,S_\imath - i\beta^\jmath_{rs}\gamma^5\,\mathrm{ad}\,P_\jmath\right)\psi_s\Big). \end{aligned} \tag{4.7}$$

The BPS monopoles are defined, as before, to have minimal energy, which implies again that they are static. To find the zero-modes, we first use the internal $SU(4)$ symmetry of the Lagrangian, so that we may assume that only the scalar field S_3 has a non-zero vacuum expectation value. (Once more, a parity-invariant vacuum would already require that the vacuum expectation values of the pseudoscalar fields $P_\jmath$ are zero. We can then still use the internal symmetry of the Lagrangian to assume that the vacuum expectation value of the $S_\imath$ fields lies in the S_3 direction.) In this case $S_3 = \Phi$ takes on the role of the Higgs field of the bosonic model, and it must satisfy the Bogomol'nyi equations (2.29). To minimise the potential energy (4.7), ψ must then satisfy the following Dirac equation in the presence of the monopole background

$$\overline{\gamma}_i D_i \psi_r + ie\overline{\gamma}_4 \alpha^3_{rs} \,\mathrm{ad}\, S_3 \psi_s \;=\; 0. \tag{4.8}$$

where the Euclidean $\overline{\gamma}$-matrices are defined in (3.8). We now define

$$\xi^+ \;=\; \psi_1 - i\psi_2, \qquad\qquad \xi^- \;=\; \psi_3 - i\psi_4, \tag{4.9}$$

which are eigenstates of α^3 in the representation (4.3). Writing

$$\xi \;=\; \begin{pmatrix} \xi^+ \\ \xi^- \end{pmatrix}, \tag{4.10}$$

the Dirac equation (4.8) becomes

$$\not{D}\xi \;\equiv\; \overline{\gamma}_{\underline{i}} D_{\underline{i}} \xi \;=\; \begin{pmatrix} \overline{\gamma}_{\underline{i}} D_{\underline{i}}\, \xi^+ \\ \overline{\gamma}_{\underline{i}} D_{\underline{i}}\, \xi^- \end{pmatrix} \;=\; 0, \tag{4.11}$$

where $D_{\underline{i}}$ is the covariant derivative in Euclidean space defined in equation (2.31). We see that ξ^+ and ξ^- both obey the same Dirac equation as ψ in the $N = 2$ supersymmetric model, equations (3.9). Therefore, they both independently correspond to a set of fermionic zero-modes equivalent to those of the $N = 2$ supersymmetric model.

4.2 Zero-modes

The bosonic zero-modes of this model are again exactly the same as those of the purely bosonic model. The fermionic zero-modes are the solutions of the Dirac equation

(4.11), and both ξ^+ and ξ^- have the same form as the zero-modes ψ in the $N = 2$ supersymmetric model (see section 3.2),

$$\xi^\pm \;=\; \xi_a^\pm \lambda_\pm^a, \tag{4.12a}$$

with

$$\xi_a^\pm \;=\; \overline{\gamma}_{\underline{i}} \delta_a W_{\underline{i}} \begin{pmatrix} 0 \\ \chi^\pm \end{pmatrix} \;=\; \begin{pmatrix} \bar{\mathsf{e}}_{\underline{i}} \delta_a W_{\underline{i}}\, \chi^\pm \\ 0 \end{pmatrix} \;=\; \begin{pmatrix} \mathsf{w}_a\, \chi^\pm \\ 0 \end{pmatrix}, \tag{4.12b}$$

where the bosonic zero-mode w_a was defined by equation (2.64), $\chi^\pm$ are fixed, normalised, commuting two-component spinors, and the $\lambda_\pm^a$ are real valued Grassmann numbers.

Again, the complex structures act on the fermionic zero-modes (4.12) via equation (2.61) or (2.66). Using hats to distinguish the action of the complex structures on fermionic zero-modes from the action of the complex structures on bosonic zero-modes, we have

$$\hat{\mathcal{I}}_i(\xi_a^\pm) \;=\; \begin{pmatrix} \mathcal{I}_i(\mathsf{w}_a)\chi^\pm \\ 0 \end{pmatrix} \;=\; \begin{pmatrix} -\mathsf{w}_a \mathsf{e}_i \chi^\pm \\ 0 \end{pmatrix}. \tag{4.13}$$

We see that we may choose $\chi^\pm$ such that there is a complex structure $\hat{\mathcal{I}}$ for which

$$\hat{\mathcal{I}}(\xi_a^\pm) \;=\; \pm i \xi_a^\pm. \tag{4.14}$$

For example, if we choose

$$\chi^+ \;=\; \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad\qquad \chi^- \;=\; \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{4.15}$$

then

$$\hat{\mathcal{I}} \;=\; \hat{\mathcal{I}}_3. \tag{4.16}$$

Having made a choice, and fixed χ^+ and χ^-, the remaining two complex structures that make up the hyperkähler structure, $\hat{\mathcal{J}}$ and $\hat{\mathcal{K}}$ act again anti-linearly, as in the $N = 2$ supersymmetric model.

The vector space V of fermionic zero-modes is now a $4k$-dimensional complex vector space, which can also be viewed as an $8k$-dimensional real vector space.

With the choice of signs given in equation (4.14), the fermionic zero-modes $\xi_a^{\pm}$ (4.12b) can be interpreted as forms on $\mathbb{R}^4$, analogous to the interpretation of fermionic zero-modes as anti-holomorphic forms in the $N = 2$ supersymmetric model. ξ_a^+ and ξ_a^- are static spinors in $(3 + 1)$-dimensional space-time. Extending space-time to $\mathbb{R} \times \mathbb{R}^4$, as we did in section 2.3, we may also view them as spinors on $\mathbb{R}^4$ that are independent of the fourth dimension. These spinors can now be identified with (anti-) holomorphic forms. For example, we can identify the fermionic zero-modes ξ_a^+ with anti-holomorphic forms $\overline{\upsilon}_a$, and ξ_a^- with holomorphic forms τ_a on $\mathbb{R}^4$ by

$$\xi_a^+ = \begin{pmatrix} a_1 \\ a_2 \\ 0 \\ 0 \end{pmatrix} \qquad \sim \qquad \overline{\upsilon}_a = a_1\overline{\alpha}_1 + a_2\overline{\alpha}_2, \tag{4.17a}$$

$$\xi_a^- = \begin{pmatrix} b_1 \\ b_2 \\ 0 \\ 0 \end{pmatrix} \qquad \sim \qquad \tau_a = b_1\alpha_1 + b_2\alpha_2, \tag{4.17b}$$

where we use the basis $\alpha_1 = \frac{1}{\sqrt{2}}(\mathrm{d}x^3 + i\mathrm{d}x^4)$ and $\alpha_2 = \frac{1}{\sqrt{2}}(\mathrm{d}x^1 + i\mathrm{d}x^2)$ of holomorphic forms (with respect to the complex structure $\mathcal{I}$) on $\mathbb{R}^4$. This identification agrees with equation (4.14).

The identification of ξ_a^+ with anti-holomorphic forms is completely independent of the identification of ξ_a^- with holomorphic forms. The relationship between an anti-holomorphic form and its complex conjugate is a natural relationship from a geometrical point of view, but the corresponding relationship between fermionic zero-modes depends on the explicit choice of the identifications (4.17a) and (4.17b).

The complex structures $\mathcal{I}, \mathcal{J}, \mathcal{K}$ act naturally on the space of all differential forms, and hence on this basis. One finds (see also appendix A)

$$\begin{aligned}
\mathcal{I}(\alpha_1) &= -i\alpha_1, & \mathcal{J}(\alpha_1) &= -i\overline{\alpha}_2, & \mathcal{K}(\alpha_1) &= \overline{\alpha}_2, \\
\mathcal{I}(\alpha_2) &= -i\alpha_2, & \mathcal{J}(\alpha_2) &= i\overline{\alpha}_1, & \mathcal{K}(\alpha_2) &= -\overline{\alpha}_1,
\end{aligned} \tag{4.18a}$$

$$\begin{aligned}
\mathcal{I}(\overline{\alpha}_1) &= i\overline{\alpha}_1, & \mathcal{J}(\overline{\alpha}_1) &= i\alpha_2, & \mathcal{K}(\overline{\alpha}_1) &= \alpha_2, \\
\mathcal{I}(\overline{\alpha}_2) &= i\overline{\alpha}_2, & \mathcal{J}(\overline{\alpha}_2) &= -i\alpha_1, & \mathcal{K}(\overline{\alpha}_2) &= -\alpha_1.
\end{aligned} \tag{4.18b}$$

These (linear) actions of the complex structures on forms are related to the (linear) action of $\hat{\mathcal{I}}$, and the (anti-linear) actions of $\hat{\mathcal{J}}$ and $\hat{\mathcal{K}}$ on spinors defined in (4.13) in the same was as in section 3.2.3. Using the identification (4.17), we pull the maps $\hat{\mathcal{I}}$, $\hat{\mathcal{J}}$ and $\hat{\mathcal{K}}$ back to maps on forms, and we denote these pull-backs by the same letters $\hat{\mathcal{I}}$, $\hat{\mathcal{J}}$ and $\hat{\mathcal{K}}$. We find, for example,

$$\hat{\mathcal{J}}(\xi_a^-) = \begin{pmatrix} \begin{pmatrix} \bar{b}_2 & b_1 \\ -\bar{b}_1 & b_2 \end{pmatrix} i\sigma_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} i\bar{b}_2 \\ -i\bar{b}_1 \\ 0 \\ 0 \end{pmatrix}. \tag{4.19}$$

Under the identification (4.17) we therefore have, for example,

$$\hat{\mathcal{J}}(\xi_a^-) \qquad \sim \qquad \hat{\mathcal{J}}(\tau_a) := i\bar{b}_2\alpha_1 - i\bar{b}_1\alpha_2 = -\overline{\mathcal{J}(\tau_a)}. \tag{4.20}$$

We may again summarise the action of the complex structures on forms by

$$\hat{\mathcal{I}} = \mathcal{I}, \qquad \hat{\mathcal{J}} = -\overline{\mathcal{J}}, \qquad \hat{\mathcal{K}} = -\overline{\mathcal{K}}, \tag{4.21}$$

as we did in the $N = 2$ supersymmetric case.

4.3 The moduli space approximation

Once more we come to the moduli space approximation. As before, we must parameterise the lowest energy states with moduli space parameters, and the effective Lagrangian can be found by inserting this parameterisation into the original Lagrangian of the model.

4.3.1 Effective Lagrangian

We view the space of fermionic zero-modes as an $8k$-dimensional real vector space, we parameterise the fermionic zero-modes using equations (4.12) with real valued $\lambda^a_\pm$, which we combine into the two-component Grassmann function

$$\lambda^a = \begin{pmatrix} \lambda^a_+ \\ \lambda^a_- \end{pmatrix}. \tag{4.22}$$

Inserting parametrisation (4.12) into the Lagrangian (4.1) and expanding to lowest non-trivial order, Blum [11] has found (see also Gauntlett [9], and Weinberg and Yi [29])

$$L_{\text{eff}} = \frac{1}{2} g_{ab} \dot{X}^a \dot{X}^b + \frac{i}{2} g_{ab} (\lambda^a)^T (D_t \lambda)^b - \frac{1}{8} R_{abcd} (\lambda^a)^T \lambda^b (\lambda^c)^T \lambda^d - \frac{4\pi a}{e} b(\vec{k}). \tag{4.23}$$

Compared to the effective Lagrangian of the $N = 2$ supersymmetric monopole, we now have two copies of the fermionic term,

$$\frac{i}{2} g_{ab} (\lambda^a)^T (D_t \lambda)^b = \frac{i}{2} g_{ab} \lambda^a_+ D_t \lambda^b_+ + \frac{i}{2} g_{ab} \lambda^a_- D_t \lambda^b_-, \tag{4.24}$$

and an extra term involving the curvature of the metric, with components R_{abcd}. The curvature term provides a coupling between the fermionic variables λ^a_+ and λ^a_-.

4.3.2 Supersymmetry

The effective action corresponding to the effective Lagrangian (4.23) is invariant under $\mathcal{N} = 8$ supersymmetry transformations [29],

$$\begin{aligned} \delta_{\mathbb{1}} X^a &= \bar{\varepsilon} \lambda^a \\ \delta_{\mathbb{1}} \lambda^a &= -i \dot{X}^a \sigma_2 \varepsilon - \bar{\varepsilon} \lambda^b \Gamma^a_{bc} \lambda^c, \end{aligned} \tag{4.25}$$

$$\begin{aligned} \delta_{\mathcal{I}_j} X^a &= \bar{\varepsilon} \, (\mathcal{I}_j)^a{}_b \lambda^b \\ \delta_{\mathcal{I}_j} \lambda^a &= i (\mathcal{I}_j)^a{}_b \dot{X}^b \sigma_2 \varepsilon - \bar{\varepsilon} (\mathcal{I}_j)^c{}_b \lambda^b \Gamma^a_{cd} \lambda^d, \end{aligned} \tag{4.26}$$

where ε are two-component Grassmann parameters, and $\bar{\varepsilon} = \varepsilon^T \sigma_2$. They are again reminiscent of the supersymmetries of the original field theory. The corresponding supercharges are

$$Q^\pm_{\mathbb{1}} = \tilde{p}_a \lambda^a_\pm, \qquad Q^\pm_{\mathcal{I}_i} = \tilde{p}_b (\mathcal{I}_i)^b{}_a \lambda^a_\pm. \tag{4.27}$$

The supercharges generate the supersymmetry transformations via Dirac brackets, and they obey the supersymmetry algebra

$$\left\{Q^{\pm}_{\mathbb{1}}, Q^{\pm}_{\mathbb{1}}\right\}_{DB} = 2iH_0, \qquad \left\{Q^{\pm}_{\mathcal{I}_i}, Q^{\pm}_{\mathcal{I}_j}\right\}_{DB} = \delta_{ij}\, 2iH_0, \tag{4.28}$$

and all other brackets vanishing.

4.4 Quantisation using forms on the moduli space

The quickest route towards quantisation of the effective Lagrangian (4.23) is to start again by introducing an orthonormal frame to parameterise the fermionic zero-modes as in (3.30). The effective Lagrangian can then be written as

$$L_{\text{eff}} = \frac{1}{2} g_{ab}\dot{X}^a\dot{X}^b + \frac{i}{2}\delta_{AB}(\lambda^A)^T(D_t\lambda)^B - \frac{1}{8}R_{abcd}(\lambda^a)^T\lambda^b(\lambda^c)^T\lambda^d - \frac{4\pi a}{e}b(\vec{k}), \tag{4.29}$$

where

$$\lambda^A_{\pm} = e^A{}_a\lambda^a_{\pm}. \tag{4.30}$$

4.4.1 Effective Hamiltonian

The curvature term has no influence on the canonical momenta, so that the canonical momenta of the effective Lagrangian (4.29) are

$$p_a = \frac{\partial L_{\text{eff}}}{\partial \dot{X}^a} = g_{ab}\dot{X}^b + \frac{i}{2}\omega_{aAB}(\lambda^A)^T\lambda^B, \tag{4.31}$$

$$\pi^{\pm}_A = \frac{\partial L_{\text{eff}}}{\partial \dot{\lambda}^A_{\pm}} = -\frac{i}{2}\delta_{AB}\lambda^B_{\pm}, \tag{4.32}$$

Again, the expression for the fermionic momenta leads to constraints, and we will have to replace Poisson brackets by Dirac brackets.

The effective Hamiltonian is given by

$$\begin{aligned} H_{\text{eff}} &= \dot{X}^a p_a + \dot{\lambda}^A_+\pi^+_A + \dot{\lambda}^A_-\pi^-_A - L_{\text{eff}} \\ &= H_0 + \frac{4\pi a}{e}b(\vec{k}), \end{aligned} \tag{4.33}$$

where we have defined

$$H_0 = \frac{1}{2}g^{ab}\tilde{p}_a\tilde{p}_b + \frac{1}{8}R_{abcd}(\lambda^a)^T\lambda^b(\lambda^c)^T\lambda^d, \tag{4.34}$$

and

$$\tilde{p}_a = p_a - \frac{i}{2}\omega_{aAB}(\lambda^A)^T\lambda^B = g_{ab}\dot{X}^b. \tag{4.35}$$

4.4.2 Quantisation

Having used an orthonormal frame to define the fermionic variables λ^A, the Dirac brackets of the bosonic and fermionic variables decouple, as for the $N = 2$ supersymmetric monopoles (see section 3.4.2). The only non-vanishing Dirac brackets are

$$\left\{p_a, X^b\right\}_{DB} = \delta_a^b, \qquad \left\{\lambda_+^A, \lambda_+^B\right\}_{DB} = i\delta^{AB}, \qquad \left\{\lambda_-^A, \lambda_-^B\right\}_{DB} = i\delta^{AB}, \tag{4.36}$$

which can be quantised as follows,

$$\left\{p_a, X^b\right\}_{DB} = \delta_a^b \qquad \mapsto \qquad \left[\hat{p}_a, \hat{X}^b\right] = -i\delta_a^b \tag{4.37}$$

$$\left\{\lambda_+^A, \lambda_+^B\right\}_{DB} = i\delta^{AB} \qquad \mapsto \qquad \left\{\hat{\lambda}_+^A, \hat{\lambda}_+^B\right\} = \delta^{AB} \tag{4.38}$$

$$\left\{\lambda_-^A, \lambda_-^B\right\}_{DB} = i\delta^{AB} \qquad \mapsto \qquad \left\{\hat{\lambda}_-^A, \hat{\lambda}_-^B\right\} = \delta^{AB} \tag{4.39}$$

We may interpret the Hilbert space of states generated by the λ_+^A and λ_-^A as the space of two spinors on the moduli space, just as the λ^A in section 3.4 gave rise to a Hilbert space of states corresponding to a single spinor on the moduli space. However, this is not a convenient interpretation of the quantum states. We are interested in a quantisation in terms of forms on the moduli space. In their review paper Weinberg and Yi [29] describe an interpretation of quantum states as forms on the moduli space. In particular, they define $\overline{\varphi}^a = \frac{1}{\sqrt{2}}\left(\lambda_+^a + i\lambda_-^a\right)$, which is then quantised as $\mathrm{d}X^a\wedge$. The supercharges can then be interpreted as the exterior derivative, its adjoint, and the related operators obtained via a twisting with the complex structures. This is a valid quantisation, although it obscures the relationship between the λ^A in the $N = 2$ theory, with the $\lambda_\pm^A$ in the $N = 4$ theory: we would like to quantise the

$N = 4$ supersymmetric model in terms of all forms on the moduli space, in such a way that the λ_+^A correspond to anti-holomorphic forms, and the λ_-^A correspond to holomorphic forms, in accordance with equation (4.14). To this end, we introduce complex coordinates once more.

Complex coordinates

We use complex coordinates Z, as defined in equation (3.67), and we define

$$\zeta_\pm^A = \lambda_\pm^A + i\lambda_\pm^{A+2k}, \qquad \zeta_\pm^{\overline{A}} = \lambda_\pm^A - i\lambda_\pm^{A+2k}, \tag{4.40}$$

We write the first two terms of the effective Lagrangian (4.29) in terms of these variables, and find

$$L_{\text{eff}} = \mathsf{g}_{\alpha\overline{\beta}}\dot{Z}^\alpha\dot{\overline{Z}}^{\beta} + i\delta_{\overline{A}B}(\zeta^{\overline{A}})^T\left(D_t\zeta^B\right) - \frac{1}{8}R_{abcd}(\lambda^a)^T\lambda^b(\lambda^c)^T\lambda^d - \frac{4\pi a}{e}b(\vec{k}). \tag{4.41}$$

The curvature term has no influence on the canonical momenta, for which we find

$$P_\alpha = \frac{\partial L_{\text{eff}}}{\partial \dot{Z}^\alpha} = \mathsf{g}_{\alpha\overline{\beta}}\dot{\overline{Z}}^{\beta} + i\omega_{\alpha\overline{A}C}(\zeta^{\overline{A}})^T\zeta^C, \tag{4.42}$$

$$P_{\overline{\alpha}} = \frac{\partial L_{\text{eff}}}{\partial \dot{\overline{Z}}^{\alpha}} = \mathsf{g}_{\beta\overline{\alpha}}\dot{Z}^\beta + i\omega_{\overline{\alpha}\overline{A}C}(\zeta^{\overline{A}})^T\zeta^C, \tag{4.43}$$

$$\Pi_A^\pm = \frac{\partial L_{\text{eff}}}{\partial \dot{\zeta}_\pm^A} = -i\delta_{\overline{B}A}\zeta_\pm^{\overline{B}}, \tag{4.44}$$

$$\Pi_{\overline{A}}^\pm = \frac{\partial L_{\text{eff}}}{\partial \dot{\zeta}_\pm^{\overline{A}}} = 0. \tag{4.45}$$

The non-vanishing Dirac brackets are then

$$\left\{P_\alpha, Z^\beta\right\}_{DB} = \delta_\alpha^\beta, \quad \left\{P_{\overline{\alpha}}, \overline{Z}^{\overline{\beta}}\right\}_{DB} = \delta_{\overline{\alpha}}^{\overline{\beta}}, \quad \left\{\zeta_\pm^A, \zeta_\pm^{\overline{B}}\right\}_{DB} = i\delta^{A\overline{B}}. \tag{4.46}$$

We quantise the Dirac brackets as usual, by

$$\left\{P_\alpha, Z^\beta\right\}_{DB} = \delta_\alpha^\beta \quad \mapsto \quad \left[P_\alpha, Z^\beta\right] = -i\delta_\alpha^\beta \tag{4.47a}$$

$$\left\{P_{\overline{\alpha}}, \overline{Z}^{\overline{\beta}}\right\}_{DB} = \delta_{\overline{\alpha}}^{\overline{\beta}} \quad \mapsto \quad \left[P_{\overline{\alpha}}, \overline{Z}^{\overline{\beta}}\right] = -i\delta_{\overline{\alpha}}^{\overline{\beta}} \tag{4.47b}$$

$$\left\{\zeta_+^A, \zeta_+^{\overline{B}}\right\}_{DB} = i\delta^{A\overline{B}} \quad \mapsto \quad \left\{\zeta_+^A, \zeta_+^{\overline{B}}\right\} = \delta^{A\overline{B}} \tag{4.47c}$$

$$\left\{\zeta_-^A, \zeta_-^{\overline{B}}\right\}_{DB} = i\delta^{A\overline{B}} \quad \mapsto \quad \left\{\zeta_-^A, \zeta_-^{\overline{B}}\right\} = \delta^{A\overline{B}} \tag{4.47d}$$

We can now interpret the Hilbert space of states as the space of square-integrable forms on $\mathcal{M}$. The bosonic coordinates act by multiplication and the bosonic momenta are represented as derivatives,

$$P_\alpha \;\mapsto\; -i\partial_\alpha \qquad\qquad P_{\overline{\alpha}} \;\mapsto\; -i\partial_{\overline{\alpha}} \tag{4.48}$$

while the quantisation of fermions is given by

$$\zeta_+^{\overline{A}} \;\mapsto\; \theta^{\overline{A}}\wedge \qquad\qquad \zeta_+^{A} \;\mapsto\; \iota(\theta^A) \tag{4.49}$$

$$\zeta_-^{A} \;\mapsto\; \theta^{A}\wedge \qquad\qquad \zeta_-^{\overline{A}} \;\mapsto\; \iota(\theta^{\overline{A}}) \tag{4.50}$$

where $\iota(\theta^A)(\theta^{\overline{B}}) = \delta^{A\overline{B}}$, $\iota(\theta^{\overline{A}})(\theta^B) = \delta^{\overline{A}B}$ and $\iota(\theta^A)(\theta^B) = \iota(\theta^{\overline{A}})(\theta^{\overline{B}}) = 0$.

The covariant momenta are

$$\tilde{P}_\alpha \;=\; P_\alpha - i\omega_{\alpha\overline{A}C}(\zeta^{\overline{A}})^T\zeta^C \;=\; \mathsf{g}_{\alpha\overline{\beta}}\dot{\overline{Z}}^{\overline{\beta}} \;=\; \frac{1}{2}\left(\tilde{p}_\alpha - i\tilde{p}_{\alpha+2k}\right), \tag{4.51}$$

$$\tilde{P}_{\overline{\alpha}} \;=\; P_{\overline{\alpha}} - i\omega_{\overline{\alpha}\overline{A}C}(\zeta^{\overline{A}})^T\zeta^C \;=\; \mathsf{g}_{\overline{\alpha}\beta}\dot{Z}^{\beta} \;=\; \frac{1}{2}\left(\tilde{p}_\alpha + i\tilde{p}_{\alpha+2k}\right), \tag{4.52}$$

and they are quantised as before, as

$$\tilde{P}_\alpha \;=\; P_\alpha - i\omega_{\alpha\overline{A}C}(\zeta^{\overline{A}})^T\zeta^C \qquad \mapsto \qquad -i\nabla_\alpha, \tag{4.53}$$

$$\tilde{P}_{\overline{\alpha}} \;=\; P_{\overline{\alpha}} - i\omega_{\overline{\alpha}\overline{A}C}(\zeta^{\overline{A}})^T\zeta^C \qquad \mapsto \qquad -i\nabla_{\overline{\alpha}}. \tag{4.54}$$

The Hamiltonian becomes, in terms of the covariant momenta,

$$H_0 \;=\; \mathsf{g}^{\alpha\overline{\beta}}\tilde{P}_\alpha\tilde{P}_{\overline{\beta}} + \frac{1}{8}R_{abcd}(\lambda^a)^T\lambda^b(\lambda^c)^T\lambda^d, \tag{4.55}$$

We define complex linear combinations of the supercharges, analogous to those in equations (3.102), by

$$\begin{aligned}
\tilde{\mathcal{Q}}^{\pm} \;&=\; \frac{i}{2}(Q^{\pm}_{\mathbb{1}} + iQ^{\pm}_{\mathcal{I}})\\
&=\; \frac{i}{2}\left(\tilde{p}_\alpha(\lambda^\alpha_\pm + i\mathcal{I}^\alpha{}_{\alpha+2k}\lambda^{\alpha+2k}_\pm) + \tilde{p}_{\alpha+2k}(\lambda^{\alpha+2k}_\pm + i\mathcal{I}^{\alpha+2k}{}_\alpha\lambda^\alpha_\pm)\right)\\
&=\; \frac{i}{2}\left(\tilde{p}_\alpha(\lambda^\alpha_\pm - i\lambda^{\alpha+2k}_\pm) + \tilde{p}_{\alpha+2k}(\lambda^{\alpha+2k}_\pm + i\lambda^\alpha_\pm)\right)\\
&=\; \frac{i}{2}\left(\tilde{p}_\alpha + i\tilde{p}_{\alpha+2k}\right)\left(\lambda^\alpha_\pm - i\lambda^{\alpha+2k}_\pm\right)\\
&=\; i\tilde{P}_{\overline{\alpha}}\zeta^{\overline{\alpha}}_\pm,
\end{aligned} \tag{4.56a}$$

$$(\tilde{\mathcal{Q}}^{\pm})^* \;=\; -\frac{i}{2}(Q^{\pm}_{\mathbb{1}} - iQ^{\pm}_{\mathcal{I}}) \;=\; -i\tilde{P}_{\alpha}\zeta^{\alpha}_{\pm}, \tag{4.56b}$$

$$\tilde{\mathcal{Q}}^{\pm}_{\mathcal{J}} \;=\; \frac{i}{2}(Q^{\pm}_{\mathcal{J}} - iQ^{\pm}_{\mathcal{K}}) \;=\; i\tilde{P}_{\alpha}\mathcal{J}^{\alpha}{}_{\overline{\alpha}}\zeta^{\overline{\alpha}}_{\pm}, \tag{4.56c}$$

$$(\tilde{\mathcal{Q}}^{\pm}_{\mathcal{J}})^* \;=\; -\frac{i}{2}(Q^{\pm}_{\mathcal{J}} + iQ^{\pm}_{\mathcal{K}}) \;=\; -i\tilde{P}_{\overline{\alpha}}\mathcal{J}^{\overline{\alpha}}{}_{\alpha}\zeta^{\alpha}_{\pm}. \tag{4.56d}$$

From (4.28) we find that the algebra they satisfy has the following non-vanishing Dirac brackets

$$\left\{\tilde{\mathcal{Q}}^{\pm}, (\tilde{\mathcal{Q}}^{\pm})^*\right\}_{DB} \;=\; iH_0, \qquad \left\{\tilde{\mathcal{Q}}^{\pm}_{\mathcal{J}}, (\tilde{\mathcal{Q}}^{\pm}_{\mathcal{J}})^*\right\}_{DB} \;=\; iH_0, \tag{4.57}$$

The supercharges are quantised as [9, 38]

$$\tilde{\mathcal{Q}}^{+} \;=\; i\tilde{P}_{\overline{\alpha}}\zeta^{\overline{\alpha}}_{+} \quad\mapsto\quad \theta^{\overline{\alpha}} \wedge \nabla_{\overline{\alpha}} \;=\; \overline{\partial}, \tag{4.58}$$

$$(\tilde{\mathcal{Q}}^{+})^* \;=\; -i\tilde{P}_{\alpha}\zeta^{\alpha}_{+} \quad\mapsto\quad -\iota(\theta^{\alpha})\nabla_{\alpha} \;=\; \overline{\partial}^{\dagger}, \tag{4.59}$$

$$\tilde{\mathcal{Q}}^{+}_{\mathcal{J}} \;=\; i\tilde{P}_{\alpha}\mathcal{J}^{\alpha}{}_{\overline{\alpha}}\zeta^{\overline{\alpha}}_{+} \quad\mapsto\quad \mathcal{J}(\theta^{\alpha}) \wedge \nabla_{\alpha} \;=\; \mathcal{J}\partial\mathcal{J}^{-1} \;=\; \overline{\partial}_{\mathcal{J}}, \tag{4.60}$$

$$(\tilde{\mathcal{Q}}^{+}_{\mathcal{J}})^* \;=\; -i\tilde{P}_{\overline{\alpha}}\mathcal{J}^{\overline{\alpha}}{}_{\alpha}\zeta^{\alpha}_{+} \quad\mapsto\quad -\iota\left(\mathcal{J}(\theta^{\overline{\alpha}})\right) \wedge \nabla_{\overline{\alpha}} \;=\; \mathcal{J}\partial^{\dagger}\mathcal{J}^{-1} \;=\; \overline{\partial}^{\dagger}_{\mathcal{J}}, \tag{4.61}$$

and

$$(\tilde{\mathcal{Q}}^{-})^* \;=\; i\tilde{P}_{\alpha}\zeta^{\alpha}_{-} \quad\mapsto\quad \theta^{\alpha} \wedge \nabla_{\alpha} \;=\; \partial, \tag{4.62}$$

$$\tilde{\mathcal{Q}}^{-} \;=\; -i\tilde{P}_{\overline{\alpha}}\zeta^{\overline{\alpha}}_{-} \quad\mapsto\quad -\iota(\theta^{\overline{\alpha}})\nabla_{\overline{\alpha}} \;=\; \partial^{\dagger}, \tag{4.63}$$

$$(\tilde{\mathcal{Q}}^{-}_{\mathcal{J}})^* \;=\; i\tilde{P}_{\overline{\alpha}}\mathcal{J}^{\overline{\alpha}}{}_{\alpha}\zeta^{\alpha}_{-} \quad\mapsto\quad \mathcal{J}(\theta^{\overline{\alpha}}) \wedge \nabla_{\overline{\alpha}} \;=\; \mathcal{J}\overline{\partial}\mathcal{J}^{-1} \;=\; \partial_{\mathcal{J}}, \tag{4.64}$$

$$\tilde{\mathcal{Q}}^{-}_{\mathcal{J}} \;=\; -i\tilde{P}_{\alpha}\mathcal{J}^{\alpha}{}_{\overline{\alpha}}\zeta^{\overline{\alpha}}_{-} \quad\mapsto\quad -\iota\left(\mathcal{J}(\theta^{\alpha})\right) \wedge \nabla_{\alpha} \;=\; \mathcal{J}\overline{\partial}^{\dagger}\mathcal{J}^{-1} \;=\; \partial^{\dagger}_{\mathcal{J}}, \tag{4.65}$$

Finally, the Hamiltonian is quantised as

$$H_0 \quad\mapsto\quad \left\{\tilde{\mathcal{Q}}^{+}, (\tilde{\mathcal{Q}}^{+})^*\right\} \;=\; \overline{\partial}\,\overline{\partial}^{\dagger} + \overline{\partial}^{\dagger}\overline{\partial} \;=\; \frac{1}{2}\Delta. \tag{4.66}$$

The action of the complex structures

As in section 3.5.6, the action of the complex structures $\hat{\mathcal{I}}$, $\hat{\mathcal{J}}$ and $\hat{\mathcal{K}}$ on the spinor zero-modes in the original field theory (3.15) can be implemented as an action on the forms on the moduli space, using their relation to $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$ given in section 4.2, equations (4.21): $\hat{\mathcal{I}} = \mathcal{I}$, $\hat{\mathcal{J}} = -\overline{\mathcal{J}}$ and $\hat{\mathcal{K}} = -\overline{\mathcal{K}}$. This way we obtain again an (anti-linear) action of the quaternion algebra on the space of forms on the moduli space, that respects the holomorphicity of those forms.

Summary

The discussion and results in this chapter can be summarised as follows. The moduli space approximation of the $N = 4$ supersymmetric Lagrangian in (3+1)-dimensions leads to an $\mathcal{N} = 8$ supersymmetric σ-model on the moduli space. The quantisation of this model can be done in terms of general differential forms on the moduli space and the supercharges corresponding to the $N = 8$ supersymmetries correspond to the Dolbeault operator and the $\mathcal{J}$-twisted Dolbeault operator, their complex conjugates, and all of their adjoints. The effective Hamiltonian in this geometrical interpretation is half the Laplacian acting on these differential forms.

Chapter 5

Angular Momentum

The spin operators for the fermionic zero-modes of the $N = 2$ and $N = 4$ supersymmetric $SU(2)$ monopole of charge 1, derived from the field theory by Osborn [39], are given by

$$\vec{S} \;=\; \frac{1}{2}\sum_{n,s,s'} a_s^{n\dagger}(\vec{\sigma})_{ss'}a_{s'}^{n}, \tag{5.1}$$

where n indicates the fermion species, and s and s' indicate the spin state of the fermion zero-mode (i.e. up or down). In the case of $N = 2$ supersymmetry there is only one fermion species. For $N = 4$ supersymmetric monopoles there are two fermion species and $n \in \{1, 2\}$. The spin operator acts on the Hilbert space generated by the $a_s^{n\dagger}$ acting on a vacuum state $\Big|0\Big\rangle$, defined by $a_s^n\Big|0\Big\rangle = 0$.

The states of the $N = 2$ supersymmetric monopole multiplet can be grouped into two singlets and a doublet under the action of the spin operator,

$$\begin{aligned} &\text{2 singlets:} && \Big|0\Big\rangle \text{ and } \Big|\uparrow\downarrow\Big\rangle, \\ &\text{1 doublet:} && \Big(\Big|\uparrow\Big\rangle, \Big|\downarrow\Big\rangle\Big), \end{aligned}$$

where

$$\Big|0\Big\rangle \;=\; \Big|0\Big\rangle, \qquad \Big|\uparrow\downarrow\Big\rangle \;=\; a_\downarrow^\dagger a_\uparrow^\dagger\Big|0\Big\rangle, \tag{5.2a}$$

$$\Big|\uparrow\Big\rangle \;=\; a_\uparrow^\dagger\Big|0\Big\rangle, \qquad \Big|\downarrow\Big\rangle \;=\; a_\downarrow^\dagger\Big|0\Big\rangle. \tag{5.2b}$$

The $N = 4$ multiplet consists of five singlets, four doublets and a triplet,

$$\begin{aligned}
&\text{5 singlets:} \quad \left|\begin{smallmatrix}0\\0\end{smallmatrix}\right\rangle, \quad \left|\begin{smallmatrix}\uparrow\downarrow\\0\end{smallmatrix}\right\rangle, \quad \frac{1}{\sqrt{2}}\left(\left|\begin{smallmatrix}\uparrow\\\downarrow\end{smallmatrix}\right\rangle - \left|\begin{smallmatrix}\downarrow\\\uparrow\end{smallmatrix}\right\rangle\right), \quad \left|\begin{smallmatrix}0\\\uparrow\downarrow\end{smallmatrix}\right\rangle, \quad \left|\begin{smallmatrix}\uparrow\downarrow\\\uparrow\downarrow\end{smallmatrix}\right\rangle \\
&\text{4 doublets:} \quad \left(\left|\begin{smallmatrix}\uparrow\\0\end{smallmatrix}\right\rangle, \left|\begin{smallmatrix}\downarrow\\0\end{smallmatrix}\right\rangle\right), \quad \left(\left|\begin{smallmatrix}0\\\uparrow\end{smallmatrix}\right\rangle, \left|\begin{smallmatrix}0\\\downarrow\end{smallmatrix}\right\rangle\right), \quad \left(\left|\begin{smallmatrix}\uparrow\downarrow\\\uparrow\end{smallmatrix}\right\rangle, \left|\begin{smallmatrix}\uparrow\downarrow\\\downarrow\end{smallmatrix}\right\rangle\right), \quad \left(\left|\begin{smallmatrix}\uparrow\\\uparrow\downarrow\end{smallmatrix}\right\rangle, \left|\begin{smallmatrix}\downarrow\\\uparrow\downarrow\end{smallmatrix}\right\rangle\right) \\
&\text{1 triplet:} \quad \left(\left|\begin{smallmatrix}\uparrow\\\uparrow\end{smallmatrix}\right\rangle, \frac{1}{\sqrt{2}}\left(\left|\begin{smallmatrix}\uparrow\\\downarrow\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}\downarrow\\\uparrow\end{smallmatrix}\right\rangle\right), \left|\begin{smallmatrix}\downarrow\\\downarrow\end{smallmatrix}\right\rangle\right)
\end{aligned}$$

where the two entries in the kets correspond to the two fermion species $n = 1, 2$, analogous to the states (5.2). For example,

$$\left|\begin{smallmatrix}\uparrow\\0\end{smallmatrix}\right\rangle = a_\uparrow^{1\dagger}\left|0\right\rangle, \qquad \left|\begin{smallmatrix}\uparrow\downarrow\\\downarrow\end{smallmatrix}\right\rangle = a_\downarrow^{2\dagger}a_\downarrow^{1\dagger}a_\uparrow^{1\dagger}\left|0\right\rangle. \tag{5.3}$$

In this chapter, we would like to translate Osborn's result into geometrical language, where we describe the quantum states corresponding to fermionic zero-modes with (anti-)holomorphic forms on the moduli space. We would like to define a differential operator acting on forms, which corresponds to the (spin) angular momentum operator acting on zero-modes [38].

In general, the spin operator for higher charge monopoles is not well defined since it is not unambiguously possible to separate orbital angular momentum from spin, due to the extended nature of monopoles. Therefore we have to look at the total angular momentum operator $\vec{J}$ instead. However, in special cases (in particular for the charge-1 monopole, and for well separated monopoles) we expect that a spin operator, corresponding to $\vec{S}$ in equation (5.1), could reappear. This would allow us to determine the spin of individual (well separated) monopoles, which should be possible if we are to compare monopole scattering with the scattering of electrically charged particles in the dual theory.

We will explicitly construct a spin-operator for the charge-1 monopole in section 6.5, and confirm that it agrees with Osborn's spin operator (5.1). We will discuss some of the issues involved in computing spins of well-separated monopoles in our outlook at the end of this thesis.

5.1 The total angular momentum operator on the moduli space

The total angular momentum operator $\vec{J}$ should act on quantum states by an infinitesimal $SU(2)$ or $SO(3)$ action. It obeys the angular momentum algebra

$$[J_i, J_j] \;=\; i\,\epsilon_{ijk}J_k, \tag{5.4}$$

and acts via a Leibniz rule on tensor products of states. We expect $\vec{J}$ to contain orbital and spin contributions, but, as explained above, neither of these needs to be separately well-defined. It would be natural to guess that J_i acts by rotating the spatial coordinates through the Lie derivative $\mathcal{L}_{Y_i}$, where the vector fields Y_i generate such rotations. To agree with equation (5.4), these vector fields must satisfy $[Y_i, Y_j] = \varepsilon_{ijk}Y_k$. However, we want the expression for $\vec{J}$ to respect our decomposition of vectors and forms into their holomorphic and anti-holomorphic parts, because in the $N = 2$ theory only the $(0,p)$-forms are identified with fermionic zero-modes. Furthermore, in the $N = 4$ theory, the geometrical interpretation of the two species of fermionic zero-modes as holomorphic and anti-holomorphic forms are independent, and an angular momentum operator should not mix between the two. Therefore we require that

$$[J_i, \operatorname{ad}\mathcal{I}_j] \;=\; 0. \tag{5.5}$$

Here we use the adjoint action of the complex structure, $\operatorname{ad}\mathcal{I}_j$, which is the extension of the complex structure that acts on p-forms following the Leibniz rule [40]. This means that the Lie derivative by itself cannot be the angular momentum operator, since Atiyah and Hitchin [3] have found that the complex structures are rotated into each other through the $SO(3)$ action: $\mathcal{L}_{Y_i}(\mathcal{I}_j) = \varepsilon_{ijk}\mathcal{I}_k$. For the action on p-forms, this becomes

$$[\mathcal{L}_{Y_i}, \operatorname{ad}\mathcal{I}_j] \;=\; \varepsilon_{ijk}\operatorname{ad}\mathcal{I}_k. \tag{5.6}$$

To correct for this unwanted rotation of the complex structures, we note that the action of the complex structures generate an $SU(2)$ action on the bundle of forms themselves [40, 41],

$$[\mathrm{ad}\,\mathcal{I}_i, \mathrm{ad}\,\mathcal{I}_j] \;=\; 2\varepsilon_{ijk}\,\mathrm{ad}\,\mathcal{I}_k, \tag{5.7}$$

and therefore we define the operator J_i by [38]

$$J_i \;=\; i\left(\mathcal{L}_{Y_i} - \frac{1}{2}\,\mathrm{ad}\,\mathcal{I}_i\right). \tag{5.8}$$

It obeys the Leibniz rule and the angular momentum algebra (equation (5.4)), and it leaves the complex structures invariant (equation (5.5)). In particular, the three generators J_i map (anti-)holomorphic forms to (anti-)holomorphic forms, even though J_1 and J_2 are made up of two operators ($\mathcal{L}_{Y_1}$ and $\mathrm{ad}\,\mathcal{I}_1$, and $\mathcal{L}_{Y_2}$ and $\mathrm{ad}\,\mathcal{I}_2$ respectively) which, individually, mix up anti-holomorphic forms and holomorphic forms.

5.2 Supercharges and angular momentum

The supercharges can be used to create spin states from bosonic states, and therefore they correspond to spin-$\frac{1}{2}$ operators. As such, the angular momentum operators must obey the appropriate algebra with the supercharges. We find that they do, and that the Dolbeault operators $\overline{\partial}$ and $\partial_{\mathcal{J}}$ increase the total angular momentum of states (with respect to J_3) by $\frac{1}{2}$, while $\overline{\partial}_{\mathcal{J}}$ and ∂ decrease it by $\frac{1}{2}$. The supercharges are therefore indeed spin-$\frac{1}{2}$ operators.

To derive the commutators of the total angular momentum operators J_i with the Dolbeault operators, we first need to compute some basic geometrical identities.

5.2.1 Geometrical identities

In this subsection (only) we do not use Einstein's summation convention; repeated indices are not summed over, unless explicitly stated using the summation symbol $\sum$.

We start off by writing the (twisted) Dolbeault operators in terms of (twisted) exterior derivatives [40, 41]

$$\partial \;=\; \frac{1}{2}(\mathrm{d} + i\mathrm{d}_{\mathcal{I}}), \qquad \partial_{\mathcal{J}} \;=\; \frac{1}{2}(\mathrm{d}_{\mathcal{J}} + i\mathrm{d}_{\mathcal{K}}),$$

$$\overline{\partial} \;=\; \frac{1}{2}(\mathrm{d} - i\mathrm{d}_{\mathcal{I}}), \qquad \overline{\partial}_{\mathcal{J}} \;=\; \frac{1}{2}(\mathrm{d}_{\mathcal{J}} - i\mathrm{d}_{\mathcal{K}}), \tag{5.9}$$

where the twisted exterior derivative[1] is defined by $\mathrm{d}_{\mathcal{I}_i} = \mathcal{I}_i \mathrm{d} \mathcal{I}_i^{-1}$. To compute the brackets of J_i with the (twisted) Dolbeault operators, using (5.9) and definition (5.8), we need to compute the brackets of $\mathrm{ad}\,\mathcal{I}_i$ and $\mathcal{L}_{Y_i}$ with the complex structures and the (twisted) exterior derivatives.

First of all, we compute the brackets between $\mathrm{ad}\,\mathcal{I}_i$ and the complex structures.

$$
\begin{aligned}
[\mathrm{ad}\,\mathcal{I}_i, \mathcal{I}_j] &= [\mathrm{ad}\,\mathcal{I}_i, (\mathcal{I}_j \otimes \mathcal{I}_j \otimes \ldots \otimes \mathcal{I}_j)] \\
&= ([\mathrm{ad}\,\mathcal{I}_i, \mathcal{I}_j] \otimes \mathcal{I}_j \otimes \ldots \otimes \mathcal{I}_j) + (\mathcal{I}_j \otimes [\mathrm{ad}\,\mathcal{I}_i, \mathcal{I}_j] \otimes \mathcal{I}_j \otimes \ldots \otimes \mathcal{I}_j) \\
&\qquad + \ldots + (\mathcal{I}_j \ldots \otimes \mathcal{I}_j \otimes [\mathrm{ad}\,\mathcal{I}_i, \otimes \mathcal{I}_j]) \\
&= \sum_k 2\varepsilon_{ijk} \big((\mathcal{I}_k \otimes \mathcal{I}_j \otimes \ldots \otimes \mathcal{I}_j) + (\mathcal{I}_j \otimes \mathcal{I}_k \otimes \mathcal{I}_j \otimes \ldots \otimes \mathcal{I}_j) \\
&\qquad + \ldots + (\mathcal{I}_j \ldots \otimes \mathcal{I}_j \otimes \mathcal{I}_k) \big) \\
&= \begin{cases} 0 & \text{if } i = j \\ 2\,\mathrm{ad}\,\mathcal{I}_i \mathcal{I}_j & \text{if } i \neq j \end{cases} \qquad (5.10) \\
&= 2(1 - \delta_{ij})\,\mathrm{ad}(\mathcal{I}_i)\mathcal{I}_j
\end{aligned}
$$

Similarly we have

$$
[\mathcal{L}_{Y_i}, \mathcal{I}_j] = \begin{cases} 0 & \text{if } i = j, \\ \mathrm{ad}(\mathcal{I}_i)\mathcal{I}_j & \text{if } i \neq j. \end{cases} \qquad (5.11)
$$

Since the action of $\mathcal{I}_j^{-1}$ on p-forms is given by $\mathcal{I}_j^{-1} = (-1)^p \mathcal{I}_j$, we also have

$$
\left[\mathrm{ad}\,\mathcal{I}_i, \mathcal{I}_j^{-1}\right] = \begin{cases} 0 & \text{if } i = j, \\ 2\,\mathrm{ad}(\mathcal{I}_i)\mathcal{I}_j^{-1} & \text{if } i \neq j, \end{cases} \qquad (5.12)
$$

$$
\left[\mathcal{L}_{Y_i}, \mathcal{I}_j^{-1}\right] = \begin{cases} 0 & \text{if } i = j, \\ \mathrm{ad}(\mathcal{I}_i)\mathcal{I}_j^{-1} & \text{if } i \neq j. \end{cases} \qquad (5.13)
$$

[1] Viewing the moduli space as a Kähler manifold $(\mathcal{M}_{\vec{k}}, \mathcal{I})$ we have $d_{\mathcal{I}} = \mathcal{I}\mathrm{d}\mathcal{I}^{-1} = d^c$.

To compute the bracket of a complex structure with the exterior derivative, we first write the latter in terms of Dolbeault operators with respect to this complex structure. Denoting the Dolbeault operators corresponding to the complex structure $\mathcal{I}_i$ by ∂_i and $\overline{\partial}_i$, we have

$$[\mathrm{ad}\,\mathcal{I}_j, \partial_j] = -i\partial_j, \tag{5.14}$$

$$\left[\mathrm{ad}\,\mathcal{I}_j, \overline{\partial}_j\right] = i\overline{\partial}_j, \tag{5.15}$$

and therefore

$$[\mathrm{ad}\,\mathcal{I}_i, \mathrm{d}] = \left[\mathrm{ad}\,\mathcal{I}_i, \partial_i + \overline{\partial}_i\right] = -i\left(\partial_i - \overline{\partial}_i\right) = \mathrm{d}_{\mathcal{I}_i}. \tag{5.16}$$

This means that the twisted exterior derivatives can be obtained from the ordinary exterior derivative by taking the bracket with the adjoint action of the complex structures. We can now compute

$$\begin{aligned}
\left[\mathrm{ad}\,\mathcal{I}_i, \mathrm{d}_{\mathcal{I}_j}\right] &= \left[\mathrm{ad}\,\mathcal{I}_i, \mathcal{I}_j \mathrm{d}\mathcal{I}_j^{-1}\right] \\
&= [\mathrm{ad}\,\mathcal{I}_i, \mathcal{I}_j]\,\mathrm{d}\mathcal{I}_j^{-1} + \mathcal{I}_j\,[\mathrm{ad}\,\mathcal{I}_i, \mathrm{d}]\,\mathcal{I}_j^{-1} + \mathcal{I}_j\mathrm{d}\left[\mathrm{ad}\,\mathcal{I}_i, \mathcal{I}_j^{-1}\right] \\
&= 2(1-\delta_{ij})\,\mathrm{ad}\,\mathcal{I}_i\mathcal{I}_j\mathrm{d}\mathcal{I}_j^{-1} + \mathcal{I}_j\mathcal{I}_i\,\mathrm{d}\,\mathcal{I}_i^{-1}\mathcal{I}_j^{-1} - 2(1-\delta_{ij})\mathcal{I}_j\mathrm{d}\mathcal{I}_j^{-1}\,\mathrm{ad}\,\mathcal{I}_i \\
&= -\delta_{ij}\mathrm{d} - \sum_k \varepsilon_{ijk}\mathrm{d}_{\mathcal{I}_k} + 2(1-\delta_{ij})\left[\mathrm{ad}\,\mathcal{I}_i, \mathrm{d}_{\mathcal{I}_j}\right] \\
&= \begin{cases} -\mathrm{d} & \text{if } i = j, \\ -\varepsilon_{ijk}\mathrm{d}_{\mathcal{I}_k} + 2\left[\mathrm{ad}\,\mathcal{I}_i, \mathrm{d}_{\mathcal{I}_j}\right] & \text{if } i \neq j. \end{cases}
\end{aligned}$$

Solving the latter equation for the case $i \neq j$ we find

$$\left[\mathrm{ad}\,\mathcal{I}_i, \mathrm{d}_{\mathcal{I}_j}\right] = \begin{cases} -\mathrm{d} & \text{if } i = j, \\ \sum_k \varepsilon_{ijk}\mathrm{d}_{\mathcal{I}_k} & \text{if } i \neq j, \end{cases}$$

which can be rewritten as

$$\left[\mathrm{ad}\,\mathcal{I}_i, \mathrm{d}_{\mathcal{I}_j}\right] = -\delta_{ij}\mathrm{d} + \sum_k \varepsilon_{ijk}\mathrm{d}_{\mathcal{I}_k}. \tag{5.17}$$

The latter equation, together with equation (5.16), suggests that we can think of the commutator with the adjoint action of a complex structure as a twisting operator for exterior derivatives on a hyperkähler manifold: when $\mathfrak{d}$ is either d or $\mathrm{d}_{\mathcal{I}_i}$,

$$[\mathrm{ad}\,\mathcal{I}_i, \mathfrak{d}] \;=\; \mathcal{I}_i \mathfrak{d} \mathcal{I}_i^{-1}. \tag{5.18}$$

Since the Dolbeault operators can be written in terms of d and $\mathrm{d}_{\mathcal{I}_i}$ (as in equation (5.9)), this relation holds for all Dolbeault operators $\mathfrak{d}$ as well.

For the Lie-derivative part of the angular momentum operator, we know that $[\mathcal{L}_{Y_i}, \mathrm{d}] = 0$, and we compute

$$\begin{aligned}
\left[\mathcal{L}_{Y_i}, \mathrm{d}_{\mathcal{I}_j}\right] &= \left[\mathcal{L}_{Y_i}, \mathcal{I}_j \mathrm{d} \mathcal{I}_j^{-1}\right] \\
&= \left([\mathcal{L}_{Y_i}, \mathcal{I}_j]\, \mathrm{d}\mathcal{I}_j^{-1} + \mathcal{I}_j \mathrm{d}\left[\mathcal{L}_{Y_i}, \mathcal{I}_j^{-1}\right]\right) \\
&= (1-\delta_{ij})\left(\mathrm{ad}\,\mathcal{I}_i \mathcal{I}_j \mathrm{d}\mathcal{I}_j^{-1} + \mathcal{I}_j \mathrm{d}\,\mathrm{ad}\,\mathcal{I}_i \mathcal{I}_j^{-1}\right) \\
&= (1-\delta_{ij})\left(\mathrm{ad}\,\mathcal{I}_i \mathcal{I}_j \mathrm{d}\mathcal{I}_j^{-1} - \mathcal{I}_j \mathrm{d}\mathcal{I}_j^{-1}\,\mathrm{ad}\,\mathcal{I}_i\right) \\
&= (1-\delta_{ij})\left[\mathrm{ad}\,\mathcal{I}_i, \mathrm{d}_{\mathcal{I}_j}\right] \\
&= \sum_k \varepsilon_{ijk} \mathrm{d}_{\mathcal{I}_k}.
\end{aligned} \tag{5.19}$$

5.2.2 Lie brackets with the angular momentum operator

We are now ready to compute the Lie brackets of the angular momentum operator with the (twisted) exterior derivatives and Dolbeault operators. First of all,

$$[J_i, \mathrm{d}] \;=\; i\left[\mathcal{L}_{Y_i} - \frac{1}{2}\,\mathrm{ad}\,\mathcal{I}_i, \mathrm{d}\right] \;=\; -\frac{i}{2}\mathrm{d}_{\mathcal{I}_i}, \tag{5.20}$$

$$\left[J_i, \mathrm{d}_{\mathcal{I}_j}\right] \;=\; i\left[\mathcal{L}_{Y_i} - \frac{1}{2}\,\mathrm{ad}\,\mathcal{I}_i, \mathrm{d}_{\mathcal{I}_j}\right] \;=\; -\frac{i}{2}\left(-\delta_{ij}\mathrm{d} - \sum_k \varepsilon_{ijk}\mathrm{d}_{\mathcal{I}_k}\right). \tag{5.21}$$

We define raising and lowering operators as usual by

$$J_\pm \;=\; J_1 \pm i J_2, \tag{5.22}$$

and the algebra satisfied by the angular momentum operators and the Dolbeault operators is the following:

$$\begin{aligned}
\left[J_3,\overline{\partial}\right] &= \frac{1}{2}\overline{\partial}, & \left[J_3,\overline{\partial}_{\mathcal{J}}\right] &= -\frac{1}{2}\overline{\partial}_{\mathcal{J}},\\
\left[J_+,\overline{\partial}\right] &= 0, & \left[J_+,\overline{\partial}_{\mathcal{J}}\right] &= i\overline{\partial},\\
\left[J_-,\overline{\partial}\right] &= -i\overline{\partial}_{\mathcal{J}}, & \left[J_-,\overline{\partial}_{\mathcal{J}}\right] &= 0,
\end{aligned} \tag{5.23}$$

and by complex conjugation

$$\begin{aligned}
[J_3,\partial] &= -\frac{1}{2}\partial, & [J_3,\partial_{\mathcal{J}}] &= \frac{1}{2}\partial_{\mathcal{J}},\\
[J_-,\partial] &= 0, & [J_-,\partial_{\mathcal{J}}] &= i\partial,\\
[J_+,\partial] &= -i\partial_{\mathcal{J}}, & [J_+,\partial_{\mathcal{J}}] &= 0.
\end{aligned} \tag{5.24}$$

We see that indeed the action of $\overline{\partial}$ or $\partial_{\mathcal{J}}$ increases the angular momentum of a state with respect to J_3 by $\frac{1}{2}$, whereas $\overline{\partial}_{\mathcal{J}}$ or ∂ decreases it by $\frac{1}{2}$. Finally, using $\mathfrak{d}^\dagger = -*\overline{\mathfrak{d}}*$ and $[\vec{J},*] = 0$, we have

$$[\vec{J},\mathfrak{d}^\dagger] = -*[\vec{J},\overline{\mathfrak{d}}]* = \left(\overline{[\vec{J},\overline{\mathfrak{d}}]}\right)^\dagger . \tag{5.25}$$

Explicitly, therefore, we find that the (twisted) adjoint Dolbeault operators and the angular momentum operators satisfy the following algebra, as expected.

$$\begin{aligned}
\left[J_3,\overline{\partial}^\dagger\right] &= -\frac{1}{2}\overline{\partial}^\dagger, & \left[J_3,\overline{\partial}^\dagger_{\mathcal{J}}\right] &= \frac{1}{2}\overline{\partial}^\dagger_{\mathcal{J}},\\
\left[J_+,\overline{\partial}^\dagger\right] &= -i\overline{\partial}^\dagger_{\mathcal{J}}, & \left[J_+,\overline{\partial}^\dagger_{\mathcal{J}}\right] &= 0,\\
\left[J_-,\overline{\partial}^\dagger\right] &= 0, & \left[J_-,\overline{\partial}^\dagger_{\mathcal{J}}\right] &= i\overline{\partial}^\dagger,
\end{aligned} \tag{5.26a}$$

and

$$
\begin{aligned}
\left[J_3, \partial^\dagger\right] &= \frac{1}{2}\partial^\dagger, & \left[J_3, \partial^\dagger_{\mathcal{J}}\right] &= -\frac{1}{2}\partial^\dagger_{\mathcal{J}}, \\
\left[J_-, \partial^\dagger\right] &= -i\partial^\dagger_{\mathcal{J}}, & \left[J_-, \partial^\dagger_{\mathcal{J}}\right] &= 0, \\
\left[J_+, \partial^\dagger\right] &= 0, & \left[J_+, \partial^\dagger_{\mathcal{J}}\right] &= i\partial^\dagger. \qquad (5.26b)
\end{aligned}
$$

Part II

Examples

Chapter 6

Charge-1 Monopoles

As a first example, we now consider a monopole of unit charge, in YMH-models with maximal symmetry breaking, for example $SU(2) \to U(1)$ or $SU(3) \to U(1) \times U(1)$. In the latter case, the charge-1 monopole is an embedding of the $SU(2)$ monopole in the $SU(3)$ model. We discuss the classical dynamics and the quantum mechanics of the bosonic, $N = 2$ supersymmetric monopole [38], and finally $N = 4$ supersymmetric monopoles. We will explicitly exhibit the equivalence between quantisation in terms of spinors and quantisation in terms of anti-holomorphic forms on the moduli space. To conclude this chapter, we discuss the angular momentum and spin of charge-1 monopole states.

In this example, and the next, we use geometrical units, in which we have scaled the coupling constant and the vacuum expectation value of the Higgs field to unity,

$$e \ = \ 1, \qquad\qquad a \ = \ 1. \tag{6.1}$$

This implies that Planck's constant $\hbar$ is dimensionless, but in general not equal to 1.

The moduli space of a single monopole in this theory is [4],

$$\mathcal{M}_1 \ = \ \mathbb{R}^3 \times S^1. \tag{6.2}$$

The factor $\mathbb{R}^3$ corresponds to translation of the monopole, and the S^1 factor to long-range gauge transformations (of the form $g(\chi) = e^{-\chi\Phi}$). The metric on $\mathcal{M}_1$ is the flat metric

$$\mathrm{d}s^2 \ = \ m\left(\mathrm{d}\vec{x}^2 \ + \ \mathrm{d}\chi^2\right) \ = \ (e^1)^2 + (e^3)^2 + (e^3)^2 + (e^4)^2, \tag{6.3}$$

where the mass of a monopole is $m = 4\pi$, the vector $\vec{x} = (x^1, x^2, x^3)$, and the vier-bein is defined by

$$e^i \;=\; \sqrt{m}\,\mathrm{d}x^i, \qquad\qquad e^4 \;=\; \sqrt{m}\,\mathrm{d}\chi. \tag{6.4}$$

The range of χ is $0 \leq \chi < 2\pi$.

6.1 Classical dynamics

The classical motion of the monopole in the moduli space approximation is given by the geodesics on $\mathcal{M}_1$ (see section 2.4). Since $\mathcal{M}_1$ is flat, these are straight lines, corresponding to uniform motion through space and a constant electric charge.

The Lagrangian corresponding to the system of a single monopole is

$$L \;=\; \frac{m}{2}\left(\dot{\vec{x}}\cdot\dot{\vec{x}} \;+\; \dot{\chi}^2\right) \;-\; m. \tag{6.5}$$

The Euler-Lagrange equations give us the conserved quantities, the momentum and the electric charge of the monopole (or dyon if the electric charge is non-zero),

$$\vec{p} \;=\; \frac{\partial L}{\partial\dot{\vec{x}}} \;=\; m\,\dot{\vec{x}}, \qquad\qquad q \;=\; \frac{\partial L}{\partial\dot{\chi}} \;=\; m\,\dot{\chi}. \tag{6.6}$$

The energy, given by the usual Legendre transformation,

$$E \;=\; \frac{\partial L}{\partial\dot{\vec{x}}}\cdot\dot{\vec{x}} + \frac{\partial L}{\partial\dot{\chi}}\dot{\chi} - L \;=\; \frac{m}{2}\left(\dot{\vec{x}}\cdot\dot{\vec{x}} \;+\; \dot{\chi}^2\right) \;+\; m, \tag{6.7}$$

is conserved, as well as the angular momentum $\vec{J} = \vec{x}\times\vec{p}$.

6.2 Quantum mechanics of bosonic monopoles

The discussion of the quantisation of supersymmetric monopoles in terms of forms on the moduli space shows that the quantum mechanics of bosonic monopoles is described by wavefunctions (0-forms) on the moduli space. The Schrödinger equation is given by

$$i\hbar\partial_t\Psi \;=\; H_{\mathrm{eff}}\Psi. \tag{6.8}$$

As we have found in section 3.5, the Hamiltonian is half the Laplacian plus the mass of the monopole. In this case the metric (6.3) is flat, so that the Schrödinger equation becomes

$$i\hbar\partial_t\Psi \;=\; \frac{\hbar^2}{2}\Delta_{\mathcal{M}_1}\Psi + m\Psi \;=\; -\frac{\hbar^2}{2m}(\partial_i^2+\partial_\chi^2)\Psi + m\Psi. \tag{6.9}$$

By separation of variables, we find plane wave solutions

$$\Psi \;=\; f(t)F(\vec{x},\chi), \qquad f(t) \;=\; Ae^{-\frac{i}{\hbar}Et}, \qquad F(\vec{x},\chi) \;=\; Be^{\frac{i}{\hbar}(\vec{p}\cdot\vec{x}+q\chi)}, \tag{6.10}$$

for arbitrary constants A, B, $\vec{p}$, and q, such that the energy of the monopole is

$$E \;=\; m + \frac{1}{2m}\left(|\vec{p}|^2 + q^2\right), \tag{6.11}$$

where $\vec{p}$ is its momentum, and q its electric charge. Since the range of χ is $0 \le \chi < 2\pi$, the electric charge q is $\hbar$ times an integer.

The probabilistic interpretation of the wavefunction is the usual one: $|\Psi(\vec{x},\chi)|^2$ is the probability density for the monopole at the point $(\vec{x},\chi)$ in the moduli space. This means that $|\Psi(\vec{x},\chi)|^2$ is the probability density for the Yang-Mills-Higgs fields of the original field theory to be in the (charge-1 monopole) configuration that corresponds to the point $(\vec{x},\chi)$ in the moduli space.

6.3 $N = 2$ supersymmetric monopoles

Starting with a solution Ψ to the bosonic Schrödinger equation, we can use supersymmetry to find other solutions. By applying the supercharges, i.e. the Dolbeault operator $\overline{\partial}$, its twisted counterpart $\overline{\partial}_{\mathcal{J}}$ and their adjoints $\overline{\partial}^\dagger$ and $\overline{\partial}^\dagger_{\mathcal{J}}$, we may generate the other states of the supermultiplet containing Ψ. Since on a Kähler manifold they commute with the Hamiltonian, the solutions that we find by applying these operators to the wavefunction Ψ also obey the Schrödinger equation (see appendix A.2.2).

BPS states of $N = 2$ supersymmetric monopoles are those states which have minimal energy for given charges of the states. They correspond to short supermultiplets, which contain two spin-0 states, and one spin-$\frac{1}{2}$ doublet. A short multiplet for a charge-1 monopole is obtained whenever the spin-0 state Ψ with which we start, is an eigenstate of the Hamiltonian. The states in a short multiplet can then be found using *only* the (twisted) Dolbeault operators $\overline{\partial}$ and $\overline{\partial}_{\mathcal{J}}$. The states we find this way are

$$\begin{array}{ll} \text{2 singlets:} & \Psi \text{ and } \overline{\partial}\overline{\partial}_J\Psi = -\overline{\partial}_{\mathcal{J}}\overline{\partial}\Psi \\ \text{1 doublet:} & \left(\overline{\partial}\Psi, \overline{\partial}_J\Psi\right) \end{array}$$

In this case, using the adjoint operators to the (twisted) Dolbeault operators give us no new independent states. For example, $\overline{\partial}^{\dagger}_{\mathcal{J}}\overline{\partial}\Psi = -\overline{\partial}\overline{\partial}^{\dagger}_{\mathcal{J}}\Psi = 0$, and $\overline{\partial}^{\dagger}\overline{\partial}\Psi = \frac{1}{2}\Delta\Psi$ is not an independent state if Ψ is an eigenstate of Δ, for any eigenvalue.

In the next two sections we will explicitly show the equivalence between anti-holomorphic forms and spinors on the moduli space. We obtain equivalent expressions for the Dirac operator and Hamiltonian for both descriptions of the fermionic zero-modes, and we give an explicit correspondence between forms and spinors.

6.3.1 Quantisation using spinors

We now construct the Dirac operator and Hamiltonian acting on spinors on $\mathcal{M}_1$. The Dirac operator is defined by

$$\not{D}_s = \frac{1}{\sqrt{m}}\gamma^a\partial_a. \tag{6.12}$$

We use the following representation for the Dirac γ-matrices,

$$\gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}, \qquad \gamma^4 = \begin{pmatrix} 0 & i\mathbb{1} \\ i\mathbb{1} & 0 \end{pmatrix}, \tag{6.13}$$

which satisfy $\{\gamma^\alpha, \gamma^\beta\} = -2\delta^{\alpha\beta}$. The Dirac operator is therefore

$$\not{D}_s = \frac{1}{\sqrt{m}}\begin{pmatrix} 0 & \sigma^j\partial_j + i\partial_\chi \\ -\sigma^j\partial_j + i\partial_\chi & 0 \end{pmatrix}. \tag{6.14}$$

The Hamiltonian is then given by

$$H_0 = \frac{1}{2}\not{D}_s^2 = -\frac{1}{2m}\left[\partial_j^2 + \partial_\chi^2\right]\mathbb{1}_4 = \frac{1}{2}\Delta_{\mathcal{M}_1}\mathbb{1}_4. \tag{6.15}$$

6.3.2 Quantisation using forms

To identify the Dirac operator acting on forms, $\not{D}_{\overline{\partial}} = \sqrt{2}(\overline{\partial} + \overline{\partial}^\dagger)$, with the Dirac operator on spinors, we need to find an appropriate matrix representation of this operator. This implies that we need to look for a matrix representation of the Dolbeault operator and its adjoint. We do this by choosing a basis of anti-holomorphic forms, and representing a general anti-holomorphic form as a vector with respect to this basis. The Dolbeault operators map anti-holomorphic forms to anti-holomorphic forms and can therefore be represented by a matrix with respect to this basis.

Kähler coordinates for the moduli space $\mathcal{M}_1 \cong \mathbb{C} \times \mathbb{C}^*$ are given by [3]

$$z^2 = x^1 + ix^2, \qquad z^1 = e^{x^3 + i\chi}. \tag{6.16}$$

We define the complex structures $\mathcal{I}_i$ on the moduli space defined by

$$\begin{aligned} \mathcal{I}_i(e^j) &= \delta_{ij} e^4 + \varepsilon_{ijk} e^k \\ \mathcal{I}_i(e^4) &= -e^i \end{aligned} \tag{6.17}$$

where the vier-bein e was defined in (6.4). The complex structure corresponding to the Kähler coordinates (6.16) is $\mathcal{I} = \mathcal{I}_3$. A convenient basis of holomorphic 1-forms, with respect to this complex structure, is

$$\alpha_2 = \sqrt{\frac{m}{2}}\,\mathrm{d}z^2 = \frac{1}{\sqrt{2}}(e^1 + ie^2), \qquad \alpha_1 = \sqrt{\frac{m}{2}}\frac{\mathrm{d}z^1}{z^1} = \frac{1}{\sqrt{2}}(e^3 + ie^4). \tag{6.18}$$

The metric on $\mathcal{M}_1$ then becomes

$$\mathrm{d}s^2 = m\left[\left|\mathrm{d}z^2\right|^2 + \left|\frac{\mathrm{d}z^1}{z^1}\right|^2\right] = 2|\alpha_2|^2 + 2|\alpha_1|^2. \tag{6.19}$$

The exterior derivative is decomposed into the Dolbeault operator and its complement, $\mathrm{d} = \partial + \overline{\partial}$, as usual. The action of the Dolbeault operator is given by the action of the exterior derivative followed by a projection onto the anti-holomophic forms,

$$\overline{\partial} = \pi^{0,\bullet} \circ \mathrm{d}, \qquad \pi^{0,\bullet} : \Omega^\bullet(M) \to \Omega^{0,\bullet}(M). \tag{6.20}$$

We choose $\{\overline{\alpha}_1, \overline{\alpha}_2, 1, \overline{\alpha}_1 \wedge \overline{\alpha}_2\}$ as an ordered basis of $\Omega^{0,\bullet}(M)$, to represent the action of the Dolbeault operator $\overline{\partial}$, and its adjoint $\overline{\partial}^\dagger = - * \partial *$, as matrices.

$$(\overline{\partial}) = \begin{pmatrix} 0 & 0 & \overline{f}_1 & 0 \\ 0 & 0 & \overline{f}_2 & 0 \\ 0 & 0 & 0 & 0 \\ -\overline{f}_2 & \overline{f}_1 & 0 & 0 \end{pmatrix} \qquad (\overline{\partial}^\dagger) = \begin{pmatrix} 0 & 0 & 0 & f_2 \\ 0 & 0 & 0 & -f_1 \\ -f_1 & -f_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.21}$$

The operators f_1 and f_2 are given by

$$f_1 = \frac{1}{\sqrt{2m}}\left(\partial_3 - i\partial_\chi\right), \qquad f_2 = \frac{1}{\sqrt{2m}}\left(\partial_1 - i\partial_2\right). \tag{6.22}$$

The Dirac operator in the chosen basis of anti-holomorphic forms is therefore the same as the Dirac operator acting on spinors, found in section 6.3.1,

$$\not{D}_{\overline{\partial}} = \sqrt{2}\left(\overline{\partial} + \overline{\partial}^\dagger\right) = \frac{1}{\sqrt{m}}\begin{pmatrix} 0 & i\partial_\chi + \sigma_k\partial_k \\ i\partial_\chi - \sigma_k\partial_k & 0 \end{pmatrix} = \not{D}_s, \tag{6.23}$$

which means that our chosen basis of anti-holomorphic forms provides an easy way to translate between spinors and anti-holomorphic forms on the moduli space:

$$\overline{\alpha}_1 \sim \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \overline{\alpha}_2 \sim \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad 1 \sim \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \overline{\alpha}_1 \wedge \overline{\alpha}_2 \sim \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{6.24}$$

The Hamiltonian $H_0 = \frac{1}{2}\not{D}^2_{\overline{\partial}} = \overline{\partial}^\dagger\overline{\partial} + \overline{\partial}\,\overline{\partial}^\dagger = \frac{1}{2}\Delta_{\mathcal{M}_1}$ is the same as the one found in section 6.3.1. Since H_0 acts diagonally, we see that the spectrum of the $N = 2$ supersymmetric monopole is simply a four-fold degenerate copy of the bosonic monopole spectrum.

We compute the matrix representation of the twisted Dolbeault operator and its adjoint, using $\overline{\partial}_{\mathcal{J}} = \mathcal{J}\partial\mathcal{J}^{-1}$ and $\overline{\partial}^\dagger_{\mathcal{J}} = \mathcal{J}\partial^\dagger\mathcal{J}^{-1}$,

$$(\overline{\partial}_{\mathcal{J}}) = i\begin{pmatrix} 0 & 0 & f_2 & 0 \\ 0 & 0 & -f_1 & 0 \\ 0 & 0 & 0 & 0 \\ f_1 & f_2 & 0 & 0 \end{pmatrix}, \qquad (\overline{\partial}^\dagger_{\mathcal{J}}) = i\begin{pmatrix} 0 & 0 & 0 & \overline{f}_1 \\ 0 & 0 & 0 & \overline{f}_2 \\ \overline{f}_2 & -\overline{f}_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6.25}$$

Now we can compute the operators $\overline{\partial}\,\overline{\partial}_{\mathcal{J}} = -\overline{\partial}_{\mathcal{J}}\overline{\partial}$ and $-\overline{\partial}^{\dagger}\overline{\partial}^{\dagger}_{\mathcal{J}} = \overline{\partial}^{\dagger}_{\mathcal{J}}\overline{\partial}^{\dagger}$.

$$\left(\overline{\partial}\,\overline{\partial}_{\mathcal{J}}\right) \;=\; \frac{i}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta_{\mathcal{M}_1} & 0 \end{pmatrix} \qquad \left(\overline{\partial}^{\dagger}\overline{\partial}^{\dagger}_{\mathcal{J}}\right) \;=\; \frac{i}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_{\mathcal{M}_1} \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.26}$$

These expressions can be verified using the Kodaira relations [41],

$$\left[L_{\overline{\Omega}}, \overline{\partial}^{\dagger}\right] \;=\; \overline{\partial}_{\mathcal{J}}, \qquad\qquad \left[L_{\overline{\Omega}}, \overline{\partial}^{\dagger}_{\mathcal{J}}\right] \;=\; -\overline{\partial}. \tag{6.27}$$

Here $L_{\overline{\Omega}}$ is an operator of exterior multiplication by $\overline{\Omega}$, and

$$\Omega \;=\; \omega_{\mathcal{J}} + i\omega_{\mathcal{K}} \;=\; i\alpha_1 \wedge \alpha_2 \tag{6.28}$$

is the canonical holomorphic symplectic form (see appendix A.1.5). Since the highest degree of an anti-holomorphic form on the 4-dimensional moduli space $\mathcal{M}_1$ is $(0,2)$, we know that $\overline{\partial}\,\overline{\partial}_{\mathcal{J}}$ must act trivially on $(0,1)$- and $(0,2)$-forms. To compute its action on functions, we note that, again because of the dimension of the moduli space,

$$\overline{\partial}L_{\overline{\Omega}}f \;=\; \overline{\partial}L_{\overline{\Omega}}f \;=\; 0, \tag{6.29}$$

and therefore, using $\overline{\partial}^{\dagger}\overline{\partial}f = \frac{1}{2}\Delta_{\mathcal{M}_1}f$,

$$\begin{aligned} \overline{\partial}\,\overline{\partial}_{\mathcal{J}}f \;=\; -\overline{\partial}_{\mathcal{J}}\overline{\partial}f \;&=\; -\left[L_{\overline{\Omega}}, \overline{\partial}^{\dagger}\right]\overline{\partial}f \\ &=\; -L_{\overline{\Omega}}\overline{\partial}^{\dagger}\overline{\partial}f \\ &=\; \frac{i}{2}\overline{\alpha}_1 \wedge \overline{\alpha}_2 \cdot \Delta_{\mathcal{M}_1}f, \end{aligned} \tag{6.30}$$

as claimed in equation (6.26). Similarly, the Kodaira relations show that acting on a $(0,2)$-form ψ,

$$L_{\overline{\Omega}}\overline{\partial}^{\dagger}\overline{\partial}^{\dagger}_{\mathcal{J}}\psi \;=\; \left[L_{\overline{\Omega}}, \overline{\partial}^{\dagger}\right]\overline{\partial}^{\dagger}_{\mathcal{J}}\psi \;=\; \overline{\partial}_{\mathcal{J}}\overline{\partial}^{\dagger}_{\mathcal{J}}\psi \;=\; \frac{1}{2}\Delta_{\mathcal{M}_1}\psi \tag{6.31}$$

which, using equations (6.28), implies the second of equations (6.26).

6.4 $N = 4$ supersymmetric monopoles

To construct the $N = 4$ supersymmetric monopole states, we start again with a solution Ψ to the bosonic Schrödinger equation. By applying the supercharges, i.e. the Dolbeault operators ∂ and $\overline{\partial}$, their twisted counterparts, $\partial_{\mathcal{J}}$ and $\overline{\partial}_{\mathcal{J}}$, and their adjoints $\partial^\dagger$, $\overline{\partial}^\dagger$, $\partial_{\mathcal{J}}^\dagger$ and $\overline{\partial}_{\mathcal{J}}^\dagger$, we may generate the other states of the supermultiplet containing Ψ.

BPS states of $N = 4$ supersymmetric monopoles are those states which have minimal energy for given charges of the states. They correspond to short supermultiplets, which contain five spin-0 states, four spin-$\frac{1}{2}$ doublets and one spin-1 triplet. A short multiplet for a charge-1 monopole is obtained whenever the spin-0 state Ψ with which we start, is an eigenstate of the Hamiltonian. The states in an $N = 4$ short multiplet can then be found using *only* the (twisted) Dolbeault operators ∂, $\overline{\partial}$, $\partial_{\mathcal{J}}$ and $\overline{\partial}_{\mathcal{J}}$. As in the $N = 2$ supersymmetric case, using the adjoint operators does not lead to independent states. The $N = 4$ short multiplet breaks down into

$$\text{5 singlets:} \qquad \Psi,\ \overline{\partial}_J\overline{\partial}\Psi,\ \partial_J\partial\Psi,\ \partial_J\partial\overline{\partial}_J\overline{\partial}\Psi,\ \frac{1}{\sqrt{2}}\left(\partial_J\overline{\partial} - \partial\overline{\partial}_J\right)\Psi$$

$$\text{4 doublets:} \qquad \left(\overline{\partial}\Psi, \overline{\partial}_J\Psi\right),\ \left(\partial\Psi, \partial_J\Psi\right),\ \left(\overline{\partial}\partial_J\partial\Psi, \overline{\partial}_J\partial_J\partial\Psi\right),\ \left(\partial\overline{\partial}_J\overline{\partial}\Psi, \partial_J\overline{\partial}_J\overline{\partial}\Psi\right)$$

$$\text{1 triplet:} \qquad \left(\partial\overline{\partial}\Psi, \frac{1}{\sqrt{2}}\left(\partial_J\overline{\partial} + \partial\overline{\partial}_J\right)\Psi, \partial_J\overline{\partial}_J\Psi\right)$$

6.5 Angular momentum and spin

The total angular momentum operator $\vec{J}$ is defined by equation (5.8), where the vector fields Y_i generating the $SO(3)$ action are defined by

$$Y_i \;=\; -\varepsilon_{ijk}x^j\partial_k. \tag{6.32}$$

They satisfy $[Y_i, Y_j] = \varepsilon_{ijk}Y_k$. Using

$$\mathcal{L}_{Y_i}(e^j) \;=\; \mathrm{d}i_{Y_i}(e^j) \;=\; \varepsilon_{ijk}e^k, \qquad\qquad \mathcal{L}_{Y_i}(e^4) \;=\; 0, \tag{6.33}$$

we find that

$$\left[\mathcal{L}_{Y_i}, \mathcal{I}_j\right](e^a) \;=\; \varepsilon_{ijk}\mathcal{I}_k(e^a) \tag{6.34}$$

Since $\mathrm{ad}\,\mathcal{I}_i(e^a) = \mathcal{I}_i(e^a)$ we immediately also have

$$[\mathcal{L}_{Y_i}, \mathrm{ad}\,\mathcal{I}_j]\,(e^a) \;=\; \varepsilon_{ijk}\,\mathrm{ad}\,\mathcal{I}_k(e^a). \tag{6.35}$$

Because $\mathcal{L}_{Y_i}$ and $\mathrm{ad}\,\mathcal{I}_j$ both satisfy the Leibnitz rule, we deduce that in general the Lie derivatives with respect to these vector fields obey indeed the commutation relations (5.6) with the complex structures (6.17): $[\mathcal{L}_{Y_i}, \mathrm{ad}\,\mathcal{I}_j] = \varepsilon_{ijk}\,\mathrm{ad}\,\mathcal{I}_k$.

We can write the Lie-derivative in terms of the exterior derivative and interior product using Cartan's formula,

$$\mathcal{L}_{Y_i} \;=\; \iota_{Y_i}\mathrm{d} + \mathrm{d}\iota_{Y_i}. \tag{6.36}$$

The complex structure $\mathrm{ad}\,\mathcal{I}_i$ and the term $\mathrm{d}\iota_{Y_i}$ act trivially on functions, as 0. Furthermore, since the 1-forms α_1, α_2, $\overline{\alpha}_1$ and $\overline{\alpha}_2$ are closed, the term $\iota_{Y_i}\mathrm{d}$ acts on them as 0. This suggests that we identify the orbital angular momentum and spin operators as

$$L_i \;=\; i\,(\iota_{Y_i}\mathrm{d})\,, \qquad\qquad S_i \;=\; i\left(\mathrm{d}\iota_{Y_i} - \frac{1}{2}\,\mathrm{ad}\,\mathcal{I}_i\right), \tag{6.37}$$

which both act on forms obeying the Leibniz rule. With these definitions,

$$\vec{L}(\alpha_m) \;=\; \vec{L}(\overline{\alpha}_m) \;= 0, \qquad\qquad \vec{S}(f) \;= 0. \tag{6.38}$$

We have also, for example,

$$\begin{aligned} J_1(\alpha_1) \;=\; S_1(\alpha_1) \;&=\; \frac{i}{\sqrt{2}}\left(\mathrm{d}\iota_{Y_1} - \frac{1}{2}\,\mathrm{ad}\,\mathcal{I}_1\right)(e^3 + ie^4) \\ &=\; \frac{i}{\sqrt{2}}\left(-e^2 - \frac{1}{2}(-e^2 - ie^1)\right) \\ &=\; -\frac{i}{2\sqrt{2}}(e^2 - ie^1) \\ &=\; -\frac{1}{2}\alpha_2, \end{aligned} \tag{6.39}$$

and in general we find that the spin operator acts on α_1, α_2, $\overline{\alpha}_1$ and $\overline{\alpha}_2$ by

$$J_i(\alpha_m) \;=\; S_i(\alpha_m) \;=\; -\frac{1}{2}(\overline{\sigma}_i)_{mn}\alpha_n, \tag{6.40a}$$

$$J_i(\overline{\alpha}_m) \;=\; S_i(\overline{\alpha}_m) \;=\; \frac{1}{2}(\sigma_i)_{mn}\overline{\alpha}_n. \tag{6.40b}$$

6.5.1 $N = 2$ supersymmetric monopoles

The pair $\{\overline{\alpha}_1, \overline{\alpha}_2\}$ form the two-dimensional fundamental representation of $SU(2)$, and we identify the basic spin-states (5.2) by

$$\Big|\uparrow\Big\rangle = \overline{\alpha}_1, \qquad \Big|\downarrow\Big\rangle = \overline{\alpha}_2. \tag{6.41}$$

Writing a general state in the basis $\{\overline{\alpha}_1, \overline{\alpha}_2, 1, \overline{\alpha}_1 \wedge \overline{\alpha}_2\}$ of anti-holomorphic forms, the orbital angular momentum and spin operators for $N = 2$ supersymmetric monopoles take the form

$$\big(L_i\big) = iY_i\,\mathbb{1}_4, \qquad \big(S_i\big) = \frac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix}, \tag{6.42}$$

where the vectors Y_i act as derivatives on functions, and 0 is the (2×2)-matrix of zeros. This result perfectly agrees with Osborn's result, equation (5.1), for $N = 2$ supersymmetric monopoles (involving only a single fermion species, $n = 1$). The states (6.41) form a doublet, while $\Big|0\Big\rangle = 1$ and $\Big|\uparrow\downarrow\Big\rangle = \overline{\alpha}_1 \wedge \overline{\alpha}_2$ are the two singlets.

6.5.2 $N = 4$ supersymmetric monopoles

The pair $\{\alpha_1, \alpha_2\}$ form the two-dimensional conjugate representation to the pair $\{\overline{\alpha}_1, \overline{\alpha}_2\}$. Since these two representations are isomorphic, we may transform the conjugate representation into the fundamental representation: we define

$$\beta_1 = -i\alpha_2, \qquad \beta_2 = i\alpha_1, \tag{6.43}$$

and the spin operator acts on these states by

$$S_i(\beta_m) = \frac{1}{2}(\sigma_i)_{mn}\beta_n. \tag{6.44}$$

We now identify the basic spin-states by

$$\Big|{}^{\uparrow}_{0}\Big\rangle = \overline{\alpha}_1, \qquad \Big|{}^{\downarrow}_{0}\Big\rangle = \overline{\alpha}_2, \tag{6.45a}$$

$$\Big|{}^{0}_{\uparrow}\Big\rangle = \beta_1, \qquad \Big|{}^{0}_{\downarrow}\Big\rangle = \beta_2, \tag{6.45b}$$

for which the spin operator acts in perfect agreement with Osborn's result for $N = 4$ supersymmetric monopoles, equation (5.1).

The decomposition of the full multiplet is then the following. The singlets are

$$\left|{}^{0}_{0}\right\rangle = 1 \tag{6.46}$$

$$\left|{}^{\uparrow\downarrow}_{0}\right\rangle = \overline{\alpha}_2 \wedge \overline{\alpha}_1 \tag{6.47}$$

$$\left|{}^{0}_{\uparrow\downarrow}\right\rangle = \alpha_1 \wedge \alpha_2 \tag{6.48}$$

$$\frac{1}{\sqrt{2}}\left(\left|{}^{\uparrow}_{\downarrow}\right\rangle - \left|{}^{\downarrow}_{\uparrow}\right\rangle\right) = \frac{1}{\sqrt{2}} i \left(\alpha_1 \wedge \overline{\alpha}_1 + \alpha_2 \wedge \overline{\alpha}_2\right) \tag{6.49}$$

$$\left|{}^{\uparrow\downarrow}_{\uparrow\downarrow}\right\rangle = \alpha_1 \wedge \alpha_2 \wedge \overline{\alpha}_2 \wedge \overline{\alpha}_1 \tag{6.50}$$

Two doublets are given by equations (6.45a) and (6.45b). The remaining two doublets are

$$\left|{}^{\uparrow\downarrow}_{\uparrow}\right\rangle = -i\alpha_2 \wedge \overline{\alpha}_2 \wedge \overline{\alpha}_1 \tag{6.51a}$$

$$\left|{}^{\uparrow\downarrow}_{\downarrow}\right\rangle = i\alpha_1 \wedge \overline{\alpha}_2 \wedge \overline{\alpha}_1 \tag{6.51b}$$

and

$$\left|{}^{\uparrow}_{\uparrow\downarrow}\right\rangle = \alpha_1 \wedge \alpha_2 \wedge \overline{\alpha}_1 \tag{6.52a}$$

$$\left|{}^{\downarrow}_{\uparrow\downarrow}\right\rangle = \alpha_1 \wedge \alpha_2 \wedge \overline{\alpha}_2 \tag{6.52b}$$

The triplet is

$$\left|{}^{\uparrow}_{\uparrow}\right\rangle = -i\alpha_2 \wedge \overline{\alpha}_1 \tag{6.53a}$$

$$\frac{1}{\sqrt{2}}\left(\left|{}^{\uparrow}_{\downarrow}\right\rangle + \left|{}^{\downarrow}_{\uparrow}\right\rangle\right) = \frac{1}{\sqrt{2}} i \left(\alpha_1 \wedge \overline{\alpha}_1 - \alpha_2 \wedge \overline{\alpha}_2\right) \tag{6.53b}$$

$$\left|{}^{\downarrow}_{\downarrow}\right\rangle = i\alpha_1 \wedge \overline{\alpha}_2 \tag{6.53c}$$

We notice that the 2-forms corresponding to singlets are anti-self-dual, and related to the hyperkähler forms (see appendix A.1.4) by

$$\omega_1 \;=\; e^4 \wedge e^1 - e^2 \wedge e^3 \;=\; -i\left(\left|\begin{smallmatrix}\uparrow\downarrow\\ 0\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}0\\ \uparrow\downarrow\end{smallmatrix}\right\rangle\right), \tag{6.54a}$$

$$\omega_2 \;=\; e^4 \wedge e^2 - e^3 \wedge e^1 \;=\; \left(\left|\begin{smallmatrix}\uparrow\downarrow\\ 0\end{smallmatrix}\right\rangle - \left|\begin{smallmatrix}0\\ \uparrow\downarrow\end{smallmatrix}\right\rangle\right), \tag{6.54b}$$

$$\omega_3 \;=\; e^4 \wedge e^3 - e^1 \wedge e^2 \;=\; -\left(\left|\begin{smallmatrix}\uparrow\\ \downarrow\end{smallmatrix}\right\rangle - \left|\begin{smallmatrix}\downarrow\\ \uparrow\end{smallmatrix}\right\rangle\right). \tag{6.54c}$$

The canonical holomorphic symplectic 2-form Ω_3 and its conjugate correspond to

$$\Omega_3 \;=\; \omega_1 + i\omega_2 \;=\; -2i\left|\begin{smallmatrix}0\\ \uparrow\downarrow\end{smallmatrix}\right\rangle, \qquad \overline{\Omega}_3 \;=\; \omega_1 - i\omega_2 \;=\; -2i\left|\begin{smallmatrix}\uparrow\downarrow\\ 0\end{smallmatrix}\right\rangle. \tag{6.55}$$

The 2-forms corresponding to the triplet states are self-dual: they can be combined in to the three forms

$$T^1 \;=\; \left(\left|\begin{smallmatrix}\uparrow\\ \uparrow\end{smallmatrix}\right\rangle - \left|\begin{smallmatrix}\downarrow\\ \downarrow\end{smallmatrix}\right\rangle\right) \;=\; e^4 \wedge e^1 + e^2 \wedge e^3, \tag{6.56a}$$

$$T^2 \;=\; -i\left(\left|\begin{smallmatrix}\uparrow\\ \uparrow\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}\downarrow\\ \downarrow\end{smallmatrix}\right\rangle\right) \;=\; e^4 \wedge e^2 + e^3 \wedge e^1, \tag{6.56b}$$

$$T^3 \;=\; -\left(\left|\begin{smallmatrix}\uparrow\\ \downarrow\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}\downarrow\\ \uparrow\end{smallmatrix}\right\rangle\right) \;=\; e^4 \wedge e^3 + e^1 \wedge e^2, \tag{6.56c}$$

which satisfy

$$S_i T^j \;=\; i\varepsilon_{ijk} T^k. \tag{6.57}$$

The doublets are dual to each other,

$$*\left|\begin{smallmatrix}\uparrow\\ 0\end{smallmatrix}\right\rangle \;=\; \alpha_2 \wedge \overline{\alpha}_2 \wedge \overline{\alpha}_1 \;=\; i\left|\begin{smallmatrix}\uparrow\downarrow\\ \uparrow\end{smallmatrix}\right\rangle, \tag{6.58a}$$

$$*\left|\begin{smallmatrix}\downarrow\\ 0\end{smallmatrix}\right\rangle \;=\; \overline{\alpha}_2 \wedge \alpha_1 \wedge \overline{\alpha}_1 \;=\; i\left|\begin{smallmatrix}\uparrow\downarrow\\ \downarrow\end{smallmatrix}\right\rangle, \tag{6.58b}$$

and

$$*\left|\begin{smallmatrix}0\\ \uparrow\end{smallmatrix}\right\rangle \;=\; i\alpha_2 \wedge \alpha_1 \wedge \overline{\alpha}_1 \;=\; -i\left|\begin{smallmatrix}\uparrow\\ \uparrow\downarrow\end{smallmatrix}\right\rangle, \tag{6.59a}$$

$$*\left|\begin{smallmatrix}0\\ \downarrow\end{smallmatrix}\right\rangle \;=\; -i\alpha_2 \wedge \overline{\alpha}_2 \wedge \alpha_1 \;=\; -i\left|\begin{smallmatrix}\downarrow\\ \uparrow\downarrow\end{smallmatrix}\right\rangle, \tag{6.59b}$$

and finally we have

$$*\left|\begin{smallmatrix}0\\ 0\end{smallmatrix}\right\rangle \;=\; \left|\begin{smallmatrix}\uparrow\downarrow\\ \uparrow\downarrow\end{smallmatrix}\right\rangle. \tag{6.60}$$

Chapter 7

Charge-$(1,1)$ Monopoles

As a second example we study the system of charge-$(1,1)$ monopoles in YMH theory with symmetry breaking $SU(3) \to U(1) \times U(1)$. These monopoles may be thought of as being composed of two constituent monopoles which are each $SU(2)$ BPS monopoles of charge 1, but embedded into $SU(2)$ subgroups associated with different simple roots of $SU(3)$. The masses of the constituents depend on the direction of the vacuum expectation value of the Higgs field in the Cartan subalgebra of $SU(3)$ and are denoted m_1 and m_2 in the following. The centre of mass dynamics and the relative motion can be separated, as we will discuss in each of the following cases. As in the previous chapter we begin with the classical dynamics, then consider the quantum mechanics of the bosonic and $N = 2$ supersymmetric monopoles and exhibit explicitly the equivalence between quantisation in terms of spinors and quantisation in terms of anti-holomorphic forms on the moduli space [38]. We discuss the quantisation of $N = 4$ supersymmetric monopoles in terms of forms on the moduli space. The short multiplet of states for $N = 4$ supersymmetric monopoles requires the existence of a unique harmonic form on the component of the moduli space corresponding to the relative motion, which is called a Sen-form. Finally we also discuss the angular momentum and spin of charge-$(1,1)$ monopole states.

The main references for this section are the papers by Gauntlett and Lowe [5], and by Lee, Weinberg and Yi [6]; they include a detailed discussion of magnetic and electric charges and a derivation of the metric on moduli space for charge-$(1,1)$ monopoles:

$$\mathcal{M}_{1,1} \;=\; \mathbb{R}^3 \times \frac{\mathbb{R}\times M_{TN}}{\mathbb{Z}}, \tag{7.1}$$

where M_{TN} is the 4-dimensional Taub-NUT manifold with a positive length parameter. Topologically $M_{TN} \cong \mathbb{R}^4$, but it has a curved metric, given below. For practical calculations it is usually convenient to work with the covering space of the moduli space, $\widetilde{\mathcal{M}}_{1,1} = \mathbb{R}^3 \times \mathbb{R} \times M_{TN}$, and impose the identification by $\mathbb{Z}$ on the results at the end. The physics behind the $\mathbb{Z}$ action is explained carefully in the main references [5, 6], and can be summarised as follows. Each of the constituent monopoles that make up a given charge-(1,1) monopole are invariant under one of the residual $U(1)$ gauge symmetries. Thus one may pick generators of $U(1)\times U(1)$ so that each monopole can carry electric charge with respect to one but not the other generator. The angular coordinates conjugate to those charges have the usual range $[0, 2\pi)$, but angular coordinates appearing in the above decomposition are related to those angles by linear transformations which depend on the masses of the constituent monopoles and therefore have a non-standard range, which we will specify below.

The metric on the centre of mass moduli space $\mathbb{R}^3 \times \mathbb{R}$ is the flat metric. Using centre of mass coordinates $\vec{R}$ and χ, the metric is analogous to (6.3), replacing the mass m with the total mass of the charge-$(1,1)$ monopole system, M.

The metric on the Taub-NUT manifold M_{TN} is given by

$$\begin{aligned} \mathrm{d}s^2 \;&=\; \mu\Big[V(\mathrm{d}\vec{r}\cdot\mathrm{d}\vec{r}) + V^{-1}(\eta_3)^2\Big] \\ &=\; \mu\Big[V\left(\mathrm{d}r^2 + r^2\left((\eta_1)^2+(\eta_2)^2\right)\right) + V^{-1}(\eta_3)^2\Big], \end{aligned} \tag{7.2}$$

where μ is the reduced mass of the monopole system, and

$$V \;=\; \left(1+\frac{1}{r}\right). \tag{7.3}$$

The right-invariant 1-forms η_i are

$$\eta_1 \;=\; -\sin\psi\,\mathrm{d}\theta + \cos\psi\sin\theta\,\mathrm{d}\phi, \tag{7.4a}$$

$$\eta_2 \;=\; \cos\psi\,\mathrm{d}\theta + \sin\psi\sin\theta\,\mathrm{d}\phi, \tag{7.4b}$$

$$\eta_3 \;=\; \mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi \;=\; \mathrm{d}\psi + \vec{A}\cdot\mathrm{d}\vec{r}, \tag{7.4c}$$

which satisfy $d\eta_i = \frac{1}{2}\varepsilon_{ijk}\eta_j \wedge \eta_k$. The coordinates $\vec{r} = (x, y, z)$ correspond to the relative position of the two monopoles. We define spherical coordinates as usual by $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$. The Euler angles θ, ϕ and ψ are coordinates on S^3 with the usual ranges: $0 \leq \theta < \pi$, $0 \leq \phi < 2\pi$ and $0 \leq \psi < 4\pi$. As explained in the main references [5, 6], the angle ψ is the conjugate variable to half the difference between the electric charges of the constituent monopoles. The range $[0, 4\pi)$ reflects the fact that half the difference necessarily is an element of $\frac{1}{2}\mathbb{Z}$. Finally, the division by $\mathbb{Z}$ on the total moduli space corresponds to identifying the points

$$(\vec{R}, \chi, \vec{r}, \psi) \;\sim\; (\vec{R}, \chi + 2\pi, \vec{r}, \psi + \tfrac{4m_2}{m_1+m_2}\pi), \tag{7.5}$$

which, as explained above, depends on the constituent monopoles' masses [5, 6].

A similar and closely related system is that of charge-2 monopoles in Yang-Mills-Higgs theory with $SU(2)$ broken to $U(1)$. In this case the moduli space is $\mathcal{M}_2 = \mathbb{R}^3 \times (S^1 \times M_{AH})/\mathbb{Z}_2$, where the relative moduli space M_{AH} is the 4-dimensional Atiyah-Hitchin manifold. Asymptotically, the metric of the Atiyah-Hitchin manifold approaches the Taub-NUT metric given by equation (7.2), but with opposite sign for the mass parameter, $V = (1 - \frac{1}{r})$. Gibbons and Manton [4] have studied the classical and quantum mechanics of this system, and to a large extent we follow their approach. We will see that the opposite sign for the mass parameter in the Taub-NUT metric for charge-$(1, 1)$ monopoles has a crucial effect on the existence of bound states in this system.

7.1 Classical dynamics

Using the product structure of the moduli space, we separate the centre of mass motion and the relative motion. The centre of mass dynamics, corresponding to motion in $\mathbb{R}^3 \times \mathbb{R}$, is analogous to the single monopole dynamics discussed in example 1. For the remaining part of this section, we will focus on the relative motion of the two monopoles, described by the moduli space M_{TN}.

The Lagrangian for the relative motion is

$$L \;=\; \frac{\mu}{2}\left[V\left(\dot{\vec{r}} \cdot \dot{\vec{r}}\right) + V^{-1}\left(\dot{\psi} + \cos\theta\dot{\phi}\right)^2\right]. \tag{7.6}$$

The Euler-Lagrange equation $\frac{\partial L}{\partial \psi} = \partial_t \frac{\partial L}{\partial \dot{\psi}}$ gives us the conserved quantity

$$q = \mu V^{-1} \left(\dot{\psi} + \cos\theta \dot{\phi} \right). \tag{7.7}$$

The energy is given by

$$E = \frac{\partial L}{\partial \dot{\vec{r}}} \cdot \dot{\vec{r}} + \frac{\partial L}{\partial \dot{\psi}} \dot{\psi} - L = \frac{\mu}{2} V \left(\dot{\vec{r}} \cdot \dot{\vec{r}} + \left(\frac{q}{\mu} \right)^2 \right). \tag{7.8}$$

Following Gibbons and Manton, we now define $\vec{p}$ by

$$\frac{\partial L}{\partial \dot{\vec{r}}} = \mu V \dot{\vec{r}} + q\vec{A} = \vec{p} + q\vec{A}, \qquad \vec{p} = \mu V \dot{\vec{r}}, \tag{7.9}$$

which is only part of the momentum canonically conjugate to $\vec{r}$. The remaining equations of motion are then found to be (see below for a detailed computation)

$$\dot{\vec{p}} = -\frac{\mu}{2} \frac{\vec{r}}{r^3} \left(\dot{\vec{r}} \cdot \dot{\vec{r}} - \left(\frac{q}{\mu} \right)^2 \right) - q \frac{\dot{\vec{r}} \times \vec{r}}{r^3} \tag{7.10}$$

Two conserved quantities are the angular momentum $\vec{J}$and a Runge-Lenz type vector $\vec{K}$ (again, see below for more details):

$$\vec{J} = \vec{r} \times \vec{p} + q\hat{r}, \qquad \vec{K} = \vec{p} \times \vec{J} - \left(\mu E - q^2 \right) \hat{r}. \tag{7.11}$$

Since $\vec{r} \times \vec{p}$ and $\hat{r}$ are orthogonal, the magnitude of the orbital angular momentum, $l = |\vec{r} \times \vec{p}| = \sqrt{J^2 - q^2}$, is also conserved.

At the end of this section, we show that the conservation laws imply that all classical orbits are unbounded.

Derivation of equation (7.10)

We first compute the derivatives of V:

$$\partial_r V = -\frac{1}{r^2} \tag{7.12}$$

$$\dot{V} = -\frac{1}{r^2} \dot{r} \tag{7.13}$$

$$\partial_r (V^{-1}) = -V^{-2} \partial_r V = \frac{1}{r^2} V^{-2} \tag{7.14}$$

We now compute the Euler-Lagrange equations corresponding to Lagrangian (7.6).

$$\begin{aligned}
\frac{\partial L}{\partial r} &= -\frac{\pi}{r^2}\left(\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)\right) \\
&\quad + 2\pi r V(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \\
&\quad + \frac{\pi}{r^2}V^{-2}\left(\dot{\psi} + \cos\theta\dot{\phi}\right)^2 \\
\partial_t\frac{\partial L}{\partial \dot{r}} &= \partial_t(2\pi V\dot{r}) \\
\text{therefore:} \quad \partial_t(V\dot{r}) &= rV(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) - \frac{1}{2r^2}\left(\dot{\vec{r}}\cdot\dot{\vec{r}} - \left(\frac{q}{2\pi}\right)^2\right) \qquad (7.15)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial \theta} &= 2\pi r^2 V\sin\theta\cos\theta\dot{\phi}^2 \\
&\quad - \sin\theta\dot{\phi}\ 2\pi V^{-1}\left(\dot{\psi} + \cos\theta\dot{\phi}\right) \\
\partial_t\frac{\partial L}{\partial \dot{\theta}} &= \partial_t(2\pi r^2 V\dot{\theta}) \\
&= \partial_t(r\cdot 2\pi r V\dot{\theta}) = \dot{r}\left(2\pi r V\dot{\theta}\right) + r\partial_t\left(2\pi r V\dot{\theta}\right) \\
\text{so that:} \quad \partial_t(Vr\dot{\theta}) &= V(r\sin\theta\cos\theta\dot{\phi}^2 - \dot{r}\dot{\theta}) - \frac{\sin\theta\dot{\phi}}{2\pi r}q \qquad (7.16)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial \phi} &= 0 \\
\partial_t\frac{\partial L}{\partial \dot{\phi}} &= \partial_t(2\pi r^2\sin^2\theta V\dot{\phi}) + \partial_t\left(2\pi V^{-1}(\dot{\psi} + \cos\theta\dot{\phi})\cos\theta\right) \\
&= \partial_t(r\sin\theta)(2\pi r\sin\theta V\dot{\phi}) + r\sin\theta\partial_t(2\pi r\sin\theta V\dot{\phi}) \\
&\quad + \partial_t(q\cos\theta) \\
\text{and:} \quad \partial_t(r\sin\theta V\dot{\phi}) &= -V\dot{\phi}(\dot{r}\sin\theta + r\cos\theta\dot{\theta}) + \frac{1}{2\pi r}\dot{\theta}q \qquad (7.17)
\end{aligned}$$

We will need the following identities.

$$\vec{r} \;=\; r\hat{r} = \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} \tag{7.18}$$

$$\dot{\vec{r}} \;=\; \dot{r}\hat{r} + r(\dot{\theta}\hat{\theta} + \sin\theta\dot{\phi}\hat{\phi}) = \begin{pmatrix} \dot{r} \\ r\dot{\theta} \\ r\sin\theta\dot{\phi} \end{pmatrix} \tag{7.19}$$

$$\begin{aligned}
\ddot{\vec{r}} \;&=\; \partial_t(\dot{r})\hat{r} + \dot{r}(\dot{\theta}\hat{\theta} + \sin\theta\dot{\phi}\hat{\phi}) \\
&\quad + \partial_t(r\dot{\theta})\hat{\theta} + r\dot{\theta}(-\dot{\theta}\hat{r} + \dot{\phi}\cos\theta\hat{\phi}) \\
&\quad + \partial_t(r\sin\theta\dot{\phi})\hat{\phi} + (r\sin\theta\dot{\phi})(-\dot{\phi}\sin\theta\hat{r} - \dot{\phi}\cos\theta\hat{\theta}) \\
&=\; \begin{pmatrix} \partial_t(\dot{r}) & - & (r\dot{\theta}^2 + r\sin^2\theta\dot{\phi}^2) \\ \partial_t(r\dot{\theta}) & + & \dot{r}\dot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2 \\ \partial_t(r\sin\theta\dot{\phi}) & + & \dot{r}\sin\theta\dot{\phi} + r\cos\theta\dot{\theta}\dot{\phi} \end{pmatrix}
\end{aligned} \tag{7.20}$$

and

$$\dot{\vec{r}} \times \vec{r} \;=\; r^2 \begin{pmatrix} 0 \\ \sin\theta\dot{\phi} \\ -\dot{\theta} \end{pmatrix} \tag{7.21}$$

$$\vec{r} \times (\dot{\vec{r}} \times \vec{r}) \;=\; r^3 \begin{pmatrix} 0 \\ \dot{\theta} \\ \sin\theta\dot{\phi} \end{pmatrix} = r^3\dot{\hat{r}} \tag{7.22}$$

Now we can calculate $\dot{\vec{p}}$.

$$\vec{p} = 2\pi V \dot{\vec{r}} = 2\pi V \begin{pmatrix} \dot{r} \\ r\dot{\theta} \\ r\sin\theta\dot{\phi} \end{pmatrix} \tag{7.23}$$

$$\dot{\vec{p}} = 2\pi \left(\dot{V}\dot{\vec{r}} + V\ddot{\vec{r}} \right)$$

$$= 2\pi \begin{pmatrix} \partial_t(V\dot{r}) & - & V\left(r\dot{\theta}^2 + r\sin^2\theta\dot{\phi}^2\right) \\ \partial_t(Vr\dot{\theta}) & + & V\left(\dot{r}\dot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2\right) \\ \partial_t(Vr\sin\theta\dot{\phi}) & + & V\left(\dot{r}\sin\theta\dot{\phi} + r\cos\theta\dot{\theta}\dot{\phi}\right) \end{pmatrix} \tag{7.24}$$

Inserting the Euler-Lagrange equations, we find:

$$\dot{\vec{p}} = \begin{pmatrix} -\frac{\pi}{r^2}\left(\dot{\vec{r}}\cdot\dot{\vec{r}} - \left(\frac{q}{2\pi}\right)^2\right) \\ -\frac{1}{r}\sin\theta\dot{\phi}q \\ \frac{1}{r}\dot{\theta}q \end{pmatrix}$$

$$= -\pi\frac{\vec{r}}{r^3}\left(\dot{\vec{r}}\cdot\dot{\vec{r}} - \left(\frac{q}{2\pi}\right)^2\right) - q\frac{\dot{\vec{r}}\times\vec{r}}{r^3} \tag{7.25}$$

Derivation of equations (7.11)

The angular momentum $\vec{J} = \vec{r}\times\vec{p} + q\hat{r}$ is conserved:

$$\begin{aligned} \partial_t\vec{J} &= \partial_t(\vec{r}\times\vec{p} + q\hat{r}) \\ &= \vec{r}\times\dot{\vec{p}} + q\dot{\hat{r}} \\ &= \vec{r}\times\left(-\frac{q}{r^3}(\dot{\vec{r}}\times\vec{r})\right) + q\dot{\hat{r}} \\ &= 0 \end{aligned} \tag{7.26}$$

and the Runge-Lenz type vector $\vec{K} = \vec{p} \times \vec{J} - (2\pi E - q^2)\,\hat{r}$ is also conserved:

$$
\begin{aligned}
2\pi E - q^2 &= 2\pi^2 \left(1 + \frac{1}{r}\right) \left(\dot{\vec{r}} \cdot \dot{\vec{r}} + \left(\frac{q}{2\pi}\right)^2\right) - q^2 \\
&= 2\pi^2 \left(1 + \frac{1}{r}\right) \dot{\vec{r}} \cdot \dot{\vec{r}} + 2\pi^2 \left(\frac{q}{2\pi}\right)^2 + \frac{1}{r} 2\pi^2 \left(\frac{q}{2\pi}\right)^2 - q^2 \\
&= 2\pi^2 \left(1 + \frac{1}{r}\right) \dot{\vec{r}} \cdot \dot{\vec{r}} - \frac{q^2}{2} + \frac{q^2}{2r} \qquad (7.27)
\end{aligned}
$$

while

$$
\begin{aligned}
\partial_t(\vec{p} \times \vec{J}) &= \dot{\vec{p}} \times \vec{J} \\
&= -\frac{\pi}{r^3} \left(\dot{\vec{r}} \cdot \dot{\vec{r}} - \left(\frac{q}{2\pi}\right)^2\right) \vec{r} \times (\vec{r} \times \vec{p}) - \frac{q}{r^3} (\dot{\vec{r}} \times \vec{r}) \times q\hat{r} \\
&= \left[-2\pi^2 \left(1 + \frac{1}{r}\right) \left(\dot{\vec{r}} \cdot \dot{\vec{r}} - \left(\frac{q}{2\pi}\right)^2\right) - \frac{q^2}{r}\right] \frac{\vec{r} \times (\vec{r} \times \dot{\vec{r}})}{r^3} \\
&= -\left[-2\pi^2 \left(1 + \frac{1}{r}\right) (\dot{\vec{r}} \cdot \dot{\vec{r}}) + \frac{q^2}{2} - \frac{q^2}{2r}\right] \dot{\hat{r}} \\
&= \left[2\pi^2 \left(1 + \frac{1}{r}\right) (\dot{\vec{r}} \cdot \dot{\vec{r}}) - \frac{q^2}{2} + \frac{q^2}{2r}\right] \dot{\hat{r}} \\
&= \left[2\pi E - q^2\right] \dot{\hat{r}} \qquad (7.28)
\end{aligned}
$$

and therefore

$$
\dot{\vec{K}} \;=\; \partial_t(\vec{p} \times \vec{J}) - \left[2\pi E - q^2\right] \dot{\hat{r}} \;=\; 0 \qquad (7.29)
$$

Classical Orbits

We now show that there are no classical bound orbits. These calculations were done in collaboration with my supervisor.

In order to determine the orbits we need to distinguish the cases $q = 0$ and $q \neq 0$. In the former case we find

$$
\vec{J} \cdot \vec{r} \;=\; 0, \qquad\qquad \vec{K} \cdot \vec{r} \;=\; J^2 - \mu E r, \qquad (7.30)
$$

as well as $\vec{J} \cdot \vec{K} = 0$. Thus the motion takes place in the plane orthogonal to $\vec{J}$, and the vector $\vec{K}$ is contained in that plane. In terms of polar coordinates (r, φ) in that plane, with $\varphi = 0$ corresponding to the direction of $\vec{K}$, the second equation in (7.30) becomes

$$r = \frac{J^2}{K\cos\varphi + \mu E}. \tag{7.31}$$

Since $K = |\vec{K}| = \sqrt{2\mu E J^2 + \mu^2 E^2} > \mu E$, this is the equation for a hyperbola.

For $q \neq 0$ the orbits are determined by the simultaneous equations

$$\vec{J}\cdot\hat{r} \;=\; q, \qquad\qquad \vec{N}\cdot\vec{r} \;=\; J^2 - q^2, \tag{7.32}$$

where $\vec{N} = \vec{K} + \frac{1}{q}(\mu E - q^2)\vec{J}$. The first of these is the equation of a cone with axis $\vec{J}$ and opening angle 2α determined by $J\cos\alpha = q$. The second is the equation of a plane orthogonal to the vector $\vec{N}$. Hence the orbits are conic sections, but to determine which kinds of conic sections occur we need to find the angle between the vectors $\vec{N}$ and $\vec{J}$. A lengthy calculation shows that

$$N = |\vec{N}| \;=\; \frac{l\mu E}{|q|}, \qquad\qquad \vec{N}\cdot\vec{J} \;=\; \frac{l^2(\mu E - q^2)}{q}. \tag{7.33}$$

Let us assume for simplicity that $q > 0$ from now on, so that the vector $\vec{J}$ is inside the cone, and that $\alpha \in [0, \frac{\pi}{2})$; the case $q < 0$ can be dealt with analogously, but using $-\vec{J}$ instead of $\vec{J}$ (the second equation in (7.33) is invariant under simultaneous sign change of q and $\vec{J}$). It then follows from (7.33) that the angle β between $\vec{N}$ and $\vec{J}$ satisfies

$$\cos\beta = \frac{\mu E - q^2}{\mu E}\frac{l}{J}. \tag{7.34}$$

On the other hand we can use $q^2 + l^2 = J^2$ to see that the complementary angle to half of the opening angle of the cone satisfies

$$\cos(\tfrac{\pi}{2} - \alpha) = \frac{l}{J}. \tag{7.35}$$

Hence $\cos\beta < \cos(\frac{\pi}{2} - \alpha)$ or $\beta > (\frac{\pi}{2} - \alpha)$. We conclude that the intersection of the plane and the cone is always hyperbolic and that all orbits are unbounded.

7.2 Quantum mechanics

The Hamiltonian for the charge-$(1,1)$ monopole system is given by half the Laplacian on the total moduli space $\mathcal{M}_{1,1}$, plus the total mass. Separating the center of mass and relative motion variables, using the product structure of the moduli space, the Schrödinger equation becomes

$$i\hbar\partial_t\Psi \;=\; \frac{\hbar^2}{2}\Delta_{\mathcal{M}_{1,1}}\Psi + \mathrm{M}\Psi \;=\; \frac{\hbar^2}{2}\left(\Delta_{\mathbb{R}^3\times\mathbb{R}} + \Delta_{M_{TN}}\right)\Psi + \mathrm{M}\Psi. \tag{7.36}$$

Now we separate variables by assuming

$$\Psi(t,\vec{R},\chi,\vec{r},\psi) \;=\; f(t)F(\vec{R},\chi)\Phi(\vec{r},\psi), \tag{7.37}$$

where $F(\vec{R},\chi)$ is a wavefunction corresponding to the centre of mass motion, and $\Phi(\vec{r},\psi)$ is a wavefunction corresponding to the relative motion. For stationary states, the Schrödinger equation reduces to the following Schrödinger equation on the relative moduli space:

$$\frac{\hbar^2}{2}\Delta_{M_{TN}}\Phi \;=\; E\Phi, \tag{7.38}$$

where E is the energy of the relative motion. The total energy is given by the sum of the total mass, the energy of the centre of mass motion and the energy of the relative motion, $\mathrm{E} = \mathrm{M} + \frac{1}{2\mathrm{M}}\left(|\vec{P}|^2 + Q^2\right) + E$, where $\vec{P}$ is the total momentum and Q is a kind of "centre of mass" electric charge. Since the constituent monopoles are charged with respect to different $U(1)$ groups, the charge Q is a linear combination of different kinds of electric charges [5, 6]; in particular it is not necessarily an integer but obeys a quantisation condition that follows from (7.5). From now onwards we shall assume that for plane wave solutions of the centre of mass wavefunction $F(\vec{R},\chi)$ analogous to (6.10) the value of Q is such that (7.5) holds. This does not affect the relative motion, to which we now turn.

The Laplacian on M_{TN} can be computed from the metric on the Taub-NUT manifold, given in equation (7.2),

$$\Delta_{M_{TN}}f \;=\; -\frac{1}{\mu}\left[\frac{1}{r^2V}\partial_r\left[r^2\partial_r f\right] + \frac{1}{r^2V}\left[\xi_1^2+\xi_2^2\right]f + V\xi_3^2 f\right]. \tag{7.39}$$

Here ξ_i are the vector fields dual to η_i, $\eta_i(\xi_j) = \delta_{ij}$:

$$\xi_1 \;=\; -\frac{\cos\theta}{\sin\theta}\cos\psi\frac{\partial}{\partial\psi} - \sin\psi\frac{\partial}{\partial\theta} + \frac{\cos\psi}{\sin\theta}\frac{\partial}{\partial\phi}, \tag{7.40a}$$

$$\xi_2 \;=\; -\frac{\cos\theta}{\sin\theta}\sin\psi\frac{\partial}{\partial\psi} + \cos\psi\frac{\partial}{\partial\theta} + \frac{\sin\psi}{\sin\theta}\frac{\partial}{\partial\phi}, \tag{7.40b}$$

$$\xi_3 \;=\; \frac{\partial}{\partial\psi}. \tag{7.40c}$$

Writing $\epsilon = \frac{2\mu E}{\hbar^2}$, and multiplying with V, the Schrödinger equation (7.38) becomes

$$\frac{1}{r^2}\partial_r \left[r^2 \partial_r \Phi\right] + \frac{1}{r^2}\left[\xi_1^2 + \xi_2^2 + \xi_3^2\right]\Phi + \left(1 + \frac{2}{r}\right)\xi_3^2\Phi + \epsilon\left(1 + \frac{1}{r}\right)\Phi \;=\; 0. \qquad (7.41)$$

7.2.1 There are no bound states for the bosonic monopole

We expand Φ in terms of the Wigner functions $D^j_{sm}(\theta,\phi,\psi) = e^{im\phi}d^j_{sm}(\theta)e^{is\psi}$ on $SU(2)$ with indices $j, s, m \in \frac{1}{2}\mathbb{Z}$ and $j \geq 0$. They are eigenfunction of ξ_3 and $(\xi_1^2+\xi_2^2+\xi_3^2)$ [42]:

$$\xi_3 D^j_{sm} \;=\; isD^j_{sm}, \qquad (\xi_1^2 + \xi_2^2 + \xi_3^2)D^j_{sm} \;=\; -j(j+1)D^j_{sm}. \qquad (7.42)$$

For $\Phi = \frac{h(r)}{r}D^j_{sm}(\theta,\phi,\psi)$ the Schrödinger equation (7.41) reduces to

$$\frac{1}{r}\left[{\partial_r}^2 - \frac{j(j+1)}{r^2} + (\varepsilon - s^2) + \frac{1}{r}(\varepsilon - 2s^2)\right]h(r) \;=\; 0. \qquad (7.43)$$

To solve this equation, we use the Ansatz $h(r) = r^{j+1}e^{ikr}F$, and compute its derivatives.

$$\begin{aligned}
\partial_r(h(r)) \;&=\; (j+1)r^j e^{ikr}F + r^{j+1}e^{ikr}\left(ik + \partial_r\right)F \\
{\partial_r}^2(h(r)) \;&=\; j(j+1)r^{j-1}e^{ikr}F \\
&\quad + 2(j+1)r^j e^{ikr}\left(ik + \partial_r\right)F \\
&\quad + r^{j+1}e^{ikr}\left(-k^2 + 2ik\partial_r + {\partial_r}^2\right)F \\
&=\; \left(\frac{j(j+1)}{r^2} + \frac{2(j+1)ik}{r} - k^2\right)r^{j+1}e^{ikr}F \\
&\quad + \left(\frac{2(j+1)}{r} + 2ik\right)r^{j+1}e^{ikr}\partial_r F \\
&\quad + r^{j+1}e^{ikr}{\partial_r}^2 F
\end{aligned}$$

We now set

$$k^2 \;=\; \left(\epsilon - s^2\right), \qquad (7.44)$$

so that the Schrödinger equation becomes

$$0 \;=\; \left[{\partial_r}^2 + \left(\frac{2(j+1)}{r} + 2ik\right)\partial_r + \frac{2(j+1)}{r}ik + \frac{1}{r}\left(\epsilon - 2s^2\right)\right]F. \qquad (7.45)$$

Multiplying by $\frac{r}{-2ik}$ and substituting $x = -2ikr$ we find

$$\begin{aligned} 0 &= \left[\frac{r}{-2ik}{\partial_r}^2 + \frac{1}{-2ik}\left(2(j+1) + 2ikr\right)\partial_r - \left((j+1) + \frac{1}{2ik}(\epsilon - 2s^2)\right)\right] F \\ &= \left[x{\partial_x}^2 + \left(2(j+1) - x\right)\partial_x - \left((j+1) + \frac{1}{2ik}(\epsilon - 2s^2)\right)\right] F, \end{aligned} \tag{7.46}$$

so F is a confluent hypergeometric function. We define

$$\lambda = -\frac{1}{2k}(\epsilon - 2s^2) = -\frac{1}{2k}(k^2 - s^2) \tag{7.47}$$

so that the solution for F is

$$F = F\left((j+1) + i\lambda, 2(j+1), -2ikr\right). \tag{7.48}$$

The solution of $h(r)$ is therefore

$$h(r) = r^{j+1}e^{ikr}F\left((j+1) + i\lambda, 2(j+1), -2ikr\right). \tag{7.49}$$

Bound states correspond to square integrable solutions of $h(r)$. For these, the exponential term must vanish for large r and the series expansion of F must terminate. For the exponential term to vanish, k must be i times a positive real number ($ik < 0$), so

$$\epsilon < s^2. \tag{7.50}$$

The expansion of $F(a, b, u)$ is

$$F(a, b, u) = 1 + \frac{a}{b}u + \frac{a(a+1)}{b(b+1)}\frac{u^2}{2!} + \ldots, \tag{7.51}$$

which terminates if a is a non-positive integer. In this case $a = (j+1) + i\lambda$, so we require

$$-i\lambda = n, \qquad n = j+1, j+2, \ldots. \tag{7.52}$$

Since $-ik > 0$ and $j \geq 0$, the only solutions occur when

$$\epsilon - 2s^2 > 0, \tag{7.53}$$

which contradicts equation (7.50). Hence there are no quantum mechanical bound states in this system, reflecting the absence of bound orbits in the corresponding classical system.

7.2.2 Scattering states

To find the scattering states we follow Gibbons and Manton's approach [4]. We first write the Schrödinger equation (7.41) as

$$\begin{aligned} 0 &= \frac{1}{r^2}\partial_r\left[r^2\partial_r\Phi\right] + \frac{1}{r^2}\nabla^2_{S^3}\Phi + \left(1+\frac{2}{r}\right)\xi_3^2\Phi + \epsilon\left(1+\frac{1}{r}\right)\Phi \\ &= \frac{1}{r^2}\partial_r\left[r^2\partial_r\Phi\right] + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta\Phi) + \frac{1}{\sin^2\theta}({\partial_\phi}^2 + {\partial_\psi}^2 - 2\cos\theta\partial_\phi\partial_\psi)\Phi\right] \\ &\quad + \left(\epsilon+\xi_3^2\right)\Phi + \frac{1}{r}\left(\epsilon + 2\xi_3^2\right)\Phi. \end{aligned} \tag{7.54}$$

We change from polar coordinates $\{r,\theta,\phi,\psi\}$ to cylindrical coordinates $\{r,z,\phi,\psi\}$, and we introduce parabolic coordinates

$$\xi = r+z, \qquad \eta = r-z, \tag{7.55}$$

so that

$$\begin{aligned} &\frac{1}{r^2}\partial_r(r^2\partial_r\Phi) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta\Phi) \\ &= \frac{1}{r}\left[\partial_r(r\partial_r\Phi) + \partial_r\Phi + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta\partial_\theta\Phi)\right] \\ &= \frac{1}{r}\left[\partial_r(r\partial_r\Phi) + \partial_r(z\partial_z\Phi) + \partial_z(z\partial_r\Phi) + \partial_z(r\partial_z\Phi)\right] \\ &= \frac{4}{\xi+\eta}\left[\partial_\xi(\xi\partial_\xi\Phi) + \partial_\eta(\eta\partial_\eta\Phi)\right] \end{aligned} \tag{7.56}$$

We now use the Ansatz

$$\Phi = e^{im\phi}e^{is\psi}\Lambda(\xi,\eta), \tag{7.57}$$

with $s \in \frac{1}{2}\mathbb{Z}$ in order to respect the range of ψ. We find, substituting $\cos\theta = \frac{\xi-\eta}{\xi+\eta}$ and $r^2\sin^2\theta = \xi\eta$, that the Schrödinger equation (7.41) reduces to

$$\begin{aligned} 0 &= \frac{4}{\xi+\eta}\left[\partial_\xi(\xi\partial_\xi\Lambda) + \partial_\eta(\eta\partial_\eta\Lambda)\right] - \frac{1}{\xi\eta}\left(m^2 + s^2 - 2ms\frac{\xi-\eta}{\xi+\eta}\right)\Lambda \\ &\quad + \left(\epsilon - s^2\right)\Lambda + \frac{2}{\xi+\eta}\left(\epsilon - 2s^2\right)\Lambda. \end{aligned} \tag{7.58}$$

To solve this equation, we separate variables as follows,

$$\Lambda(\xi,\eta) \;=\; f(\xi)g(\eta), \tag{7.59}$$

and we find that f and g must satisfy

$$\frac{4}{f}\partial_\xi(\xi\partial_\xi f) - \frac{1}{\xi}(m+s)^2 + k^2\xi + 2\left(2\epsilon - s^2\right) \;=\; C, \tag{7.60}$$

$$-\frac{4}{g}\partial_\eta(\eta\partial_\eta g) + \frac{1}{\eta}(m-s)^2 - k^2\eta \;=\; C. \tag{7.61}$$

We use the following trial solutions:

$$f(\xi) \;=\; \xi^{\frac{1}{2}|m+s|}e^{-ik\xi/2}F_1(\xi), \qquad g(\eta) \;=\; \eta^{\frac{1}{2}|m-s|}e^{-ik\eta/2}F_2(\eta), \tag{7.62}$$

and find

$$\begin{aligned}
\partial_\xi f \;&=\; \frac{1}{2}|m+s|\xi^{\frac{1}{2}|m+s|-1}e^{-ik\xi/2}F_1(\xi) + \xi^{\frac{1}{2}|m+s|}e^{-ik\xi/2}\left(-\frac{1}{2}ik + \partial_\xi\right)F_1(\xi), \\
\xi\partial_\xi f \;&=\; \frac{1}{2}|m+s|\xi^{\frac{1}{2}|m+s|}e^{-ik\xi/2}F_1(\xi) + \xi^{\frac{1}{2}|m+s|+1}e^{-ik\xi/2}\left(-\frac{1}{2}ik + \partial_\xi\right)F_1(\xi), \\
\partial_\xi\xi\partial_\xi f \;&=\; \left(\frac{1}{2}|m+s|\right)^2\xi^{\frac{1}{2}|m+s|-1}e^{-ik\xi/2}F_1(\xi) \\
&\quad + \left((\frac{1}{2}|m+s|) + (\frac{1}{2}|m+s|+1)\right)\xi^{\frac{1}{2}|m+s|}e^{-ik\xi/2}\left(-\frac{1}{2}ik + \partial_\xi\right)F_1(\xi) \\
&\quad + \xi^{\frac{1}{2}|m+s|+1}e^{-ik\xi/2}\left(-\frac{1}{2}ik + \partial_\xi\right)^2 F_1(\xi) \\
&=\; \xi^{\frac{1}{2}|m+s|}e^{-ik\xi/2}\left\{\frac{1}{4}(m+s)^2\xi^{-1}\right. \\
&\qquad + (|m+s|+1)\left(-\frac{1}{2}ik + \partial_\xi\right) \\
&\qquad \left. + \xi\left(-\frac{k^2}{4} - ik\partial_\xi + {\partial_\xi}^2\right)\right\}F_1(\xi) \\
&=\; \xi^{\frac{1}{2}|m+s|}e^{-ik\xi/2}\Big\{\xi{\partial_\xi}^2 + \left(|m+s|+1-ik\xi\right)\partial_\xi \\
&\qquad - \frac{k^2}{4}\xi - \frac{1}{2}ik\left(|m+s|+1\right) + \frac{1}{4}(m+s)^2\xi^{-1}\Big\}F_1(\xi).
\end{aligned}$$

Similarly

$$\partial_\eta \eta \partial_\eta g \;=\; \eta^{\frac{1}{2}|m-s|} e^{-ik\eta/2} \Big\{ \eta {\partial_\eta}^2 + (|m-s|+1-ik\eta)\,\partial_\eta$$
$$- \frac{k^2}{4}\eta - \frac{1}{2} ik\,(|m-s|+1) + \frac{1}{4}(m-s)^2\eta^{-1} \Big\} F_2(\eta).$$

From equations (7.60) and (7.61) we find then that F_1 and F_2 have to satisfy

$$\frac{4}{F_1}\left\{ \xi {\partial_\xi}^2 + (|m+s|+1-ik\xi)\,\partial_\xi - \frac{1}{2}ik\,(|m+s|+1) \right\} F_1(\xi) + 2\left(\epsilon - 2s^2\right) - C \;=$$
$$\frac{4}{F_1}\left\{ \xi {\partial_\xi}^2 + (|m+s|+1-ik\xi)\,\partial_\xi - \frac{1}{2}ik\,(|m+s|+1) + \frac{1}{2}\left(\epsilon - 2s^2\right) - \frac{C}{4} \right\} F_1(\xi) \;=\; 0 \tag{7.63}$$

and

$$-\frac{4}{F_2}\left\{ \eta {\partial_\eta}^2 + (|m-s|+1-ik\eta)\,\partial_\eta - \frac{1}{2}ik\,(|m-s|+1) \right\} F_2(\eta) - C \;=$$
$$-\frac{4}{F_2}\left\{ \eta {\partial_\eta}^2 + (|m-s|+1-ik\eta)\,\partial_\eta - \frac{1}{2}ik\,(|m-s|+1) + \frac{C}{4} \right\} F_2(\eta) \;=\; 0. \tag{7.64}$$

Introducing

$$x \;=\; ik\xi, \qquad\qquad y \;=\; ik\eta, \tag{7.65}$$

and dividing by $\frac{4}{F_1}ik$ and $-\frac{4}{F_2}ik$ respectively, we find

$$\left\{ x{\partial_x}^2 + (|m+s|+1-x)\,\partial_x - \frac{1}{2}\,(|m+s|+1) + \frac{1}{2ik}\left(\epsilon - 2s^2\right) - \frac{C}{4ik} \right\} F_1 \;=\; 0 \tag{7.66}$$

and

$$\left\{ y{\partial_y}^2 + (|m-s|+1-y)\,\partial_y - \frac{1}{2}\,(|m-s|+1) + \frac{C}{4ik} \right\} F_2 \;=\; 0 \tag{7.67}$$

and we see that F_1 and F_2 are again confluent hypergeometric functions:

$$F_1 \;=\; F(c_1, |m+s|+1, ik\xi), \qquad F_2 \;=\; F(c_2, |m-s|+1, ik\eta), \tag{7.68}$$

where c_1 and c_2 are constants that must satisfy

$$c_1 + c_2 \;=\; 1 + \frac{1}{2}|m+s| + \frac{1}{2}|m-s| - i\lambda. \tag{7.69}$$

We can use the remaining freedom to specify the scattering situation we want to describe. The constituent monopoles are distinguishable particles (with different magnetic charges and, in general, different masses) so we can label them 1 and 2. Then we can assume without loss of generality that $\vec{r}$ is the position vector of monopole 1 relative to monopole 2, and ψ the phase of monopole 1 relative to that of monopole 2 (see [5, 6] for details), so that $s = \frac{1}{2}\times$(electric charge of monopole 1 minus electric charge of monopole 2). We would like to consider scattering where monopole 1 comes in along the negative z-axis, and monopole 2 along the positive z axis. This can be achieved by setting $m = -s$ and $c_1 = 1$:

$$\Phi \;=\; e^{is(\psi-\phi)}\,(r-z)^{|s|}\,e^{+ikz}\,F(|s|-i\lambda, 2|s|+1, ik(r-z)), \tag{7.70}$$

where we have used the fact that $F(1,1,x) = e^x$. To compute the scattering cross section, we will need to find the asymptotic form of Φ. For large $|x|$,

$$F(a,b,x) \approx \qquad \frac{\Gamma(b)}{\Gamma(b-a)}\frac{1}{(-x)^a}\left\{1 - \frac{a(a-b+1)}{x} + \frac{\Gamma(b-a)}{\Gamma(a)}\frac{(-1)^a e^x}{x^{b-2a}}\right\} \tag{7.71}$$

so that we find

$$\Phi \;\approx\; e^{is(\psi-\phi)}K\left\{\left(1+\frac{(s^2+\lambda^2)}{2ikr\sin^2\left(\frac{\theta}{2}\right)}\right)e^{i(kz+\lambda\log(k(r-z)))}+\right.$$

$$\left.\frac{(|s|-i\lambda)}{2ikr\sin^2\left(\frac{\theta}{2}\right)}e^{i(\tau+\pi|s|)}e^{i(kr-\lambda\log(k(r-z)))}\right\}, \tag{7.72}$$

where

$$K \;=\; \frac{\Gamma(2|s|+1)}{\Gamma(|s|+1+i\lambda)}\frac{1}{(-i)^{|s|-i\lambda}k^{|s|}}, \qquad \tau \;=\; \arg\frac{\Gamma(|s|+1+i\lambda)}{\Gamma(|s|+1-i\lambda)}. \tag{7.73}$$

The differential cross section is thus the same as found by Gibbons and Manton [4] in the Taub-NUT approximation to dyon scattering in $SU(2)$ Yang-Mills-Higgs theory:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \;=\; \frac{1}{4}\left(1+\frac{s^2}{4k^2}\right)^2\sin^{-4}\left(\frac{\theta}{2}\right). \tag{7.74}$$

It is interesting that a very similar cross section is found for the scattering of a charged particle off a BPS monopole [43] when exponential terms in the fields are neglected.

In the case where the relative electric charge s vanishes, which includes the case of pure monopole scattering, one obtains the purely geometric (energy independent) expression

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{4}\sin^{-4}\left(\frac{\theta}{2}\right). \tag{7.75}$$

It was shown by Schroers [8] that the symmetrised version of this formula is also a very good approximation to the differential cross section for pure monopole scattering in the $SU(2)$ theory: the s-wave phase shift correction to the Taub-NUT approximation is zero for all energies in that case. Remarkably, two identical $SU(2)$ monopoles and the two distinct $SU(3)$ monopoles that make up the charge-$(1,1)$ configuration thus have the same scattering behaviour in the quantum theory at low energies, apart from symmetrisation effects.

7.3 $N = 2$ supersymmetric monopoles

As before, we use the product structure of the moduli space to separate centre of mass and relative motion variables. This procedure is most transparent using the quantisation in terms of forms on the moduli space, although the equivalence between anti-holomorphic forms and spinors assures us that it can be done for the latter as well.

Anti-holomorphic forms on the total moduli space can be written as wedge products of anti-holomorphic forms on the centre of mass and relative moduli spaces. The supercharges, the (twisted) Dolbeault operators and their adjoints, decompose into a sum of (twisted) Dolbeault operators on the centre of mass and relative moduli spaces. The Laplacian is then seen to decompose into the sum of centre of mass and relative moduli space components as well. For an anti-holomorphic form $\overline{v} = \overline{v}_1 \wedge \overline{v}_2$, where $\overline{v}_1$ and $\overline{v}_2$ are anti-holomorphic forms on $\mathbb{R}^3 \times \mathbb{R}$ and M_{TN} respectively,

$$\overline{\partial}_{M_{1,1}}\overline{v} = (\overline{\partial}_{\mathbb{R}^3\times\mathbb{R}}\overline{v}_1) \wedge \overline{v}_2 + (-1)^{\deg(\overline{v}_1)}\overline{v}_1 \wedge (\overline{\partial}_{M_{TN}}\overline{v}_2), \tag{7.76}$$

$$\Delta_{M_{1,1}}\overline{v} = (\Delta_{\mathbb{R}^3\times\mathbb{R}}\overline{v}_1) \wedge \overline{v}_2 + \overline{v}_1 \wedge (\Delta_{M_{TN}}\overline{v}_2). \tag{7.77}$$

See appendix A.3 for more details.

The centre of mass dynamics are equivalent to the charge-1 monopole dynamics, and again we refer back to the previous chapter. Focussing on the moduli space for the relative motion of the monopoles, we can as before generate multiplets of states, starting with a wavefunction Φ on M_{TN}, and applying the (twisted) Dolbeault operators on the Taub-NUT manifold. When Φ is an eigenstate of the Laplacian $\Delta_{M_{TN}}$ we obtain, in general, four independent states. By taking the wedge product of these states with a multiplet of four centre of mass states, we obtain a multiplet of 16 states with the same energy.

BPS states in the original field correspond to the short multiplet of 4 states on the total moduli space. Therefore, these would correspond to a normalisable harmonic form on the relative moduli space, which does not generate a multiplet of 4 independent states on the moduli space. However, the only normalisable harmonic form on the Taub-NUT manifold is of degree $(1,1)$ [5, 6]. It is therefore not an anti-holomorphic state corresponding to any fermionic or bosonic state of the system. The $N=2$ supersymmetric charge-$(1,1)$ monopole system therefore has no BPS states in the moduli space approximation.

7.3.1 Quantisation using spinors

Having described how we may separate the centre of mass motion and relative motion in the previous section, we now focus our attention on the relative moduli space. We construct the Dirac operator and Hamiltonian acting on spinors on M_{TN}, which we will compare to the corresponding operators acting on forms in section 7.3.2, and show that they are equivalent.

The Taub-NUT metric can be rewritten as

$$\mathrm{d}s^2 \;=\; \mu\Big[V(\mathrm{d}\vec{r})^2 + V^{-1}\left(\mathrm{d}\psi + \vec{A}\cdot\mathrm{d}\vec{r}\right)^2\Big] \;=\; (e^1)^2+(e^2)^2+(e^3)^2+(e^4)^2, \quad (7.78)$$

where as before $\vec{A}\cdot\mathrm{d}\vec{r} = \cos\theta\mathrm{d}\phi$, and we have defined the vier-bein

$$e^i \;=\; e^i{}_{\underline{j}}\mathrm{d}x^{\underline{j}} \;=\; \sqrt{\mu V}\mathrm{d}x^i, \qquad e^4 \;=\; e^4{}_{\underline{j}}\mathrm{d}x^{\underline{j}} \;=\; \sqrt{\frac{\mu}{V}}\left(\mathrm{d}\psi + \vec{A}\cdot\mathrm{d}\vec{r}\right). \quad (7.79)$$

The Dirac operator on a general manifold is defined by

$$\not{D}_s \;=\; \hat{\gamma}^a\left(\partial_a + \Gamma_a\right), \quad (7.80)$$

where

$$\hat{\gamma}^a \;=\; (e^a{}_b)^{-1}\gamma^b, \qquad \Gamma_a \;=\; -\frac{1}{8}[\gamma^b,\gamma^c]\omega_{bc}(\partial_a), \quad (7.81)$$

and the spin connection ω corresponding to the viel-bein e is defined through

$$\mathrm{d}e^a + \omega^a{}_b \wedge e^b \;=\; 0, \qquad\qquad \omega_{ab} \;=\; -\omega_{ba}. \tag{7.82}$$

A solution to these equations for the Taub-NUT manifold is

$$\frac{1}{2}\varepsilon_{ijk}\omega^{ij} \;=\; \omega^4{}_k \;=\; -\left[\frac{1}{2V\sqrt{V}}(\partial_k V)e^4 - \frac{1}{2V\sqrt{V}}(\partial_k\omega^j - \partial_j\omega^k)e^j\right] \tag{7.83}$$

where

$$\omega^j \;=\; \frac{r+z}{z}A^j \tag{7.84}$$

which satisfies

$$\varepsilon_{ijk}\partial_j\omega^k \;=\; -\frac{1}{r^3}x^i \;=\; \partial_i V. \tag{7.85}$$

Furthermore, we have:

$$\frac{1}{2}[\gamma^i,\gamma^j] = \gamma^i\gamma^j \;=\; \begin{pmatrix} -i\varepsilon_{ijk}\sigma^k & 0 \\ 0 & -i\varepsilon_{ijk}\sigma^k \end{pmatrix} \tag{7.86}$$

$$\frac{1}{2}[\gamma^4,\gamma^j] = \gamma^4\gamma^j \;=\; \begin{pmatrix} -i\sigma^j & 0 \\ 0 & i\sigma^j \end{pmatrix} \tag{7.87}$$

and we find

$$\Gamma_\alpha \;=\; \begin{pmatrix} i\sigma^k\omega^4{}_k(\partial_\alpha) & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} i\varepsilon_{ijk}\sigma^k\omega^{ij}(\partial_\alpha) & 0 \\ 0 & 0 \end{pmatrix} \tag{7.88}$$

where

$$\omega^4{}_k(\partial_\alpha) \;=\; -\frac{1}{2V}\varepsilon_{ijk}(\partial_i V)\delta^j_\alpha - \frac{1}{2V^2}(\partial_k V)\left(\delta^4_\alpha + A^j\delta^j_\alpha\right) \tag{7.89}$$

We use the same representation for the Dirac γ-matrices as before (6.13), so that

$$\hat{\gamma}^j \;=\; \frac{1}{\sqrt{\mu}}\begin{pmatrix} 0 & \frac{1}{\sqrt{V}}\sigma^j \\ -\frac{1}{\sqrt{V}}\sigma^j & 0 \end{pmatrix}, \tag{7.90}$$

$$\hat{\gamma}^4 \;=\; \frac{1}{\sqrt{\mu}}\begin{pmatrix} 0 & i\sqrt{V} - \frac{1}{\sqrt{V}}\sigma^j A^j \\ i\sqrt{V} + \frac{1}{\sqrt{V}}\sigma^j A^j & 0 \end{pmatrix}. \tag{7.91}$$

Now we note that

$$-\frac{1}{V}\sigma^j\pi^j(\sqrt{V}\chi) = i\frac{1}{V}\sigma^j\left[(\partial_j\sqrt{V})\chi + \sqrt{V}(\partial_j\chi) - \sqrt{V}A^j\partial_\psi\chi\right]$$
$$= i\frac{1}{2V\sqrt{V}}\sigma^j(\partial_j V)\chi - \frac{1}{\sqrt{V}}\sigma^j\pi^j\chi \quad (7.92)$$

while

$$\hat{\gamma}^\mu\partial_\mu = \begin{pmatrix} 0 & \frac{1}{\sqrt{V}}\sigma^j\partial_j + (i\sqrt{V} - \frac{1}{\sqrt{V}}\sigma^j A^j)\partial_\psi \\ -\frac{1}{\sqrt{V}}\sigma^j\partial_j + (i\sqrt{V} + \frac{1}{\sqrt{V}}\sigma^j A^j)\partial_\psi & 0 \end{pmatrix}$$
$$= i\begin{pmatrix} 0 & \frac{1}{\sqrt{V}}\sigma^j\pi^j + \sqrt{V}\partial_\psi \\ -\frac{1}{\sqrt{V}}\sigma^j\pi^j + \sqrt{V}\partial_\psi & 0 \end{pmatrix} \quad (7.93)$$

and

$$\hat{\gamma}^j\Gamma_j = \begin{pmatrix} 0 & 0 \\ (-\frac{1}{\sqrt{V}}\sigma^j)\left(-i\sigma^k\left(\frac{1}{2V}\varepsilon_{ijk}(\partial_i V) + \frac{1}{2V^2}(\partial_k V)A^j\right)\right) & 0 \end{pmatrix}$$
$$= i\begin{pmatrix} 0 & 0 \\ (\delta_{jk} + i\varepsilon_{jkl}\sigma^l)\left(\frac{1}{2V\sqrt{V}}\varepsilon_{ijk}(\partial_i V)\right) + (\sigma^j\sigma^k)\left(\frac{1}{2V^2\sqrt{V}}(\partial_k V)A^j\right) & 0 \end{pmatrix}$$
$$= i\begin{pmatrix} 0 & 0 \\ i\frac{1}{V\sqrt{V}}\sigma^j(\partial_j V) + \sigma^j\sigma^k\frac{1}{2V^2\sqrt{V}}(\partial_k V)A^j & 0 \end{pmatrix} \quad (7.94)$$

$$\hat{\gamma}^4\Gamma_4 = \begin{pmatrix} 0 & 0 \\ (i\sqrt{V} + \frac{1}{\sqrt{V}}\sigma^j A^j)\left(-i\frac{1}{2V^2}\sigma^k(\partial_k V)\right) & 0 \end{pmatrix}$$
$$= i\begin{pmatrix} 0 & 0 \\ -i\frac{1}{2V\sqrt{V}}\sigma^k(\partial_k V) - \sigma^j\sigma^k\frac{1}{2V^2\sqrt{V}}(\partial_k V)A^j & 0 \end{pmatrix} \quad (7.95)$$

The Dirac operator is therefore

$$\not{D}_s = \frac{1}{\sqrt{\mu}}\begin{pmatrix} 0 & i\frac{1}{\sqrt{V}}\sigma^j\pi^j + i\sqrt{V}\partial_\psi \\ -i\frac{1}{V}\sigma^j\pi^j\sqrt{V} + i\sqrt{V}\partial_\psi & 0 \end{pmatrix} =: \frac{1}{\sqrt{\mu}}\begin{pmatrix} 0 & T^\dagger \\ T & 0 \end{pmatrix}, \quad (7.96)$$

where we have defined the operator T and its adjoint $T^\dagger$,

$$T = -i\frac{1}{V}\sigma^j\pi^j\sqrt{V} + i\sqrt{V}\partial_\psi, \tag{7.97}$$

$$T^\dagger = i\frac{1}{\sqrt{V}}\sigma^j\pi^j + i\sqrt{V}\partial_\psi, \tag{7.98}$$

and

$$\pi^j = -i(\partial_j - A^j\partial_\psi). \tag{7.99}$$

We check that $T^\dagger$ is the adjoint operator of T with respect to the given metric:

$$\begin{aligned}
< \alpha, T\beta > &= \int_M \alpha^\dagger(T\beta)\ \mathrm{dvol}_M \\
&= \int_M \alpha^\dagger\left(-i\frac{1}{V}\sigma^j\pi^j(\sqrt{V}\beta) + i\sqrt{V}\partial_\psi\beta\right)V\mathrm{d}x^4 \\
&= \int_M -\alpha^\dagger\sigma^j(\partial_j - A^j\partial_\psi)(\sqrt{V}\beta) + iV\sqrt{V}\alpha^\dagger(\partial_\psi\beta)\mathrm{d}x^4 \\
&= \int_M -\alpha^\dagger\sigma^j\partial_j(\sqrt{V}\beta) + \alpha^\dagger\sigma^jA^j\partial_\psi(\sqrt{V}\beta) + iV\sqrt{V}\alpha^\dagger(\partial_\psi\beta)\mathrm{d}x^4 \\
&= \int_M (\partial_j\alpha^\dagger)\sigma^j(\sqrt{V}\beta) - (\partial_\psi\alpha^\dagger)\sigma^jA^j(\sqrt{V}\beta) - iV\sqrt{V}(\partial_\psi\alpha^\dagger)\beta\mathrm{d}x^4 \\
&= \int_M \left[\left(\frac{1}{\sqrt{V}}\sigma^j\partial_j\alpha\right)^\dagger\beta - \left(\frac{1}{\sqrt{V}}\sigma^jA^j\partial_\psi\alpha\right)^\dagger\beta + \left(i\sqrt{V}\partial_\psi\alpha\right)^\dagger\beta\right]V\mathrm{d}x^4 \\
&= \int_M \left(i\frac{1}{\sqrt{V}}\sigma^j\pi^j\alpha + i\sqrt{V}\partial_\psi\alpha\right)^\dagger\beta\ \mathrm{dvol}_M \\
&= \int_M \left(T^\dagger\alpha\right)^\dagger\beta\ \mathrm{dvol}_M \\
&= < T^\dagger\alpha, \beta >
\end{aligned} \tag{7.100}$$

To derive the Hamiltonian, we need to compute $TT^\dagger$ and $T^\dagger T$. We find

$$\begin{aligned}
TT^\dagger\chi &= \left(\frac{1}{V}\sigma^j\pi^j - \partial_\psi\right)\left(\sigma^k\pi^k + V\partial_\psi\right)\chi \\
&= \frac{1}{V}\sigma^j\sigma^k\pi^j\pi^k\chi - V\partial_\psi^2\chi \\
&\quad + \frac{1}{V}\sigma^j\pi^j(V\partial_\psi\chi) - \partial_\psi\sigma^k\pi^k\chi \\
&= \frac{1}{V}(\delta_{jk} + i\varepsilon_{jkl}\sigma^l)\pi^j\pi^k\chi - V\partial_\psi^2\chi - \frac{i}{V}\sigma^j(\partial_j V)\partial_\psi\chi \\
&= \frac{1}{V}\pi^j\pi^j\chi - V\partial_\psi^2\chi
\end{aligned} \tag{7.101}$$

and

$$\begin{aligned}
T^\dagger T\chi &= \left(\frac{1}{\sqrt{V}}\sigma^j\pi^j + \sqrt{V}\partial_\psi\right)\left(\frac{1}{V}\sigma^k\pi^k\sqrt{V} - \sqrt{V}\partial_\psi\right)\chi \\
&= \frac{1}{\sqrt{V}}\sigma^j\pi^j\frac{1}{V}\sigma^k\pi^k\sqrt{V}\chi - V\partial_\psi^2\chi \\
&= \frac{1}{\sqrt{V}}\sigma^j\sigma^k\pi^j\frac{1}{V}\left(-i\partial_k\sqrt{V}\right)\chi + \frac{1}{\sqrt{V}}\sigma^j\sigma^k\pi^j\frac{1}{\sqrt{V}}\pi^k\chi - V\partial_\psi^2\chi \\
&= \frac{1}{\sqrt{V}}\sigma^j\sigma^k\pi^j\frac{1}{V}\left(-i\partial_k\sqrt{V}\right)\chi + \\
&\quad + \frac{1}{\sqrt{V}}\sigma^j\sigma^k\left(-i\partial_j\frac{1}{\sqrt{V}}\right)\pi^k\chi + \frac{1}{V}\sigma^j\sigma^k\pi^j\pi^k\chi - V\partial_\psi^2\chi \\
&= -i\frac{1}{\sqrt{V}}\sigma^j\sigma^k\pi^j\frac{1}{2V\sqrt{V}}\left(\partial_k V\right)\chi + \\
&\quad i\frac{1}{2V^2}\sigma^j\sigma^k\left(\partial_j V\right)\pi^k\chi + \left(H_0 + \frac{i}{V}\sigma^j(\partial_j V)\partial_\psi\right)\chi \\
&= TT^\dagger\chi\ - i\frac{1}{2V^2}\sigma^j\sigma^k\left(\partial_k V\right)\pi^j\chi - \frac{1}{\sqrt{V}}\sigma^j\sigma^k\left(\partial_j\frac{1}{2V\sqrt{V}}\left(\partial_k V\right)\right)\chi + \\
&\quad i\frac{1}{2V^2}\sigma^j\sigma^k\left(\partial_j V\right)\pi^k\chi + \frac{i}{V}\sigma^j(\partial_j V)\partial_\psi\chi
\end{aligned}$$

$$= TT^\dagger\chi \; - \frac{1}{2\sqrt{V}}\sigma^j\sigma^k\left(\partial_j\frac{1}{V\sqrt{V}}\right)(\partial_k V)\,\chi - \frac{1}{2V^2}\sigma^j\sigma^k\,(\partial_j\partial_k V)\,\chi +$$

$$- \frac{1}{V^2}(\varepsilon_{jkl}\sigma^l)\,(\partial_j V)\,\pi^k\chi + \frac{i}{V}\sigma^j(\partial_j V)\partial_\psi\chi$$

$$= TT^\dagger\chi \; + \frac{3}{4V^3}\sigma^j\sigma^k\,(\partial_j V)\,(\partial_k V)\,\chi - \frac{1}{2V^2}\,(\partial_j\partial_j V)\,\chi +$$

$$- \frac{1}{V^2}(\varepsilon_{jkl}\sigma^l)\,(\partial_j V)\,\pi^k\chi + \frac{i}{V}\sigma^j(\partial_j V)\partial_\psi\chi$$

$$= TT^\dagger\chi \; + \frac{3}{4V^3}\frac{1}{r^4}\chi + \varepsilon_{jkl}\frac{1}{V^2}\frac{x^j}{r^3}\sigma^l\pi^k\chi - \frac{i}{V}\frac{x^j}{r^3}\sigma^j\partial_\psi\chi \tag{7.102}$$

or

$$T^\dagger T \;=\; TT^\dagger + \frac{3}{4V^3r^4} + \frac{1}{V^2r^3}\vec{\sigma}\cdot(\vec{r}\times\vec{\pi}) - i\frac{1}{Vr^3}\vec{r}\cdot\vec{\sigma}\partial_\psi$$

$$= TT^\dagger + \frac{3}{4V^3r^4} + \frac{1}{V^2r^3}\vec{\sigma}\cdot\vec{L}_0 - i\frac{1}{V^2r^3}\vec{\sigma}\cdot\vec{r}\partial_\psi, \tag{7.103}$$

where

$$\vec{L}_0 \;=\; \vec{r}\times\vec{\pi} - i\hat{r}\partial_\psi. \tag{7.104}$$

This disagrees with the result found by Comtet and Horváthy [10], who studied the Dirac equation in Taub-NUT space in the context of gravitational instantons.

The Hamiltonian is then given by

$$H \;=\; \frac{1}{2}\not{D}_s^2 \;=\; \begin{pmatrix} H_2 & 0 \\ 0 & H_1 \end{pmatrix}, \tag{7.105}$$

where

$$H_1 \;=\; \frac{1}{2\mu}TT^\dagger \;=\; \frac{1}{2\mu}\left[\frac{1}{V}\pi^j\pi^j - V\partial_\psi^2\right], \tag{7.106}$$

$$H_2 \;=\; \frac{1}{2\mu}T^\dagger T \;=\; H_1 + \frac{1}{2\mu}\left[\frac{3}{4V^3r^4} + \frac{1}{V^2r^3}\vec{\sigma}\cdot\vec{L}_0 - i\frac{1}{V^2r^3}\vec{\sigma}\cdot\vec{r}\partial_\psi\right]. \tag{7.107}$$

Notice that H_1 acts diagonally - a fact we will come to appreciate further when we compute the Dirac operator acting on forms in the following section.

7.3.2 Quantisation using forms

Once again we will construct a matrix representation of the Dirac operator action on anti-holomorphic forms, by finding a matrix representation of the Dolbeault operators with respect to a suitable basis of anti-holomorphic forms.

A set of Kähler coordinates on the Taub-NUT manifold M_{TN} is defined by [44]

$$w \;=\; r\sin\theta e^{i\phi}, \qquad\qquad v \;=\; r(1+\cos\theta)e^{r\cos\theta+i(\psi+\phi)}. \tag{7.108}$$

We define the 1-forms α_2 and α_1, which form a convenient basis of holomorphic 1-forms with respect to the complex structure corresponding to the hermitian coordinates w and v, by

$$\alpha_2 \;=\; \frac{1}{\sqrt{2}}(e^1+ie^2) \;=\; \sqrt{\frac{\mu V}{2}}\,\mathrm{d}w, \tag{7.109a}$$

$$\alpha_1 \;=\; \frac{1}{\sqrt{2}}(e^3+ie^4) \;=\; \sqrt{\frac{\mu}{2V}}\left(\frac{\mathrm{d}v}{v}+(\cos\theta-1)\frac{\mathrm{d}w}{w}\right). \tag{7.109b}$$

The Taub-NUT metric can then be written in terms of Kähler coordinates as

$$\mathrm{d}s^2 \;=\; \mu\left[V\,|\mathrm{d}w|^2+V^{-1}\left|\frac{\mathrm{d}v}{v}+(\cos\theta-1)\frac{\mathrm{d}w}{w}\right|^2\right] \;=\; 2|\alpha_2|^2+2|\alpha_1|^2. \tag{7.110}$$

Again we choose $\{\overline{\alpha}_1,\overline{\alpha}_2,1,\overline{\alpha}_1\wedge\overline{\alpha}_2\}$ as and ordered basis of $\Omega^{0,\bullet}(M)$, and using the same procedure as before, we represent the action of the Dolbeault operator $\overline{\partial}$, and its adjoint $\overline{\partial}^\dagger=-*\partial*$, as a matrices. The calculations involved are lengthy, but straightforward. As an example, we compute $\overline{\partial}\varphi$ for a function φ which corresponds to the third column of the matrix of $\overline{\partial}$. The final results, the matrices for $\overline{\partial}$ and $\overline{\partial}^\dagger$ are given below in equations (7.119) and (7.120).

To compute $\overline{\partial}\varphi$, we first observe the following. On the one hand, the exterior derivative of a function φ is given by

$$\mathrm{d}\varphi \;=\; (\partial_i\varphi)\mathrm{d}x^i+(\partial_\psi\varphi)\mathrm{d}\psi, \tag{7.111}$$

while on the other,

$$\mathrm{d}\varphi \;=\; \partial\varphi+\overline{\partial}\varphi \;=\; f_1(\varphi)\,\alpha_1+\overline{f}_1(\varphi)\,\overline{\alpha}_1+f_2(\varphi)\,\alpha_2+\overline{f}_2(\varphi)\,\overline{\alpha}_2, \tag{7.112}$$

where

$$\partial\varphi = f_1\alpha_1 + f_2\alpha_2, \qquad \overline{\partial}\varphi = \overline{f}_1\overline{\alpha}_1 + \overline{f}_2\overline{\alpha}_2. \tag{7.113}$$

We now compute

$$\begin{aligned} f_1\alpha_1 + \overline{f}_1\overline{\alpha}_1 &= f_1\left(\frac{1}{\sqrt{2}}(e^3 + ie^4)\right) + \overline{f}_1\left(\frac{1}{\sqrt{2}}(e^3 - ie^4)\right) \\ &= \frac{\sqrt{\mu}}{\sqrt{2}}\left((f_1 + \overline{f}_1)\sqrt{V}\mathrm{d}x^3 + i(f_1 - \overline{f}_1)\frac{1}{\sqrt{V}}(\mathrm{d}\psi + \vec{A}\cdot\mathrm{d}\vec{r})\right), \end{aligned} \tag{7.114}$$

$$\begin{aligned} f_2\alpha_2 + \overline{f}_2\overline{\alpha}_2 &= f_2\left(\frac{1}{\sqrt{2}}(e^1 + ie^2)\right) + \overline{f}_2\left(\frac{1}{\sqrt{2}}(e^1 - ie^2)\right) \\ &= \frac{\sqrt{\mu}}{\sqrt{2}}\left((f_2 + \overline{f}_2)\sqrt{V}\mathrm{d}x^1 + i(f_2 - \overline{f}_2)\sqrt{V}\mathrm{d}x^2\right). \end{aligned} \tag{7.115}$$

Therefore, comparing equation (7.111) with (7.112), we have

$$\partial_\psi\varphi = i\frac{\sqrt{\mu}}{\sqrt{2}}(f_1 - \overline{f}_1)\frac{1}{\sqrt{V}} \tag{7.116}$$

$$\partial_3\varphi = \frac{\sqrt{\mu}}{\sqrt{2}}(f_1 + \overline{f}_1)\sqrt{V} \tag{7.117}$$

which can be solved for f_1 by

$$f_1(\varphi) = \frac{1}{\sqrt{2\mu}}\, i\left(\frac{1}{\sqrt{V}}\pi^3 - \sqrt{V}\partial_\psi\right)(\varphi), \tag{7.118}$$

where π^3 is defined in (7.99). Similarly, comparing the components of $\mathrm{d}x^1$ and $\mathrm{d}x^2$ of equations (7.111) and (7.112) gives us an expression for f_2.

As a final result we find the following matrices for the Dolbeault operator and its adjoint.

$$(\overline{\partial}) = \begin{pmatrix} 0 & 0 & \overline{f}_1 & 0 \\ 0 & 0 & \overline{f}_2 & 0 \\ 0 & 0 & 0 & 0 \\ \overline{g}_1 - \overline{f}_2 & \overline{g}_2 + \overline{f}_1 & 0 & 0 \end{pmatrix} \tag{7.119}$$

$$(\overline{\partial}^\dagger) = \begin{pmatrix} 0 & 0 & 0 & f_2 \\ 0 & 0 & 0 & -f_1 \\ -g_2 - f_1 & g_1 - f_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.120}$$

The functions g_1 and g_2, and operators f_1 and f_2, are given by

$$g_1 = -\frac{1}{\sqrt{\mu}}\frac{1}{2V\sqrt{2V}}(\partial_1 - i\partial_2)V, \qquad f_1 = i\frac{1}{\sqrt{\mu}}\frac{1}{\sqrt{2V}}\left(\pi^3 - V\partial_\psi\right),$$

$$g_2 = \frac{1}{\sqrt{\mu}}\frac{1}{2V\sqrt{2V}}(\partial_3 V), \qquad f_2 = \frac{1}{\sqrt{\mu}}\frac{1}{\sqrt{2V}}\left(\pi^2 + i\pi^1\right), \tag{7.121}$$

where π^j is defined in (7.99). The Dirac operator in the chosen basis of anti-holomorphic forms is therefore the same as the Dirac operator acting on spinors, found in section 7.3.1.

$$\not{D}_{\bar\partial} = \sqrt{2}\left(\bar\partial + \bar\partial^\dagger\right) = \frac{1}{\sqrt{\mu}}\begin{pmatrix} 0 & i\sqrt{V}\partial_\psi + i\frac{1}{\sqrt{V}}\sigma^k\pi^k \\ i\sqrt{V}\partial_\psi - i\frac{1}{V}\sigma^k\pi^k\sqrt{V} & 0 \end{pmatrix} = \not{D}_s \tag{7.122}$$

This means that our chosen basis of anti-holomorphic forms gives an easy way to translate between spinors on the moduli space and anti-holomorphic forms on the moduli space, as in equation (6.24).

The Hamiltonian $H = \frac{1}{2}\not{D}_{\bar\partial}^2 = \left(\bar\partial^\dagger\bar\partial + \bar\partial\,\bar\partial^\dagger\right) = \frac{1}{2}\Delta_{M_{TN}}$ is again the same as the Hamiltonian for spinors on the moduli space, (7.105). We see that the effective Hamiltonian for the bosonic fields in the original field theory (the 0-forms and 2-forms on the moduli space) is H_1. Since it is diagonal, it acts on functions and 2-forms independently. The Hamiltonian for fermionic fields H_2, however, mixes the two different fermionic modes through the terms involving the Pauli-matrices. Since the Hamiltonian commutes with the Dolbeault operators, the spectrum for the fermionic sector of the theory must be the same as the spectrum for the bosonic sector. This agrees with Comtet and Horváthy's argument using supersymmetry on the moduli space to argue that the bosonic and fermionic spectra are the same.

Finally, we compute the matrix representation of the twisted Dolbeault operator and its adjoint.

$$\bar\partial_{\mathcal{J}} = i\begin{pmatrix} 0 & 0 & f_2 & 0 \\ 0 & 0 & -f_1 & 0 \\ 0 & 0 & 0 & 0 \\ g_2 + f_1 & -(g_1 - f_2) & 0 & 0 \end{pmatrix} \tag{7.123}$$

$$\overline{\partial}^{\dagger}_{\mathcal{J}} = i \begin{pmatrix} 0 & 0 & 0 & \overline{f}_1 \\ 0 & 0 & 0 & \overline{f}_2 \\ -(\overline{g}_1 - \overline{f}_2) & -\overline{g_2} - \overline{f}_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.124}$$

We can again directly compute the operators $\overline{\partial}\,\overline{\partial}_{\mathcal{J}} = -\overline{\partial}_{\mathcal{J}}\overline{\partial}$ and $\overline{\partial}^{\dagger}\overline{\partial}^{\dagger}_{\mathcal{J}} = -\overline{\partial}^{\dagger}_{\mathcal{J}}\overline{\partial}^{\dagger}$, but it is easier to use the Kodaira relations (6.27). This gives once more the simple results

$$\overline{\partial}\,\overline{\partial}_{\mathcal{J}} = \frac{i}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta_{M_{TN}} & 0 \end{pmatrix}, \quad \overline{\partial}^{\dagger}\overline{\partial}^{\dagger}_{\mathcal{J}} = \frac{i}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_{M_{TN}} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{7.125}$$

analogous to equations (6.26).

7.4 $N = 4$ supersymmetric monopoles

BPS states of $N = 4$ supersymmetric monopoles correspond to short supermultiplets which contain five spin-0 states, four spin-$\frac{1}{2}$ doublets and one spin-1 triplet. To construct the $N = 4$ supersymmetric monopole states, we start again with a solution Ψ to the bosonic Schrödinger equation, and generate the remaining states in the multiplet using the (twisted) Dolbeault operators and their adjoints. As discussed above, we may separate these operators into their centre of mass and relative moduli space components.

A wavefunction on the centre of mass moduli space generates a full short $N = 4$ multiplet on the centre of mass moduli space. The full wavefunctions correspond to wedge products of these forms with normalisable forms on the relative moduli space (see also section 7.3 and appendix A.3). A multiplet of BPS states can therefore only exist if there is a unique normalisable harmonic form on the moduli space, as first argued by Sen [13]. We will call this form the Sen-form. It must necessarily be anti-self-dual or self-dual.

The discussion about the structure of the short multiplet of $N = 4$ supersymmetric monopole states for charge-1 monopoles carries over straightforwardly from the multiplet of states on the centre of mass moduli space. We will now discuss the Sen-form, required to build the wavefunctions of BPS monopoles on the total moduli space.

7.4.1 The Sen-form

As pointed out by Gauntlett and Lowe [5] there exists a normalisable harmonic 2-form on the Taub-NUT manifold,

$$\omega_S \;=\; \frac{r}{r+1}\eta_1 \wedge \eta_2 + \frac{1}{(r+1)^2}\mathrm{d}r \wedge \eta_3 \;=\; \mathrm{d}(V\eta_3) \tag{7.126}$$

which can be rewritten as

$$\begin{aligned}
\omega_S \;&=\; V^{-1}\left(-\frac{1}{r^3}\right)\frac{1}{2}\epsilon_{ijk}x^i\mathrm{d}x^j \wedge \mathrm{d}x^k + \frac{r^2}{(r+1)^2\, r^2}\frac{x^i}{r}\mathrm{d}x^i \wedge \eta_3^R \\
&=\; -\frac{1}{\mu}V^{-2}\,\frac{x^i}{r^3}\left(\frac{1}{2}\epsilon_{ijk}e^j \wedge e^k - e^i \wedge e^4\right) \\
&=\; -\frac{1}{\mu}V^{-2}\left(\partial_i V\right)T^i
\end{aligned} \tag{7.127}$$

where we have defined the triplet T^i as before by

$$T^i \;=\; e^4 \wedge e^i + \frac{1}{2}\varepsilon_{ijk}e^j \wedge e^k. \tag{7.128}$$

They are self-dual and therefore the Sen-form is self-dual as well.

The Sen-form is normalisable, since

$$\int \omega_S \wedge \omega_S \;=\; 8\pi^3 \int \frac{r}{(r+1)^3}\mathrm{d}r \;=\; 4\pi^3, \tag{7.129}$$

which is finite.

7.5 Angular momentum and spin

The total angular momentum operator $\vec{J}$ is once again defined by equation (5.8). As usual we decompose the total moduli space into the centre of mass and relative moduli spaces. The vector fields generating the $SU(2)$ action on the Taub-NUT manifold are denoted ξ_i^L (see also Gibbons and Manton [4]). They are given by

$$\xi_1^L = -\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\psi} + \sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\theta}{\sin\theta}\cos\phi\frac{\partial}{\partial\phi} \tag{7.130a}$$

$$\xi_2^L = -\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\psi} - \cos\phi\frac{\partial}{\partial\theta} + \frac{\cos\theta}{\sin\theta}\sin\phi\frac{\partial}{\partial\phi} \tag{7.130b}$$

$$\xi_3^L = -\frac{\partial}{\partial\phi} \tag{7.130c}$$

They satisfy

$$\left[\xi_i^L, \xi_j^L\right] = \varepsilon_{ijk}\xi_k^L. \tag{7.131}$$

Again, the Lie derivatives with respect to these vector fields obey indeed the commutation relations (5.6) with the complex structures. Furthermore, we have

$$\mathcal{L}_{\xi_i^L}(V) = \xi_i^L\left(1 + \frac{1}{r}\right) = 0 \tag{7.132}$$

so that

$$\mathcal{L}_{\xi_i^L}(e^j) = V^{\frac{1}{2}}\mathcal{L}_{\xi_i^L}(\mathrm{d}x^j) \tag{7.133}$$

$$\mathcal{L}_{\xi_i^L}(e^4) = V^{-\frac{1}{2}}\mathcal{L}_{\xi_i^L}(\eta_3) \tag{7.134}$$

Now we use the fact that

$$\begin{pmatrix} \mathrm{d}x^1 \\ \mathrm{d}x^2 \\ \mathrm{d}x^3 \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \mathrm{d}r \\ \mathrm{d}\theta \\ \mathrm{d}\phi \end{pmatrix} \tag{7.135}$$

to compute

$$\mathcal{L}_{\xi_i^L}(\mathrm{d}x^j) = \varepsilon_{ijk}\mathrm{d}x^k, \tag{7.136}$$

and

$$\mathcal{L}_{\xi_i^L}(\eta_3) = \mathcal{L}_{\xi_i^L}(\mathrm{d}\psi + \cos\theta\mathrm{d}\phi) = 0, \tag{7.137}$$

Therefore

$$\mathcal{L}_{\xi_i^L}(e^j) = \varepsilon_{ijk}e^k \qquad \mathcal{L}_{\xi_i^L}(e^4) = 0 \tag{7.138}$$

and as in section 6.5 we find that the Lie derivatives with respect to these vector fields obey indeed the commutation relations (5.6) with the complex structures (6.17): $\left[\mathcal{L}_{\xi_i^L}, \mathrm{ad}\,\mathcal{I}_j\right] = \varepsilon_{ijk}\,\mathrm{ad}\,\mathcal{I}_k$.

We now have, for example,

$$\begin{aligned} J_1(\alpha_1) &= \frac{i}{\sqrt{2}}\left(\mathcal{L}_{\xi_i^L} - \frac{1}{2}\,\mathrm{ad}\,\mathcal{I}_1\right)(e^3 + ie^4) \\ &= \frac{i}{\sqrt{2}}\left(-e^2 - \frac{1}{2}(-e^2 - ie^1)\right) \\ &= -\frac{i}{2\sqrt{2}}(e^2 - ie^1) \\ &= -\frac{1}{2}\alpha_2, \end{aligned} \tag{7.139}$$

and in general we find that the total angular momentum operator acts on $\overline{\alpha}_1$ and $\overline{\alpha}_2$ as

$$J_i(\alpha_m) = -\frac{1}{2}(\overline{\sigma}_i)_{mn}\alpha_n, \tag{7.140a}$$

$$J_i(\overline{\alpha}_m) = \frac{1}{2}(\sigma_i)_{mn}\overline{\alpha}_n. \tag{7.140b}$$

Again we have that $\{\overline{\alpha}_1, \overline{\alpha}_2\}$ and $\{-i\alpha_2, i\alpha_1\}$ form an angular momentum doublet. The multiplet decomposition on the Taub-NUT manifold is the same as that on the flat moduli space in section 6.5.

The Sen-form, written in the form of equation (7.127), is a singlet under the angular momentum operator. It is of the form $\omega_S = U_j T^j$, for a vector U_j so that

$$J_i\omega_S = J_i(U_jT^j) = (\varepsilon_{ijk}U_k)T^j + U_j(\varepsilon_{ijk}T^k) = 0, \tag{7.141}$$

which agrees with the uniqueness of the Sen-form.

Chapter 8

Outlook

By applying supercharges to the bosonic scattering wavefunctions Φ on the Taub-NUT manifold found in section 7.2.2 we generate multiplets of scattering states. Wedging these with the multiplet of centre of mass states as outlined in chapter 6 one obtains the full multiplets of supersymmetric monopole scattering states. In general, for the $N = 4$ supersymmetric monopoles this gives a 256-dimensional supermultiplet of states. The total angular momentum of these states can be computed using the general formula (5.8) and the expressions for the angular momentum action on the centre-of-mass states and on the relative wavefunction, as discussed in sections 6.5 and 7.5.

It remains a challenge to interpret the resulting multiplets of scattering states in terms of monopole-monopole (or dyon-dyon) scattering. One would like to be able to compute spin-polarised differential scattering cross sections, where spins of the in- and out-going monopoles or dyons are specified. To do this in practice one needs to relate the individual monopoles' spin degrees of freedom to the differential forms on the centre of mass and relative moduli space. As explained in the opening paragraphs of chapter 5, this is only possible in the asymptotic region of the moduli space, where the corresponding monopoles are well-separated. The starting point for such a calculation would thus be the decomposition of the asymptotic region of the total moduli space $\mathcal{M}_{1,1}$ (equation (7.1)) into copies of the constituent monopoles' moduli spaces, as discussed in [5, 6]. Using this decomposition one needs to decompose scattering states on $\mathcal{M}_{1,1}$ into states which have definite values of the spin for the constituent monopoles.

An important motivation for carrying out a detailed study of spin-polarised monopole-monopole scattering cross sections comes from the electric-magnetic duality conjecture, summarised in the introduction. An interesting continuation of this research project would therefore be to see if the duality conjecture continues to hold for the non-BPS states, and to investigate further which predictions for strongly interacting particle scattering may be formulated from these calculations on monopole-monopole scattering.

Part III

Appendices

Appendix A

Hyperkähler Manifolds

For most purposes of this thesis, we shall only need to consider 4-dimensional hyperkähler manifolds $\mathcal{M}$. The extension to $4k$-dimensional hyperkähler manifolds is straightforward. In particular we are interested in $\mathbb{R}^4$, $\mathcal{M}_1 = \mathbb{R}^3 \times S^1$ and the Taub-NUT manifold M_{TN}. As explained in chapter 7, the (cover of the) 8-dimensional moduli space $\mathcal{M}_{1,1}$ naturally decomposes into the flat 4-dimensional manifold corresponding to the centre of mass motion, and the Taub-NUT manifold corresponding to the relative motion of monopoles. In section A.3 we discuss the behaviour of differential operators on product manifolds, such as $\mathcal{M}_{1,1}$.

We write the metric on a 4-dimensional manifold $\mathcal{M}$ in terms of an orthonormal frame e, as

$$\mathrm{d}s^2 \;=\; (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2, \tag{A.1}$$

where e^a form the vier-bein corresponding to e. We denote the vectors dual to the e^a with e_b:

$$e^a(e_b) \;=\; \delta^a_b \tag{A.2}$$

Indices at the beginning of the alphabet, $a, b, c, \ldots$, run from 1 to $\dim \mathcal{M} = 4$, while indices in the middle of the alphabet, $i, j, k, \ldots$, run only from 1 to 3.

A.1 Hyperkähler structure

A complex structure on a manifold is a map $\mathcal{I}$ from the tangent bundle to itself, such that $\mathcal{I}^2 = -\mathbb{1}$. The hyperkähler structure is generated by a set of 3 complex structures $\mathcal{I}_i$ that act on the tangent bundle of $\mathcal{M}$, and satisfy the quaternion algebra

$$\mathcal{I}_i \mathcal{I}_j = -\delta_{ij}\mathbb{1} + \varepsilon_{ijk}\mathcal{I}_k. \tag{A.3}$$

This implies that the action of the complex structures on vectors satisfies

$$[\mathcal{I}_i, \mathcal{I}_j] = 2\varepsilon_{ijk}\mathcal{I}_k. \tag{A.4}$$

A.1.1 Complex structures

We define the action of the complex structures on vectors by[1]

$$\mathcal{I}_i(e_j) = \delta_{ij} e_4 + \varepsilon_{ijk} e_k, \tag{A.6a}$$

$$\mathcal{I}_i(e_4) = -e_i, \tag{A.6b}$$

which satisfies the quaternion algebra A.3.

The action of the complex structures on 1-forms

The action of the complex structures on 1-forms is defined by

$$\left(\mathcal{I}_i(e^a)\right)(e_b) = -e^a\left(\mathcal{I}_i(e_b)\right). \tag{A.7}$$

The minus-sign is there so that the action of the complex structure on 1-forms is compatible with the identification of vectors and co-vectors using the metric, $g_{ab} = \delta_{ab}$.

[1] An alternative choice would be

$$\mathcal{I}_i(e_j) = -\delta_{ij} e_4 + \varepsilon_{ijk} e_k,$$

$$\mathcal{I}_i(e_4) = e_i,$$

However, this choice does not correspond with our chosen definition of the complex structures in section 2.6.

In more detail:

$$
\begin{aligned}
(\mathcal{I}_i(e^a))\,(e_b) &= (\mathcal{I}_i(\delta^{ac}e_c))\,(e_b) \\
&= (\delta^{ac}(\delta_{ic}e_4 - e_i\delta_{4c} + \varepsilon_{ick}e_k))\,(e_b) \\
&= ((\delta^a_i e_4 - \delta^a_4 e_i + \delta^{ac}\varepsilon_{ick}e_k))\,(e_b) \\
&= \big((\delta^a_i g_{4d}e^d - \delta^a_4 g_{id}e^d + \delta^{ac}\varepsilon_{ick}g_{kd}e^d)\big)\,(e_b) \\
&= ((\delta^a_i\delta_{4b} - \delta^a_4\delta_{ib} + \delta^{ac}\varepsilon_{ick}\delta_{kb})) \\
&= ((-\delta^a_4\delta_{ib} + \delta^a_i\delta_{4b} + \varepsilon_{iab}) \\
&= -e^a\,(\delta_{ib}e_4 - \delta_{b4}e_i + \varepsilon_{ibk}e_k) \\
&= -e^a\,(\mathcal{I}_i(e_b))\,. \qquad \text{(A.8)}
\end{aligned}
$$

We now compute

$$(\mathcal{I}_i(e^j))(e_k) \;=\; -e^j(\mathcal{I}_i(e_k)) \;=\; -e^j(\delta_{ik}e_4 + \varepsilon_{ikl}e_l) \;=\; \varepsilon_{ijk}, \qquad \text{(A.9)}$$

$$(\mathcal{I}_i(e^j))(e_4) \;=\; -e^j(\mathcal{I}_i(e_4)) \;=\; -e^j(-e_i) \;=\; \delta_{ij}, \qquad \text{(A.10)}$$

$$(\mathcal{I}_i(e^4))(e_k) \;=\; -e^4(\mathcal{I}_i(e_k)) \;=\; -e^4(\delta_{ik}e_4 + \varepsilon_{ikl}e_l) \;=\; -\delta_{ik}, \qquad \text{(A.11)}$$

$$(\mathcal{I}_i(e^4))(e_4) \;=\; -e^4(\mathcal{I}_i(e^4)) \;=\; 0, \qquad \text{(A.12)}$$

and from this, we can read off the following action on forms:

$$\mathcal{I}_i(e^j) \;=\; \delta_{ij}e^4 + \varepsilon_{ijk}e^k, \qquad \text{(A.13a)}$$

$$\mathcal{I}_i(e^4) \;=\; -e^i, \qquad \text{(A.13b)}$$

For composition of the action of the complex structures on forms we find again

$$\mathcal{I}_i\mathcal{I}_j \;=\; -\delta_{ij}\mathbb{1} + \varepsilon_{ijk}\mathcal{I}_k. \qquad \text{(A.14)}$$

The components of the complex structures are given by

$$(\mathcal{I}_i(e^a))(e_b) \;=\; (\mathcal{I}_i)_c{}^a e^c(e_b) \;=\; (\mathcal{I}_i)_b{}^a \;= \tag{A.15}$$

$$-e^a(\mathcal{I}_i(e_b)) \;=\; -e^a((\mathcal{I}_i)^c{}_b e_c) \;=\; -(\mathcal{I}_i)^a{}_b \tag{A.16}$$

The action of the complex structures on p-forms

$\mathcal{I}_j$ acts on p-forms by extension via

$$\mathcal{I}_j(\phi \wedge \psi) \;=\; \mathcal{I}_j(\phi) \wedge \mathcal{I}_j(\psi). \tag{A.17}$$

On functions, the complex structure acts trivially

$$\mathcal{I}_j(f) \;=\; f. \tag{A.18}$$

Acting on p-forms, the composition of complex structures satisfies

$$\mathcal{I}_i\mathcal{I}_j \;=\; (-1)^p\delta_{ij} + \varepsilon_{ijk}\mathcal{I}_k. \tag{A.19}$$

The adjoint action of the complex structures

We define the operator $\mathrm{ad}\,\mathcal{I}_j$ as follows. Its action on 1-forms is defined by

$$\mathrm{ad}\,\mathcal{I}_j(\phi) \;=\; \mathcal{I}_j(\phi), \tag{A.20}$$

and its action on k-forms is given by extending the action using a Leibniz rule

$$\mathrm{ad}\,\mathcal{I}_j(\phi \wedge \psi) \;=\; \mathrm{ad}\,\mathcal{I}_j(\phi) \wedge \psi + \phi \wedge \mathrm{ad}\,\mathcal{I}_j(\psi). \tag{A.21}$$

The $\mathrm{ad}\,\mathcal{I}_j$ obey the same commutation relations as the $\mathcal{I}_j$:

$$[\mathrm{ad}\,\mathcal{I}_i, \mathrm{ad}\,\mathcal{I}_j] \;=\; 2\varepsilon_{ijk}\,\mathrm{ad}\,\mathcal{I}_k, \tag{A.22}$$

while, using equation (A.4), we find (*here we do not sum over the indices*)

$$
\begin{aligned}
[\mathrm{ad}\,\mathcal{I}_i, \mathcal{I}_j] &= \big[(\mathcal{I}_i \otimes 1 \otimes \ldots \otimes 1) + (1 \otimes \mathcal{I}_i \otimes 1 \otimes \ldots \otimes 1) + \ldots \; + (1 \otimes \ldots \otimes 1 \otimes \mathcal{I}_i)\,, \\
&\qquad (\mathcal{I}_j \otimes \mathcal{I}_j \otimes \ldots \otimes \mathcal{I}_j) \,\big] \\
&= ([\mathcal{I}_i, \mathcal{I}_j] \otimes \mathcal{I}_j \otimes \ldots \otimes \mathcal{I}_j) + (\mathcal{I}_j \otimes [\mathcal{I}_i, \mathcal{I}_j] \otimes \mathcal{I}_j \otimes \ldots \otimes \mathcal{I}_j) \\
&\qquad + \ldots \; + (\mathcal{I}_j \otimes \ldots \otimes \mathcal{I}_j \otimes [\mathcal{I}_i, \mathcal{I}_j]) \\
&= \begin{cases} 0 & \text{if } i = j, \\ 2\,\mathrm{ad}\,\mathcal{I}_i \mathcal{I}_j & \text{if } i \neq j. \end{cases}
\end{aligned} \tag{A.23}
$$

Inverse action of the complex structures

On 1-forms, the complex structures satisfy $\mathcal{I}_i^2 = -\mathbb{1}$, so that $\mathcal{I}_i^{-1} = -\mathcal{I}_i$. For p-forms, however, this doesn't hold. In general, $\mathcal{I}_i^4 = \mathbb{1}$, and $\mathcal{I}_i^3 = \mathcal{I}_i^{-1}$, from which we deduce that the inverse action of the complex structures generalises to p-forms as

$$\mathcal{I}_i^{-1} \;=\; (-1)^p \mathcal{I}_i. \tag{A.24}$$

Products of complex structures and inverse complex structures can now be derived from the composition of the complex structures. Acting on p-forms,

$$\mathcal{I}_i^{-1}\mathcal{I}_j \;=\; \mathcal{I}_i\mathcal{I}_j^{-1} \;=\; (-1)^p \mathcal{I}_i \mathcal{I}_j \;=\; (-1)^p((-1)^p \delta_{ij} + \varepsilon_{ijk}\mathcal{I}_k) \;=\; \delta_{ij} + \varepsilon_{ijk}\mathcal{I}_k^{-1} \tag{A.25}$$

which gives

$$\left[\mathcal{I}_i, \mathcal{I}_j^{-1}\right] \;=\; 2\varepsilon_{ijk}\mathcal{I}_k^{-1}. \tag{A.26}$$

A.1.2 Decomposition of complex forms

The holomorphic forms with respect to the complex structure $\mathcal{I} = \mathcal{I}_3$ are α_1 and α_2:

$$\alpha_1 \;=\; \frac{1}{\sqrt{2}}(e^3 + ie^4) \qquad\qquad \alpha_2 \;=\; \frac{1}{\sqrt{2}}(e^1 + ie^2) \tag{A.27}$$

Using equations (A.13) we find that

$$
\begin{aligned}
\mathcal{I}(\alpha_1) &= -i\alpha_1 & \qquad \mathcal{I}(\alpha_2) &= -i\alpha_2 \\
\mathcal{I}(\overline{\alpha}_1) &= i\overline{\alpha}_1 & \qquad \mathcal{I}(\overline{\alpha}_2) &= i\overline{\alpha}_2
\end{aligned} \tag{A.28}
$$

and the action of the remaining complex structures, $\mathcal{J} = \mathcal{I}_1$ and $\mathcal{K} = \mathcal{I}_2$, on these forms is given by

$$\begin{aligned} \mathcal{J}(\alpha_1) &= -i\overline{\alpha}_2 & \mathcal{J}(\alpha_2) &= i\overline{\alpha}_1 \\ \mathcal{J}(\overline{\alpha}_1) &= i\alpha_2 & \mathcal{J}(\overline{\alpha}_2) &= -i\alpha_1 \end{aligned} \tag{A.29}$$

$$\begin{aligned} \mathcal{K}(\alpha_1) &= \overline{\alpha}_2 & \mathcal{K}(\alpha_2) &= -\overline{\alpha}_1 \\ \mathcal{K}(\overline{\alpha}_1) &= \alpha_2 & \mathcal{K}(\overline{\alpha}_2) &= -\alpha_1 \end{aligned} \tag{A.30}$$

If there exist holomorphic coordinates w^1 and w^2 such that α_1 and α_2 are holomorphic linear combinations of $\mathrm{d}w^1$ and $\mathrm{d}w^2$ (and vice versa), as is the case for the manifolds $\mathbb{R}^4$, $\mathcal{M}_1$ and M_{TN}, then

$$\mathcal{I}(\mathrm{d}w^\alpha) = -i\mathrm{d}w^\alpha, \qquad \mathcal{I}(\mathrm{d}w^{\overline{\alpha}}) = i\mathrm{d}w^{\overline{\alpha}}. \tag{A.31}$$

In components,

$$\mathcal{I}_\alpha{}^\beta = -i(\mathbb{1})_\alpha{}^\beta, \qquad \mathcal{I}_{\overline{\alpha}}{}^{\overline{\beta}} = i(\mathbb{1})_{\overline{\alpha}}{}^{\overline{\beta}}, \qquad (\mathcal{I}_3)_{\overline{\alpha}}{}^\beta = (\mathcal{I}_3)_\alpha{}^{\overline{\beta}} = 0. \tag{A.32}$$

and $\mathcal{J} = \mathcal{I}_1$ and $\mathcal{K} = \mathcal{I}_2$ map holomorphic forms (φ) into anti-holomorphic forms, and vice-versa. For example,

$$\mathcal{I}(\mathcal{J}\varphi) = -\mathcal{J}(\mathcal{I}\varphi) = i\mathcal{J}(\varphi), \tag{A.33}$$

Therefore,

$$\mathcal{J}_\alpha{}^\beta = \mathcal{J}_{\overline{\alpha}}{}^{\overline{\beta}} = 0 \qquad \mathcal{K}_\alpha{}^\beta = \mathcal{K}_{\overline{\alpha}}{}^{\overline{\beta}} = 0 \tag{A.34}$$

Finally, some useful identities are

$$\alpha_1 \wedge \overline{\alpha}_1 = -ie^3 \wedge e^4 \tag{A.35a}$$

$$\alpha_1 \wedge \alpha_2 = -\frac{1}{2}(e^1 \wedge e^3 - e^2 \wedge e^4) - \frac{i}{2}(e^1 \wedge e^4 + e^2 \wedge e^3) \tag{A.35b}$$

$$\alpha_1 \wedge \overline{\alpha}_2 = -\frac{1}{2}(e^1 \wedge e^3 + e^2 \wedge e^4) - \frac{i}{2}(e^1 \wedge e^4 - e^2 \wedge e^3) \tag{A.35c}$$

$$\alpha_2 \wedge \overline{\alpha}_2 = -ie^1 \wedge e^2 \tag{A.35d}$$

A.1.3 The Hodge star operator

The Hodge star operator, $*$, maps p-forms to $(4-p)$-forms, where 4 is the dimension of the manifold $\mathcal{M}$. It is defined by

$$\psi \wedge *\psi \;=\; \text{dvol}, \tag{A.36}$$

where dvol is a volume form on the manifold $\mathcal{M}$. The natural definition of the volume form on a Kähler manifold is

$$\text{dvol} \;=\; \alpha_1 \wedge \overline{\alpha}_1 \wedge \alpha_2 \wedge \overline{\alpha}_2 \;=\; -e^1 \wedge e^2 \wedge e^3 \wedge e^4. \tag{A.37}$$

Using the expressions for holomorphic and anti-holomorphic forms in terms of the orthonormal frame, we find the following Hodge-duals:

$$\begin{aligned} *\alpha_1 &= \frac{1}{\sqrt{2}} * (e^3 + ie^4) \\ &= -\frac{1}{\sqrt{2}}(e^1 \wedge e^2 \wedge e^4 - ie^1 \wedge e^2 \wedge e^3) \\ &= -\alpha_2 \wedge \overline{\alpha}_2 \wedge \alpha_1, \end{aligned} \tag{A.38a}$$

$$\star\alpha_2 \;=\; -\alpha_2 \wedge \alpha_1 \wedge \overline{\alpha}_1, \tag{A.38b}$$

and using equations (A.35) we find

$$*(\alpha_1 \wedge \overline{\alpha}_1) \;=\; -\alpha_2 \wedge \overline{\alpha}_2, \tag{A.39a}$$

$$*(\alpha_1 \wedge \alpha_2) \;=\; -\alpha_1 \wedge \alpha_2, \tag{A.39b}$$

$$*(\alpha_1 \wedge \overline{\alpha}_2) \;=\; \alpha_1 \wedge \overline{\alpha}_2. \tag{A.39c}$$

The remaining identities can be found using

$$*\overline{\psi} \;=\; \overline{*\psi}, \tag{A.40}$$

$$*(*\psi) \;=\; (-1)^{\deg\psi}\psi. \tag{A.41}$$

A.1.4 Hyperkähler forms

The hyperkähler forms are defined by

$$\omega_i(X,Y) \;=\; g(X,\mathcal{I}_i Y). \tag{A.42}$$

The requirement for the moduli space to be a Kähler manifold with respect to the complex structure $\mathcal{I}_i$ is that ω_i is closed:

$$\mathrm{d}\omega_i \;=\; 0. \tag{A.43}$$

Using equation (A.7), we compute

$$\begin{aligned} \omega_i &= -e^j \otimes \mathcal{I}_i(e^j) - e^4 \otimes \mathcal{I}_i(e^4) \\ &= -e^j \otimes (\delta_{ij} e^4 + \varepsilon_{ijk} e^k) + e^4 \otimes e^i \\ &= e^4 \wedge e^i - \frac{1}{2}\varepsilon_{ijk} e^j \wedge e^k. \end{aligned} \tag{A.44}$$

The hyperkähler forms are of degree (1,1) with respect to their corresponding complex structure. For example,

$$\omega_3 \;=\; -i(\alpha_1 \wedge \overline{\alpha}_1 + \alpha_2 \wedge \overline{\alpha}_2). \tag{A.45}$$

They are of mixed degree $(2,0)$ and $(0,2)$ with respect to the other complex structures, as we will show below in the following section. The hyperkähler forms are anti-self-dual:

$$*\omega_i \;=\; -\omega_i. \tag{A.46}$$

A.1.5 The canonical holomorphic symplectic form

The canonical holomorphic symplectic form of a hyperkähler manifold $\mathcal{M}$ with respect to the complex structure $\mathcal{I}_3$ is defined by

$$\Omega_3 \;=\; \omega_1 + i\omega_2. \tag{A.47}$$

Taking cyclic permutations of the indices we can also define the canonical holomorphic symplectic forms with respect to the other complex structures. In terms of our basis of holomorphic forms, we have

$$\begin{aligned}\Omega_3 &= e^4 \wedge e^1 + e^3 \wedge e^2 + i(e^4 \wedge e^2 + e^1 \wedge e^3) \\ &= -i(e^3 + ie^4) \wedge (e^1 + ie^2) \\ &= -2i\alpha_1 \wedge \alpha_2. \end{aligned} \tag{A.48}$$

It is of degree (2,0) with respect to the complex structure $\mathcal{I}_3$, and closed since the hyperkähler forms ω_1 and ω_2 are closed. These two hyperkähler forms are the real and imaginary part of Ω_3 and are therefore forms of mixed type $(2,0)$ and $(0,2)$ with respect to the complex structure $\mathcal{I}_3$.

A.2 Differential operators

A.2.1 Dolbeault operators

The exterior derivative maps p-forms into $(p+1)$-forms. On a Kähler manifold, it can be separated into the Dolbeault operator (with respect to a complex structure $\mathcal{I}$) and its conjugate, via

$$\mathrm{d} = \partial + \overline{\partial}, \tag{A.49}$$

where

$$\partial : \Omega^{p,q} \rightarrow \Omega^{p+1,q}, \qquad \overline{\partial} : \Omega^{p,q} \rightarrow \Omega^{p,q+1}. \tag{A.50}$$

The space $\Omega^{p,q}$ is the space of forms of degree (p,q) with respect to the complex structure. They are eigenstates of the complex structure $\mathcal{I}$ with eigenvalue $i(q-p)$.

On a hyperkähler manifold, we have three complex structures, $\mathcal{I}_j$, which all have their own corresponding Dolbeault operators, ∂_j and $\overline{\partial}_j$, and we can split up the exterior derivative as

$$\mathrm{d} = \partial_j + \overline{\partial}_j, \tag{A.51}$$

where

$$\partial_j : \Omega_j^{p,q} \rightarrow \Omega_j^{p+1,q}, \qquad \overline{\partial}_j : \Omega_j^{p,q} \rightarrow \Omega_j^{p,q+1}. \tag{A.52}$$

We will often pick a complex structure, $\mathcal{I} = \mathcal{I}_3$, and work with (p, q)-forms with respect to this complex structure. We will then omit the subscript 3 for the Dolbeault operators, $\partial = \partial_3$ and $\overline{\partial} = \overline{\partial}_3$.

A.2.2 The Laplacian

The Laplacian Δ is given in terms of the exterior derivative by

$$\Delta \;=\; \nabla^2 \;=\; (\mathrm{d} + \mathrm{d}^\dagger)^2. \tag{A.53}$$

On a Kähler manifold it becomes, in terms of the Dolbeault operator,

$$\Delta \;=\; 2\Delta_\partial \;=\; 2(\partial + \partial^\dagger)^2 \;=\; 2\Delta_{\overline{\partial}} \;=\; 2(\overline{\partial} + \overline{\partial}^\dagger)^2. \tag{A.54}$$

$\mathrm{d}^2 \;=\; \partial^2 \;=\; \overline{\partial}^2 \;=\; 0$, which implies that the exterior derivative and the Dolbeault operators commute with the Laplacian. For example,

$$\Delta\partial \;=\; 2(\partial\partial^\dagger + \partial^\dagger\partial)\partial \;=\; 2\partial\partial^\dagger\partial \;=\; 2\partial(\partial\partial^\dagger + \partial^\dagger\partial) \;=\; \partial\Delta, \tag{A.55}$$

and similarly for d and $\overline{\partial}$.

A.2.3 Twisted exterior derivatives

On a Kähler manifold, we can define a twisted exterior derivative:

$$\mathrm{d}^c \;=\; \mathrm{d}_\mathcal{I} \;=\; -\mathcal{I}^{-1}\mathrm{d}\mathcal{I} \;=\; \mathcal{I}\mathrm{d}\mathcal{I}^{-1}. \tag{A.56}$$

We have, for a (p, q)-form ψ,

$$\begin{aligned} \mathrm{d}_\mathcal{I}\psi &= \mathcal{I}\mathrm{d}\mathcal{I}^{-1}\psi \\ &= (i)^{p-q}\mathcal{I}\mathrm{d}\psi \\ &= (i)^{p-q}\left((-i)^{p+1-q}\partial\psi + (-i)^{p-(q+1)}\overline{\partial}\psi\right) \\ &= i\left(\overline{\partial} - \partial\right)\psi. \end{aligned} \tag{A.57}$$

On a hyperkähler manifold we have three complex structures, and we define twisted exterior derivatives by

$$\mathrm{d}_{\mathcal{I}_i} \;=\; -(\mathcal{I}_i)^{-1}\mathrm{d}(\mathcal{I}_i) \;=\; (\mathcal{I}_i)\mathrm{d}(\mathcal{I}_i)^{-1}. \tag{A.58}$$

We choose a particular complex structure $\mathcal{I} = \mathcal{I}_3$, so that $\mathrm{d}_{\mathcal{I}_3} = \mathrm{d}_{\mathcal{I}}$. Furthermore we define $\mathcal{J} = \mathcal{I}_1$ and $\mathcal{K} = \mathcal{I}_2$ as usual. We now also define twisted Dolbeault operators by

$$\partial_{\mathcal{J}} \;=\; -\mathcal{J}^{-1}\overline{\partial}\mathcal{J} \;=\; \mathcal{J}\overline{\partial}\mathcal{J}^{-1}, \qquad \overline{\partial}_{\mathcal{J}} \;=\; -\mathcal{J}^{-1}\partial\mathcal{J} \;=\; \mathcal{J}\partial\mathcal{J}^{-1}. \tag{A.59}$$

Since $\mathcal{J}$ (and its inverse) transforms a (p,q)-form into a (q,p)-form, $\partial_{\mathcal{J}}$ increases the holomorphic degree by one, while $\overline{\partial}_{\mathcal{J}}$ increases the anti-holomorphic degree by one:

$$\partial_{\mathcal{J}} : \Lambda^{p,q}M \;\rightarrow\; \Lambda^{p+1,q}M, \qquad \overline{\partial}_{\mathcal{J}} : \Lambda^{p,q}M \;\rightarrow\; \Lambda^{p,q+1}M. \tag{A.60}$$

We could also define twisted Dolbeault operators using the complex structures $\mathcal{I}$ and $\mathcal{K}$, but those can then be simply reexpressed in terms of the Dolbeault operators and the $\mathcal{J}$-twisted Dolbeault operators respectively.

* * *

The twisted Dolbeault operators $\partial_{\mathcal{J}}$ should not be confused with Dolbeault operators corresponding to the different complex structure ∂_j. The twisted Dolbeault operators have a capital index indicating the complex structure that does the 'twisting', whereas a lowercase index indicates that we work with a Dolbeault operator corresponding to a different complex structure from the chose complex structure $\mathcal{I}$.

* * *

Since $\mathcal{J}_{\overline{\alpha}}{}^{\alpha} \;=\; \overline{\mathcal{J}_{\alpha}{}^{\overline{\alpha}}}$, the twisted Dolbeault operator satisfies

$$\overline{\overline{\partial}_{\mathcal{J}}\psi} \;=\; \overline{\mathcal{J}\partial\mathcal{J}^{-1}\psi} \;=\; \mathcal{J}\;\overline{\partial\;\mathcal{J}^{-1}\psi} \;=\; \mathcal{J}\;\overline{\partial}\;\overline{\mathcal{J}^{-1}\psi} \;=\; \mathcal{J}\;\overline{\partial}\;\mathcal{J}^{-1}\overline{\psi} \;=\; \partial_{\mathcal{J}}\overline{\psi}. \tag{A.61}$$

We find, using that for a (p,q)-form ψ we have $\mathcal{J}\psi = (-i)^{p-q}\mathcal{K}\psi$,

$$\mathrm{d}_{\mathcal{J}} \;=\; \partial_{\mathcal{J}} + \overline{\partial}_{\mathcal{J}}, \qquad \mathrm{d}_{\mathcal{K}} \;=\; -i(\partial_{\mathcal{J}} - \overline{\partial}_{\mathcal{J}}), \tag{A.62}$$

and

$$\partial_{\mathcal{J}} \;=\; \frac{1}{2}(\mathrm{d}_{\mathcal{J}} + i\mathrm{d}_{\mathcal{K}}), \qquad \overline{\partial}_{\mathcal{J}} \;=\; \frac{1}{2}(\mathrm{d}_{\mathcal{J}} - i\mathrm{d}_{\mathcal{K}}). \tag{A.63}$$

* * *

The complex structures are integrable, so

$$\nabla(\mathcal{I}_i) \;=\; 0, \tag{A.64}$$

from which we deduce that

$$\mathcal{I}_i(\nabla_X\psi) \;=\; \nabla_X(\mathcal{I}_i(\psi)) - (\nabla_X\mathcal{I}_i)\psi \;=\; \nabla_X(\mathcal{I}_i(\psi)). \tag{A.65}$$

Now, using

$$\mathrm{d} \;=\; \sum_j e^j \wedge \nabla_{e_j} \tag{A.66}$$

we find

$$\mathrm{d}_{\mathcal{I}_i} \;=\; (\mathcal{I}_i)\mathrm{d}(\mathcal{I}_i)^{-1} \;=\; (\mathcal{I}_i)\left(\sum_j e^j \wedge \nabla_{e_j}\right)(\mathcal{I}_i)^{-1} \;=\; \left(\sum_j (\mathcal{I}_i)e^j \wedge \nabla_{e_j}\right). \tag{A.67}$$

Similarly, using

$$\partial \;=\; \sum_\alpha e^\alpha \wedge \nabla_{e_\alpha}, \qquad \overline{\partial} \;=\; \sum_{\overline{\alpha}} e^{\overline{\alpha}} \wedge \nabla_{e_{\overline{\alpha}}}, \tag{A.68}$$

we find

$$\overline{\partial}_{\mathcal{J}} \;=\; \sum_\alpha \mathcal{J}e^\alpha \wedge \nabla_{e_\alpha}, \qquad \partial_{\mathcal{J}} \;=\; \sum_{\overline{\alpha}} \mathcal{J}e^{\overline{\alpha}} \wedge \nabla_{e_{\overline{\alpha}}}. \tag{A.69}$$

* * *

Finally, we can write the Dolbeault operators with respect to the other complex structures in terms of the exterior derivative and complex structures,

$$\mathrm{d} \;=\; \left(\partial_j + \overline{\partial}_j\right), \qquad \mathrm{d}_{\mathcal{I}_j} \;=\; i\left(\overline{\partial}_j - \partial_j\right), \tag{A.70}$$

so that

$$\partial_j \;=\; \frac{1}{2}(\mathrm{d} + i\mathrm{d}_{\mathcal{I}_j}), \qquad \overline{\partial}_j \;=\; \frac{1}{2}(\mathrm{d} - i\mathrm{d}_{\mathcal{I}_j}). \tag{A.71}$$

A.2.4 Adjoint operators

Adjoint operators are defined using the natural inner product for forms,

$$(\psi, \phi) \;=\; \int \psi \wedge *\phi. \tag{A.72}$$

$A^\dagger$ is defined to be the adjoint operator of A if

$$(A^\dagger \psi, \phi) \;=\; (\psi, A\phi). \tag{A.73}$$

The adjoint operators to the exterior derivative and Dolbeault operators are

$$\mathrm{d}^\dagger \;=\; -*\mathrm{d}* \;=\; -\iota(e^i)\nabla_i \;=\; -\iota(e^\alpha)\nabla_\alpha - \iota(e^{\overline{\alpha}})\nabla_{\overline{\alpha}}, \tag{A.74}$$

and

$$\partial^\dagger \;=\; -\iota(e^{\overline{\alpha}})\nabla_{\overline{\alpha}}, \qquad\qquad \overline{\partial}^\dagger \;=\; -\iota(e^\alpha)\nabla_\alpha, \tag{A.75}$$

where

$$\iota(e^a)(e^b) \;=\; g^{ab} \qquad \iota(e^\alpha)(e^{\overline{\beta}}) \;=\; g^{\alpha\overline{\beta}}, \qquad \iota(e^{\overline{\alpha}})(e^\beta) \;=\; g^{\overline{\alpha}\beta}, \tag{A.76}$$

so that indeed $\partial^\dagger$ lowers the holomorphic degree of a form by 1 (via contraction with a (0,1)-form) and $\overline{\partial}^\dagger$ lowers the anti-holomorphic degree of a form by 1 (via contraction with a (1,0)-form).

The adjoint action of the complex structure is

$$\mathcal{I}_i^\dagger \;=\; \mathcal{I}_i^{-1}, \tag{A.77}$$

which allows us to compute the adjoint operator to the twisted exterior derivatives. We find

$$\mathrm{d}_{\mathcal{J}}^\dagger \;=\; \mathcal{J}\mathrm{d}^\dagger\mathcal{J}^{-1}. \tag{A.78}$$

A similar result holds for the adjoint operator to the twisted Dolbeault operators:

$$\begin{aligned}\partial_{\mathcal{J}}^\dagger \;&=\; \mathcal{J}\overline{\partial}^\dagger\mathcal{J}^{-1} \;=\; -\mathcal{J}\iota(e^\alpha)\nabla_\alpha\mathcal{J}^{-1} \\ &=\; -\iota(\mathcal{J}e^\alpha)\nabla_\alpha, \end{aligned}\tag{A.79}$$

$$\overline{\partial}_{\mathcal{J}}^\dagger \;=\; \mathcal{J}\partial^\dagger\mathcal{J}^{-1} \;=\; -\iota(\mathcal{J}e^{\overline{\alpha}})\nabla_{\overline{\alpha}}. \tag{A.80}$$

A.2.5 Commutation relations

In general

$$\mathrm{d}^2 = (\partial + \overline{\partial})^2 = \partial^2 + (\partial\overline{\partial} + \overline{\partial}\partial) + \overline{\partial}^2 = 0, \tag{A.81}$$

so that, by separating out operators of different degree, we have

$$\partial^2 = \left\{\partial, \overline{\partial}\right\} = \overline{\partial}^2 = 0. \tag{A.82}$$

From the formulae for the adjoint operators, e.g. $\partial^\dagger = - * \overline{\partial} *$, we immediately have

$$(\mathrm{d}^\dagger)^2 = (\partial^\dagger)^2 = \left\{\partial^\dagger, \overline{\partial}^\dagger\right\} = (\overline{\partial}^\dagger)^2 = 0. \tag{A.83}$$

On a Kähler manifold

$$\left\{\mathrm{d}, \mathrm{d}^\dagger\right\} = 2\left\{\partial, \partial^\dagger\right\} = 2\left\{\overline{\partial}, \overline{\partial}^\dagger\right\} = \Delta, \tag{A.84}$$

$$\left\{\partial, \overline{\partial}^\dagger\right\} = \left\{\overline{\partial}, \partial^\dagger\right\} = 0. \tag{A.85}$$

Commutation relations of the twisted operators

Since on a hyperkähler manifold $\partial_{\mathcal{I}} = -i\partial$ and $\partial_{\mathcal{K}} = -i\partial_{\mathcal{J}}$ we need only concern ourselves with the $\mathcal{J}$-twisted operators. We immediately have

$$\mathrm{d}_{\mathcal{J}}^2 = \mathcal{J}\mathrm{d}^2\mathcal{J}^{-1} = 0, \tag{A.86}$$

$$\partial_{\mathcal{J}}^2 = \mathcal{J}\overline{\partial}^2\mathcal{J}^{-1} = 0, \tag{A.87}$$

$$\overline{\partial}_{\mathcal{J}}^2 = \mathcal{J}\partial^2\mathcal{J}^{-1} = 0, \tag{A.88}$$

$$\left\{\partial_{\mathcal{J}}, \overline{\partial}_{\mathcal{J}}\right\} = \mathcal{J}\left\{\overline{\partial}, \partial\right\}\mathcal{J}^{-1} = 0. \tag{A.89}$$

Similarly

$$(\mathrm{d}_{\mathcal{J}}^\dagger)^2 = (\partial_{\mathcal{J}}^\dagger)^2 = \left\{\partial_{\mathcal{J}}^\dagger, \overline{\partial}_{\mathcal{J}}^\dagger\right\} = (\overline{\partial}_{\mathcal{J}}^\dagger)^2 = 0, \tag{A.90}$$

and also

$$\left\{\partial_{\mathcal{J}}, \overline{\partial}_{\mathcal{J}}^\dagger\right\} = \left\{\overline{\partial}_{\mathcal{J}}, \partial_{\mathcal{J}}^\dagger\right\} = 0. \tag{A.91}$$

Since the Laplacian commutes with the complex structures,

$$\left\{\mathrm{d}_{\mathcal{J}}, \mathrm{d}^\dagger_{\mathcal{J}}\right\} = \mathcal{J}\left\{\mathrm{d}, \mathrm{d}^\dagger\right\}\mathcal{J}^{-1} = \Delta, \tag{A.92}$$

$$2\left\{\partial_{\mathcal{J}}, \partial^\dagger_{\mathcal{J}}\right\} = 2\mathcal{J}\left\{\overline{\partial}, \overline{\partial}^\dagger\right\}\mathcal{J}^{-1} = \Delta, \tag{A.93}$$

$$2\left\{\overline{\partial}_{\mathcal{J}}, \overline{\partial}^\dagger_{\mathcal{J}}\right\} = 2\mathcal{J}\left\{\partial, \partial^\dagger\right\}\mathcal{J}^{-1} = \Delta. \tag{A.94}$$

Commutation relations of twisted operators with untwisted operators

Since $\mathcal{I}$, $\mathcal{J}$, and $\mathcal{K}$ are integrable, d, $\mathrm{d}_{\mathcal{I}}$, $\mathrm{d}_{\mathcal{J}}$, and $\mathrm{d}_{\mathcal{K}}$ pairwise anti-commute. Using equations (A.82) we find

$$\begin{aligned}\left\{\mathrm{d}, \mathrm{d}_{\mathcal{I}_j}\right\} &= \left\{\partial_j + \overline{\partial}_j, i(\overline{\partial}_j - \partial_j)\right\} \\ &= i\left\{\partial_j, \overline{\partial}_j\right\} - i\left\{\partial_j, \partial_j\right\} + i\left\{\overline{\partial}_j, \overline{\partial}_j\right\} - i\left\{\overline{\partial}_j, \partial_j\right\} \\ &= 0,\end{aligned} \tag{A.95}$$

and therefore we also have

$$\left\{\mathrm{d}_{\mathcal{I}_i}, \mathrm{d}_{\mathcal{I}_j}\right\} = \mathcal{I}_i\left\{\mathrm{d}, \mathrm{d}_{\mathcal{I}_i^{-1}\mathcal{I}_j}\right\}\mathcal{I}_i^{-1} = 0. \tag{A.96}$$

Finally, therefore, $\partial = \frac{1}{2}(\mathrm{d} + i\mathrm{d}_{\mathcal{I}})$, $\overline{\partial} = \frac{1}{2}(\mathrm{d} - i\mathrm{d}_{\mathcal{I}})$, $\partial_{\mathcal{J}} = \frac{1}{2}(\mathrm{d}_{\mathcal{J}} + i\mathrm{d}_{\mathcal{K}})$ and $\overline{\partial}_{\mathcal{J}} = \frac{1}{2}(\mathrm{d}_{\mathcal{J}} - i\mathrm{d}_{\mathcal{K}})$ anti-commute as well.

A.3 Product manifolds

We start out with two complex manifolds, M_i $(i = 1, 2)$, with coordinates X_i and the usual operators $\mathrm{d}_i, \partial_i, *_i$ and $\mathrm{d}_i^\dagger = - *_i \mathrm{d}_i *_i$ and $\partial_i^\dagger = - *_i \overline{\partial}_i *_i$, and complex structures $\mathcal{I}_i$. The product manifold $\mathcal{M} = M_1 \times M_2$ then has induced operators $\mathrm{d}, \partial, *, \partial^\dagger$ and $\mathcal{I}$. For example,

$$\mathrm{d} = \mathrm{d}X_1^i \otimes \frac{\partial}{\partial X_1^i} + \mathrm{d}X_2^j \otimes \frac{\partial}{\partial X_2^j} = \mathrm{d}_1 + \mathrm{d}_2. \tag{A.97}$$

For $\omega = \omega_1 \wedge \omega_2$ with ω_i a form on M_i,

$$\begin{aligned} \omega \wedge (*\omega) &= \mathrm{dvol}_M \\ &= \mathrm{dvol}_{M_1} \wedge \mathrm{dvol}_{M_2} \\ &= \omega_1 \wedge (*_1\omega_1) \wedge \omega_2 \wedge (*_2\omega_2), \end{aligned} \tag{A.98}$$

so that

$$*\omega = (-1)^{\deg(*_1\omega_1)\cdot\deg(\omega_2)}(*_1\omega_1) \wedge (*_2\omega_2). \tag{A.99}$$

If we assume separation of variables for a form $\omega \in \Omega^{\bullet,0}M$, i.e. $\omega = \omega_1 \wedge \omega_2$ where ω_1 is independent of the coordinates X_2 on M_2, and ω_2 is independent of the coordinates X_1 on M_1, then we have

$$\mathrm{d}\omega = (\mathrm{d}_1 + \mathrm{d}_2)(\omega_1 \wedge \omega_2) = (\mathrm{d}_1\omega_1) \wedge \omega_2 + (-1)^{\deg(\omega_1)}\omega_1 \wedge (\mathrm{d}_2\omega_2) \tag{A.100}$$

$$\partial\omega = (\partial_1 + \partial_2)(\omega_1 \wedge \omega_2) = (\partial_1\omega_1) \wedge \omega_2 + (-1)^{\deg(\omega_1)}\omega_1 \wedge (\partial_2\omega_2) \tag{A.101}$$

This allows us, for example, to compute the action of $\partial^\dagger$ on a form:

$$\begin{aligned} \partial^\dagger\omega &= - \; *\overline{\partial} * \omega \\ &= -(-1)^{\deg(*_1\omega_1)\cdot\deg(\omega_2)} \; *\overline{\partial}(*_1\omega_1) \wedge (*_2\omega_2) \\ &= -(-1)^{\deg(\omega_1)\cdot\deg(\omega_2)} \; *(\overline{\partial}_1 *_1 \omega_1) \wedge (*_2\omega_2) + \\ &\quad -(-1)^{\deg(\omega_1)\cdot\deg(\omega_2)}(-1)^{\deg(*_1\omega_1)} \; *(*_1\omega_1) \wedge (\overline{\partial}_2 *_2 \omega_2) \\ &= (-1)^{\deg(\omega_1)\cdot\deg(\omega_2)}(-1)^{(\deg(\omega_1)-1)\cdot\deg(*_2\omega_2)}(-1)^{\deg(\omega_2)} \; (\partial_1^\dagger\omega_1) \wedge (\omega_2) + \\ &\quad (-1)^{\deg(\omega_1)\cdot\deg(\omega_2)}(-1)^{2\deg(\omega_1)}(-1)^{\deg(\omega_1)\cdot\deg(\overline{\partial}_2 *_2 \omega_2)} \; (\omega_1) \wedge (\partial_2^\dagger\omega_2) \\ &= (\partial_1^\dagger\omega_1) \wedge (\omega_2) + (-1)^{\deg(\omega_1)}(\omega_1) \wedge (\partial_2^\dagger\omega_2), \end{aligned} \tag{A.102}$$

and similarly for other differential operators, such as the exterior derivative d.

$$\mathrm{d}^\dagger\omega = (\mathrm{d}_1^\dagger\omega_1) \wedge (\omega_2) + (-1)^{\deg(\omega_1)}(\omega_1) \wedge (\mathrm{d}_2^\dagger\omega_2). \tag{A.103}$$

Here we have used that M_1 and M_2 are even dimensional, so that

$$(-1)^{\deg(*_i\omega_i)} = (-1)^{\deg(\omega_i)}, \tag{A.104}$$

$$(-1)^{\deg(\overline{\partial}_i *_i\omega_i)} = (-1)^{\deg(\omega_i)+1}, \tag{A.105}$$

$$*_i *_i \omega_i = (-1)^{\deg(\omega_i)}\omega_i, \tag{A.106}$$

and

$$\deg(\partial_i^\dagger\omega_i) = \deg(\omega_i) - 1. \tag{A.107}$$

We can now express the Laplacian as follows:

$$\begin{aligned}
\Delta\omega &= (\mathrm{d} + \mathrm{d}^\dagger)^2\omega = (\mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger\mathrm{d})\omega \\
&= (\mathrm{d}_1\mathrm{d}_1^\dagger\omega_1)\wedge\omega_2 + (-1)^{\deg(\mathrm{d}_1^\dagger\omega_1)}(\mathrm{d}_1^\dagger\omega_1)\wedge(\mathrm{d}_2\omega_2) + \\
&\quad (-1)^{\deg(\omega_1)}(\mathrm{d}_1\omega_1)\wedge(\mathrm{d}_2^\dagger\omega_2) + \omega_1\wedge(\mathrm{d}_2\mathrm{d}_2^\dagger\omega_2) + \\
&\quad + (\mathrm{d}_1^\dagger\mathrm{d}_1\omega_1)\wedge\omega_2 + (-1)^{\deg(\mathrm{d}_1\omega_1)}(\mathrm{d}_1\omega_1)\wedge(\mathrm{d}_2^\dagger\omega_2) + \\
&\quad (-1)^{\deg(\omega_1)}(\mathrm{d}_1^\dagger\omega_1)\wedge(\mathrm{d}_2\omega_2) + \omega_1\wedge(\mathrm{d}_2^\dagger\mathrm{d}_2\omega_2) + \\
&= (\Delta_1\omega_1)\wedge\omega_2 + \omega_1\wedge(\Delta_2\omega_2).
\end{aligned} \tag{A.108}$$

We see that if ω_i are eigenstates of Δ_i with eigenvalues E_i (i.e. $\Delta_i\omega_i = E_i\omega_i$, without summing over the index i), then $\omega = \omega_1\wedge\omega_2$ is an eigenstate of Δ with eigenvalue $E = E_1 + E_2$.

Appendix B

Supersymmetric Lagrangians By Dimensional Reduction

In this appendix we derive the $N = 2$ and $N = 4$ supersymmetric extensions of the Georgi-Glashow Lagrangian (2.13) in 4 dimensions. We follow the approach of Brink, Schwarz and Scherk [31].

B.1 The $N = 2$ supersymmetric Lagrangian

The $N = 2$ supersymmetric Lagrangian in 4 dimensions, equation (3.1), can be derived from the $N = 1$ supersymmetric Lagrangian in 6 dimensions,

$$L_6 \;=\; \int \mathrm{d}^5x \; \mathcal{L}_6 \;=\; \int \mathrm{d}^5x \left(-\frac{1}{4} F^{mn} \cdot F_{mn} + i\overline{\Psi} \cdot \Gamma^m D_m \Psi \right). \tag{B.1}$$

The Lorentzian metric $g = \eta$ in 6 dimension has signature $(+, -, -, -, -, -)$. Ψ is a complex Weyl spinor,

$$\Gamma_7 \Psi \;=\; -\,\Psi, \tag{B.2}$$

where $\Gamma_7 = \Gamma_0 \cdots \Gamma_5$, and the 8×8 Γ-matrices satisfy

$$\{\Gamma_m, \Gamma_n\} \;=\; 2\eta_{mn} \mathbb{1}_8. \tag{B.3}$$

The action, $S = \int \mathrm{d}t\ L_6$, is invariant under the supersymmetry transformation defined by

$$\begin{aligned}
\delta A_m &= i\left(\bar{\epsilon}\ \Gamma_m \Psi - \overline{\Psi}\ \Gamma_m \epsilon\right), \\
\delta\Psi &= \frac{1}{4}\left[\Gamma^m, \Gamma^n\right] F_{mn}\ \epsilon, \\
\delta\overline{\Psi} &= -\frac{1}{4}\ \bar{\epsilon}\ \left[\Gamma^m, \Gamma^n\right] F_{mn},
\end{aligned} \tag{B.4}$$

where ϵ is a spinor of the same chirality as Ψ.

We dimensionally reduce to 4 dimensions by making the fields independent of the 5th and 6th dimensions, so that

$$\partial_4 \ = \ \partial_5 \ = \ 0. \tag{B.5}$$

Some of the components of the gauge fields become independent scalar fields by this procedure. We define

$$P \ = \ A_4 \ = \ -A^4, \qquad\qquad S \ = \ A_5 \ = \ -A^5. \tag{B.6}$$

The six matrices Γ^m can be decomposed as

$$\begin{aligned}
\Gamma^\mu &= \gamma^\mu \otimes \mathbb{1} = \begin{pmatrix} \gamma^\mu & 0 \\ 0 & \gamma^\mu \end{pmatrix}, \\
\Gamma^4 &= \pm\gamma_5 \otimes i\sigma_1 = \pm i \begin{pmatrix} 0 & \gamma_5 \\ \gamma_5 & 0 \end{pmatrix}, \\
\Gamma^5 &= \pm\gamma_5 \otimes i\sigma_2 = \pm \begin{pmatrix} 0 & \gamma_5 \\ -\gamma_5 & 0 \end{pmatrix}.
\end{aligned} \tag{B.7}$$

The 4-dimensional γ-matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} \ = \ 2\eta^{\mu\nu}, \tag{B.8}$$

and we have

$$\Gamma_7 \ = \ \Gamma_0 \cdots \Gamma_5 \ = \ -\ \gamma_5 \otimes \sigma_3 \ = \ \begin{pmatrix} -\gamma_5 & 0 \\ 0 & \gamma_5 \end{pmatrix}. \tag{B.9}$$

The Weyl condition on the spinors,

$$\Gamma_7\Psi \;=\; \begin{pmatrix} -\gamma_5 & 0 \\ 0 & \gamma_5 \end{pmatrix}\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \;=\; \begin{pmatrix} -\gamma_5\psi_+ \\ \gamma_5\psi_- \end{pmatrix} \;=\; \begin{pmatrix} -\psi_+ \\ -\psi_- \end{pmatrix}, \tag{B.10}$$

therefore implies that

$$\Psi \;=\; \begin{pmatrix} L\psi \\ R\psi \end{pmatrix}, \qquad L \;=\frac{1}{2}(1+\gamma_5), \qquad R \;=\frac{1}{2}(1-\gamma_5), \tag{B.11}$$

where ψ is a Dirac spinor in 4 dimensions. The operators L and R satisfy

$$L\gamma_5 \;=\; \gamma_5 L \;= L, \qquad R\gamma_5 \;=\; \gamma_5 R \;=\; -R, \tag{B.12}$$

and

$$\begin{aligned} L^2 \;&= L, & L+R \;&= 1, \\ R^2 \;&= R, & L-R \;&= \gamma_5. \end{aligned} \tag{B.13}$$

We can now derive the dimensionally reduced Lagrangian. First of all, we note that

$$D_4 \;=\; -e\,\mathrm{ad}\,P, \qquad D_5 \;=\; -e\,\mathrm{ad}\,S, \tag{B.14}$$

so that the first term in the Lagrangian (B.1) becomes

$$\begin{aligned} -\frac{1}{4}F^{mn}\cdot F_{mn} \;=\; & -\frac{1}{4}F^{\mu\nu}\cdot F_{\mu\nu} && (m=\mu, n=\nu) \\ & +2\left(\frac{1}{4}D^\mu P\cdot D_\mu P\right) && (m=\mu, n=4 \text{ or } m=4, n=\mu) \\ & +2\left(\frac{1}{4}D^\mu S\cdot D_\mu S\right) && (m=\mu, n=5 \text{ or } m=5, n=\mu) \\ & +2\left(-\frac{1}{4}e^2||\,[P,S]\,||^2\right) && (m=4, n=5 \text{ or } m=5, n=4). \end{aligned}$$

To compute the second term, we note that, since $L\gamma_\mu = \gamma_\mu R$,

$$\overline{\Psi} \;=\; \left(\psi^\dagger L \;\; \psi^\dagger R\right)\left(\gamma^0\otimes \mathbb{1}\right) \;=\; \left(\psi^\dagger\gamma^0 R \;\; \psi^\dagger\gamma^0 L\right) \;=\; \left(\overline{\psi}R \;\; \overline{\psi}L\right), \tag{B.15}$$

and, using equations (B.12), we have

$$
\begin{aligned}
i\overline{\Psi}\cdot\Gamma^m D_m\Psi &= i\overline{\Psi}\cdot\Gamma^\mu D_\mu\Psi - ie\overline{\Psi}\cdot\Gamma^4\,\mathrm{ad}\,P\Psi - ie\overline{\Psi}\cdot\Gamma^5\,\mathrm{ad}\,S\Psi \\
&= i\left(\overline{\psi}R\ \ \overline{\psi}L\right)\cdot\left(\gamma^\mu\otimes\mathbb{1}\right)D_\mu\begin{pmatrix} L\psi \\ R\psi \end{pmatrix} \\
&\quad - ie\left(\overline{\psi}R\ \ \overline{\psi}L\right)\cdot\left(\pm\gamma_5\otimes i\sigma_1\right)\mathrm{ad}\,P\begin{pmatrix} L\psi \\ R\psi \end{pmatrix} \\
&\quad - ie\left(\overline{\psi}R\ \ \overline{\psi}L\right)\cdot\left(\pm\gamma_5\otimes i\sigma_2\right)\mathrm{ad}\,S\begin{pmatrix} L\psi \\ R\psi \end{pmatrix} \\
&= i\left(\overline{\psi}\gamma^\mu L\ \ \overline{\psi}\gamma^\mu R\right)\cdot D_\mu\begin{pmatrix} L\psi \\ R\psi \end{pmatrix} \\
&\quad \pm e\left(-\overline{\psi}R\ \ \overline{\psi}L\right)\cdot\mathrm{ad}\,P\begin{pmatrix} R\psi \\ L\psi \end{pmatrix} \\
&\quad \pm e\left(-\overline{\psi}R\ \ \overline{\psi}L\right)\cdot\mathrm{ad}\,S\begin{pmatrix} -iR\psi \\ iL\psi \end{pmatrix} \\
&= i\overline{\psi}\cdot\gamma^\mu(L^2+R^2)D_\mu\psi \\
&\quad \pm e\overline{\psi}\cdot(L^2-R^2)\,\mathrm{ad}\,P\psi \\
&\quad \pm ie\overline{\psi}\cdot(L^2+R^2)\,\mathrm{ad}\,S\psi \\
&= i\overline{\psi}\cdot\gamma^\mu D_\mu\psi \pm e\overline{\psi}\cdot\gamma_5\,\mathrm{ad}\,P\psi \pm ie\overline{\psi}\cdot\mathrm{ad}\,S\psi \\
&= i\overline{\psi}\cdot\gamma^\mu D_\mu\psi \pm ie\overline{\psi}\cdot(\mathrm{ad}\,S - i\gamma_5\,\mathrm{ad}\,P)\psi.
\end{aligned}
$$

Therefore, the dimensionally reduced Lagrangian is given by equation (3.1). Furthermore we find that the supersymmetry transformations of the bosonic fields after dimensional reduction give

$$
\begin{aligned}
\delta A_\mu &= i\ \left(\bar\epsilon\ \Gamma_\mu\Psi - \overline{\Psi}\ \Gamma_\mu\epsilon\right) \\
&= i\left(\left(\overline{\alpha}R\ \ \overline{\alpha}L\right)\left(\gamma_\mu\otimes\mathbb{1}\right)\begin{pmatrix} L\psi \\ R\psi \end{pmatrix} - \left(\overline{\psi}R\ \ \overline{\psi}L\right)\left(\gamma_\mu\otimes\mathbb{1}\right)\begin{pmatrix} L\alpha \\ R\alpha \end{pmatrix}\right) \\
&= i\left(\left(\overline{\alpha}\gamma_\mu L\ \ \overline{\alpha}\gamma_\mu R\right)\begin{pmatrix} L\psi \\ R\psi \end{pmatrix} - \left(\overline{\psi}\gamma_\mu L\ \ \overline{\psi}\gamma_\mu R\right)\begin{pmatrix} L\alpha \\ R\alpha \end{pmatrix}\right) \\
&= i\left(\overline{\alpha}\gamma_\mu\psi - \overline{\psi}\gamma_\mu\alpha\right),
\end{aligned}
$$

$$
\begin{aligned}
\delta A_4 = \delta P &= i\ \left(\bar\epsilon\ \Gamma_4\Psi - \overline{\Psi}\ \Gamma_4\epsilon\right) \\
&= i\left(\left(\overline{\alpha}R\ \ \overline{\alpha}L\right)\left(\pm\gamma_5\otimes i\sigma_1\right)\begin{pmatrix} L\psi \\ R\psi \end{pmatrix} - \left(\overline{\psi}R\ \ \overline{\psi}L\right)\left(\pm\gamma_5\otimes i\sigma_1\right)\begin{pmatrix} L\alpha \\ R\alpha \end{pmatrix}\right) \\
&= \pm\ i\left(\left(-\overline{\alpha}R\ \ \overline{\alpha}L\right)\begin{pmatrix} iR\psi \\ iL\psi \end{pmatrix} - \left(-\overline{\psi}R\ \ \overline{\psi}L\right)\begin{pmatrix} iR\alpha \\ iL\alpha \end{pmatrix}\right) \\
&= \pm\ \left(\overline{\psi}\gamma_5\alpha - \overline{\alpha}\gamma_5\psi\right),
\end{aligned}
$$

$$
\begin{aligned}
\delta A_5 = \delta S &= i\ \left(\bar\epsilon\ \Gamma_5\Psi - \overline{\Psi}\ \Gamma_5\epsilon\right) \\
&= i\left(\left(\overline{\alpha}R\ \ \overline{\alpha}L\right)\left(\pm\gamma_5\otimes i\sigma_2\right)\begin{pmatrix} L\psi \\ R\psi \end{pmatrix} - \left(\overline{\psi}R\ \ \overline{\psi}L\right)\left(\pm\gamma_5\otimes i\sigma_2\right)\begin{pmatrix} L\alpha \\ R\alpha \end{pmatrix}\right) \\
&= \pm\ i\left(\left(-\overline{\alpha}R\ \ \overline{\alpha}L\right)\begin{pmatrix} R\psi \\ -L\psi \end{pmatrix} - \left(-\overline{\psi}R\ \ \overline{\psi}L\right)\begin{pmatrix} R\alpha \\ -L\alpha \end{pmatrix}\right) \\
&= \pm\ i\left(\overline{\psi}\alpha - \overline{\alpha}\psi\right).
\end{aligned}
$$

Finally,

$$\begin{aligned}
\Gamma^m \Gamma^n F_{mn} \;=\; & \quad \gamma^\mu \gamma^\nu \otimes \mathbb{1}_2 \, (F_{\mu\nu}) && m = \mu, n = \nu \\
& \pm 2\, i\gamma^\mu \gamma_5 \otimes \sigma_1 \, (D_\mu P) && m = \mu, n = 4 \\
& \pm 2\, i\gamma^\mu \gamma_5 \otimes \sigma_2 \, (D_\mu S) && m = \mu, n = 5 \\
& + 2\, \mathbb{1} \otimes -i\sigma_3 \, (-e\, [P, S]) && m = 4, n = 5
\end{aligned}$$

so that the supersymmetry transformation of the fermionic field Ψ is given by

$$\begin{aligned}
\delta\Psi \;&=\; \frac{1}{2}\Gamma^m \Gamma^n F_{mn}\, \epsilon \\
&= \Big(\frac{1}{2}\gamma^\mu \gamma^\nu F_{\mu\nu} \otimes \mathbb{1}_2 \;+\; \mathbb{1} \otimes -i\sigma_3 \, (-e\,[P,S]) \\
&\qquad \pm\, \gamma^\mu \gamma_5 \otimes i\sigma_1 \, (D_\mu P) \;\pm\, \gamma^\mu \gamma_5 \otimes i\sigma_2 \, (D_\mu S) \Big) \begin{pmatrix} L\alpha \\ R\alpha \end{pmatrix} \\
&= \frac{1}{2}\gamma^\mu \gamma^\nu F_{\mu\nu} \begin{pmatrix} L\alpha \\ R\alpha \end{pmatrix} \;+\; ie\,[P,S] \begin{pmatrix} L\alpha \\ -R\alpha \end{pmatrix} \\
&\qquad \pm\, i\gamma^\mu D_\mu P \begin{pmatrix} -R\alpha \\ L\alpha \end{pmatrix} \;\pm\, \gamma^\mu D_\mu S \begin{pmatrix} -R\alpha \\ -L\alpha \end{pmatrix} \\
&= \begin{pmatrix} L\; (\frac{1}{2}\gamma^\mu \gamma^\nu F_{\mu\nu})\; \alpha \\ R\; (\frac{1}{2}\gamma^\mu \gamma^\nu F_{\mu\nu})\; \alpha \end{pmatrix} + \begin{pmatrix} L\; (ie\gamma_5\,[P,S])\; \alpha \\ R\; (ie\gamma_5\,[P,S])\; \alpha \end{pmatrix} \\
&\qquad + \begin{pmatrix} L\; (\mp i\gamma^\mu D_\mu P)\; \alpha \\ R\; (\pm i\gamma^\mu D_\mu P)\; \alpha \end{pmatrix} - \begin{pmatrix} L\; (\pm \gamma^\mu D_\mu S)\; \alpha \\ R\; (\pm \gamma^\mu D_\mu S)\; \alpha \end{pmatrix} \\
&= \begin{pmatrix} L\; \left(\frac{1}{2}\gamma^\mu \gamma^\nu F_{\mu\nu} + ie\gamma_5\,[P,S] \;\mp\; (\gamma^\mu D_\mu S + i\gamma_5 \gamma^\mu D_\mu P)\right)\; \alpha \\ R\; \left(\frac{1}{2}\gamma^\mu \gamma^\nu F_{\mu\nu} + ie\gamma_5\,[P,S] \;\mp\; (\gamma^\mu D_\mu S + i\gamma_5 \gamma^\mu D_\mu P)\right)\; \alpha \end{pmatrix}.
\end{aligned}$$

Therefore, the supersymmetry transformation of the dimensionally reduced fermionic field ψ is given by

$$\delta\psi \;=\; \left(\frac{1}{2}\gamma^\mu \gamma^\nu F_{\mu\nu} + ie\gamma_5\,[P,S] \;\mp\; \gamma^\mu D_\mu (S - i\gamma_5 P) \right)\; \alpha. \qquad \text{(B.16)}$$

We have confirmed that the supersymmetries of the 4-dimensional Lagrangian (3.1) are given by (3.2).

B.2 The $N = 4$ supersymmetric Lagrangian

The $N = 4$ supersymmetric Lagrangian in 4 dimensions (4.1) can be derived from the $N = 1$ supersymmetric Lagrangian in 10 dimensions,

$$L_{10} \;=\; \int \mathrm{d}^9x \; \mathcal{L}_{10} \;=\; \int \mathrm{d}^9x \left(-\frac{1}{4} F^{AB} \cdot F_{AB} + \frac{i}{2} \overline{\Psi} \cdot \Gamma^A D_A \Psi \right). \qquad \text{(B.17)}$$

The Lorentzian metric $g = \eta$ in 10 dimension has a signature that is mostly minus, $(+, -, -, -, -, -, -, -, -, -)$. Ψ is a Majorana-Weyl spinor:

$$\Gamma_{11}\Psi \;=\; -\,\Psi \qquad \text{(Weyl)} \qquad \text{(B.18)}$$

$$\overline{\Psi} \;=\; \Psi^\dagger \Gamma_0 \;=\; \Psi^t C \qquad \text{(Majorana)} \qquad \text{(B.19)}$$

where $\Gamma_{11} = \Gamma_0 \cdots \Gamma_9$, and the 32×32 Γ-matrices satisfy

$$\{\Gamma_m, \Gamma_n\} \;=\; 2\eta_{mn} \mathbb{1}_{32}. \qquad \text{(B.20)}$$

The Majorana condition can be rewritten as follows

$$\begin{aligned} \Psi^* \;=\; (\Psi^\dagger)^t \;&=\; \left(\Psi^t C \Gamma_0\right)^t \\ &=\; (C\Gamma_0)^t \,\Psi \\ &=\; C\Gamma_0\Psi, \end{aligned} \qquad \text{(B.21)}$$

and therefore it can be interpreted as a reality condition on Ψ. The action, $S = \int \mathrm{d}t \; L_{10}$, is invariant under the supersymmetry transformation defined by

$$\begin{aligned} \delta A_m \;&=\; i\bar{\epsilon}\;\Gamma_m \Psi \;=\; -i\overline{\Psi}\;\Gamma_m \epsilon, \\ \delta\Psi \;&=\; \frac{1}{4}\left[\Gamma^m, \Gamma^n\right] F_{mn}\;\epsilon, \\ \delta\overline{\Psi} \;&=\; -\,\frac{1}{4}\;\bar{\epsilon}\;\left[\Gamma^m, \Gamma^n\right] F_{mn}, \end{aligned} \qquad \text{(B.22)}$$

where ϵ is an anti-commuting Majorana-Weyl spinor of the same chirality as Ψ.

We dimensionally reduce to 4 dimensions by making the fields independent of the 5th – 10th dimensions,

$$\partial_4 \;=\; \ldots \;=\; \partial_9 \;=\; 0 \tag{B.23}$$

Some of the components of the gauge fields become independent scalar fields by this procedure. We define

$$S_\imath \;= A_{3+\imath} \;=\; -A^{3+\imath}, \qquad\qquad P_\imath \;= A_{6+\imath} \;=\; -A^{6+\imath}. \tag{B.24}$$

The ten matrices Γ^m can be decomposed as

$$\Gamma^\mu \;= \gamma^\mu \otimes \mathbb{1}_4 \otimes \sigma_3, \qquad \Gamma^{3+\imath} \;= \mathbb{1}_4 \otimes \alpha^\imath \otimes \sigma_1, \qquad \Gamma^{6+\jmath} \;= \gamma_5 \otimes \beta^\jmath \otimes \sigma_3, \tag{B.25}$$

where $\alpha^\imath$ and $\beta^\jmath$ are 4×4 real anti-symmetric matrices, satisfying

$$\begin{aligned} [\alpha^\imath, \alpha^\jmath] \;&=\; -\,2\varepsilon^{\imath\jmath\kappa}\alpha^\kappa & \{\alpha^\imath, \alpha^\jmath\} \;&=\; -\,2\delta^{ij}\mathbb{1}_4 \\ [\beta^\imath, \beta^\jmath] \;&=\; -\,2\varepsilon^{\imath\jmath\kappa}\beta^\kappa & \{\beta^\imath, \beta^\jmath\} \;&=\; -\,2\delta^{ij}\mathbb{1}_4 \\ [\alpha^\imath, \beta^\jmath] \;&=\; \;\;0 \end{aligned} \tag{B.26}$$

The 4-dimensional γ-matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} \;=\; 2\eta^{\mu\nu}. \tag{B.27}$$

The decomposition of the Γ-matrices gives us

$$\Gamma_{11} \;=\; \Gamma_0 \ldots \Gamma_9 \;=\; -\mathbb{1}_4 \otimes \mathbb{1}_4 \otimes \sigma_2 \tag{B.28}$$

and the requirement on the charge conjugation matrix C,

$$C\Gamma^t_\mu C^{-1} \;=\; -\Gamma_\mu \tag{B.29}$$

leads us to define

$$C \;=\; \mathcal{C} \otimes \mathbb{1}_4 \otimes \mathbb{1}_2 \tag{B.30}$$

where $\mathcal{C}$ is the charge conjugation matrix in four dimensions.

The Weyl condition on the spinors,

$$\Gamma_{11}\Psi \;=\; -\Psi \tag{B.31}$$

suggests that we decompose Ψ as

$$\Psi \;=\; \boldsymbol{\psi}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} \tag{B.32}$$

The decomposition of the Γ-matrices (B.25) suggests that we can further decompose $\boldsymbol{\psi}$ as

$$\boldsymbol{\psi} \;=\; \psi\otimes\theta. \tag{B.33}$$

Now the Majorana condition, $\Psi^* = C\Gamma_0\Psi$, implies

$$\begin{aligned}\Psi^* \;=\; \psi^*\otimes\theta^*\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix} \;&=\; (\mathcal{C}\gamma_0\otimes\mathbb{1}_4\otimes\sigma_3)\left(\psi\otimes\theta\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}\right)\\ &=\; \left(\mathcal{C}\gamma_0\psi\otimes\theta\otimes\frac{1}{\sqrt{2}}\sigma_3\begin{pmatrix}1\\ i\end{pmatrix}\right)\\ &=\; \left(\mathcal{C}\gamma_0\psi\otimes\theta\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}\right)\end{aligned} \tag{B.34}$$

In other words, ψ satisfies the Majorana condition in 4 dimensions, $\psi^* = \mathcal{C}\gamma_0\psi$, and θ is a vector with real components, $\theta^* = \theta$. We use the standard orthonormal basis $\{e_r\}$ in $\mathbb{R}^4$,

$$\theta \;=\; \theta_r e_r \tag{B.35}$$

to define ψ_r by

$$\psi\otimes\theta \;=\; \psi\otimes\theta_r e_r \;=\; \theta_r\psi\otimes e_r \;\equiv\; \psi_r\otimes e_r. \tag{B.36}$$

We now interpret ψ_r as a quartet of Majorana fermions in four dimensions.

It remains to derive the dimensionally reduced lagrangian density. First of all,

$$\begin{aligned} D_m \;&=\; -\,e\;\mathrm{ad}\,S_\imath && \text{for } m = 3+\imath\\ D_m \;&=\; -\,e\;\mathrm{ad}\,P_\jmath && \text{for } m = 6+\jmath \end{aligned} \tag{B.37}$$

so that the first term in the Lagrangian (B.17) becomes

$$
\begin{aligned}
-\frac{1}{4}F^{mn}\cdot F_{mn} \;=\; & -\frac{1}{4}F^{\mu\nu}\cdot F_{\mu\nu} && (m=\mu, n=\nu)\\
& +2\left(\frac{1}{4}D^{\mu}S_{\imath}\cdot D_{\mu}S_{\imath}\right) && (m=\mu, n=3+\imath \text{ or } m=3+\imath, n=\mu)\\
& +2\left(\frac{1}{4}D^{\mu}P_{\jmath}\cdot D_{\mu}P_{\jmath}\right) && (m=\mu, n=6+\jmath \text{ or } m=6+\jmath, n=\mu)\\
& -\frac{1}{4}e^2||\,[S_{\imath},S_{\jmath}]\,||^2 && (m=3+\imath, n=3+\jmath)\\
& +2\left(-\frac{1}{4}e^2||\,[S_{\imath},P_{\jmath}]\,||^2\right) && (m=3+\imath, n=6+\jmath \text{ or } m=6+\imath, n=3+\jmath)\\
& -\frac{1}{4}e^2||\,[P_{\imath},P_{\jmath}]\,||^2 && (m=6+\imath, n=6+\jmath)
\end{aligned}
$$

To compute the second term, we note that

$$
\overline{\Psi} \;=\; \left(\psi_r^{\dagger}\otimes e_r^t\otimes\frac{1}{\sqrt{2}}\,(1\ \ -i)\right)\left(\gamma^0\otimes \mathbb{1}_4\otimes\sigma_3\right) \;=\; \left(\overline{\psi}_r\otimes e_r^t\otimes\frac{1}{\sqrt{2}}\,(1\ i)\right), \quad \text{(B.38)}
$$

so that

$$
\begin{aligned}
\frac{i}{2}\overline{\Psi}\cdot\Gamma^A D_A\Psi &= \frac{i}{2}\overline{\Psi}\cdot\Gamma^\mu D_\mu\Psi - \frac{ie}{2}\overline{\Psi}\cdot\Gamma^{3+\imath}\,\mathrm{ad}\,S_\imath\Psi - \frac{ie}{2}\overline{\Psi}\cdot\Gamma^{6+\jmath}\,\mathrm{ad}\,P_\jmath\Psi \\
&= \frac{i}{2}\overline{\Psi}\cdot\left(\gamma^\mu\otimes \mathbb{1}_4\otimes\sigma_3\right)D_\mu\left(\psi_s\otimes e_s\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}\right) \\
&\quad - \frac{ie}{2}\overline{\Psi}\cdot\left(\mathbb{1}_4\otimes\alpha^\imath\otimes\sigma_1\right)\mathrm{ad}\,S_\imath\left(\psi_s\otimes e_s\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}\right) \\
&\quad - \frac{ie}{2}\overline{\Psi}\cdot\left(\gamma_5\otimes\beta^\jmath\otimes\sigma_3\right)\mathrm{ad}\,P_\jmath\left(\psi_s\otimes e_s\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}\right) \\
&= \frac{i}{2}\left(\overline{\psi}_r\otimes e_r^t\otimes\frac{1}{\sqrt{2}}\,(1\ i)\right)\cdot D_\mu\left(\gamma^\mu\psi_s\otimes e_s\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}\right) \\
&\quad - \frac{ie}{2}\left(\overline{\psi}_r\otimes e_r^t\otimes\frac{1}{\sqrt{2}}\,(1\ i)\right)\cdot\mathrm{ad}\,S_\imath\left(\psi_s\otimes\alpha^\imath e_s\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}i\\ 1\end{pmatrix}\right) \\
&\quad - \frac{ie}{2}\left(\overline{\psi}_r\otimes e_r^t\otimes\frac{1}{\sqrt{2}}\,(1\ i)\right)\cdot\mathrm{ad}\,P_\jmath\left(\gamma_5\psi_s\otimes\beta^\imath e_s\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}\right) \\
&= \frac{i}{2}\left(\overline{\psi}_r\cdot\gamma^\mu D_\mu\psi_s\otimes(e_r)^t e_s\otimes 1\right) \\
&\quad - \frac{ie}{2}\left(\overline{\psi}_r\cdot\mathrm{ad}\,S_\imath\psi_s\otimes(e_r)^t\alpha^\imath e_s\otimes i\right) \\
&\quad - \frac{ie}{2}\left(\overline{\psi}_r\cdot\gamma_5\,\mathrm{ad}\,P_\jmath\psi_s\otimes(e_r)^t\beta^\jmath e_s\otimes 1\right) \\
&= \frac{i}{2}\overline{\psi}_r\cdot\gamma^\mu D_\mu\psi_r + \frac{e}{2}\left(\overline{\psi}_r\cdot\alpha^\imath_{rs}\ \mathrm{ad}\,S_\imath\,\psi_s\right) - \frac{ie}{2}\left(\overline{\psi}_r\cdot\beta^\jmath_{rs}\gamma_5\ \mathrm{ad}\,P_\jmath\,\psi_s\right).
\end{aligned}
$$

Here we have used that $(e_r)^t e_s = \delta_{rs}$, $(e_r)^t\alpha^\imath e_s = \alpha^\imath_{rs}$ and $(e_r)^t\beta^\imath e_s = \beta^\imath_{rs}$, where $\alpha^\imath_{rs}$ and $\beta^\imath_{rs}$ are the matrix elements of $\alpha^\imath$ and $\beta^\imath$. Therefore, we find that the dimensionally reduced Lagrangian is given by equation (4.1). Furthermore we find that the supersymmetry transformations of the bosonic fields after dimensional reduction give

$$\delta A_\mu = i\left(\bar{\epsilon}_r \otimes e_r^t \otimes \frac{1}{\sqrt{2}}\,(1\ i)\right)(\gamma^\mu \otimes \mathbb{1}_4 \otimes \sigma_3)\left(\psi_s \otimes e_s \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}\right)$$

$$= i\bar{\epsilon}_r \gamma_\mu \psi_r,$$

$$\delta S_\imath = i\left(\bar{\epsilon}_r \otimes e_r^t \otimes \frac{1}{\sqrt{2}}\,(1\ i)\right)(\mathbb{1}_4 \otimes \alpha^\imath \otimes \sigma_1)\left(\psi_s \otimes e_s \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}\right)$$

$$= -\,\bar{\epsilon}_r \alpha^\imath_{rs} \psi_s,$$

$$\delta P_\imath = i\left(\bar{\epsilon}_r \otimes e_r^t \otimes \frac{1}{\sqrt{2}}\,(1\ i)\right)(\gamma_5 \otimes \beta^\jmath \otimes \sigma_3)\left(\psi_s \otimes e_s \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}\right)$$

$$= i\ \bar{\epsilon}_r \gamma_5 \beta^\jmath_{rs} \psi_s,$$

and using

$$\begin{aligned}
\Gamma^m \Gamma^n F_{mn} &= \gamma^\mu \gamma^\nu \otimes \mathbb{1}_4 \otimes \mathbb{1}_2\ F_{\mu\nu} && (m=\mu, n=\nu) \\
&+ 2\left(\gamma^\mu \otimes \alpha^\imath \otimes i\sigma_2\ D_\mu S_\imath\right) && (m=\mu, n=3+\imath) \\
&+ 2\left(\gamma^\mu \gamma_5 \otimes \beta^\jmath \otimes \mathbb{1}_2\ D_\mu P_\jmath\right) && (m=\mu, n=6+\jmath) \\
&+ \mathbb{1}_4 \otimes \alpha^\imath \alpha^\jmath \otimes \mathbb{1}_2 \cdot -e\,[S_\imath, S_\jmath] && (m=3+\imath, n=3+\jmath) \\
&+ 2\left(\gamma_5 \otimes \alpha^\imath \beta^\jmath \otimes -i\sigma_2 \cdot -e\,[S_\imath, P_\jmath]\right) && (m=3+\imath, n=6+\jmath) \\
&+ \mathbb{1}_4 \otimes \beta^\imath \beta^\jmath \otimes \mathbb{1}_2 \cdot -e\,[P_\imath, P_\jmath] && (m=6+\imath, n=6+\jmath)
\end{aligned}$$

we find

$$
\begin{aligned}
\delta\Psi &= \frac{1}{2}\Gamma^m\Gamma^n F_{mn}\left(\epsilon_s \otimes e_s \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}\right) \\
&= \frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}\left(\epsilon_s \otimes e_s \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}\right) \\
&\quad - i\gamma^\mu D_\mu S_\imath\left(\epsilon_s \otimes \alpha^\imath e_s \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}\right) \\
&\quad + \gamma^\mu\gamma_5 D_\mu P_\jmath\left(\epsilon_s \otimes \beta^\jmath e_s \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}\right) \\
&\quad - \frac{e}{2}\left[S_\imath, S_\jmath\right]\left(\epsilon_s \otimes \alpha^\imath\alpha^\jmath e_s \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}\right) \\
&\quad - ie\gamma_5\left[S_\imath, P_\jmath\right]\left(\epsilon_s \otimes \alpha^\imath\beta^\jmath e_s \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}\right) \\
&\quad - \frac{e}{2}\left[P_\imath, P_\jmath\right]\left(\epsilon_s \otimes \beta^\imath\beta^\jmath e_s \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}\right)
\end{aligned}
$$

Therefore, using $\alpha^\imath e_s = e_r\alpha^\imath_{rs}$ and $\beta^\imath e_s = e_r\beta^\imath_{rs}$,

$$
\begin{aligned}
\delta\psi_r &= \Big(\frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}\delta_{rs} - i\gamma^\mu D_\mu S_\imath\alpha^\imath_{rs} + \gamma^\mu\gamma_5 D_\mu P_\jmath\beta^\jmath_{rs} \\
&\quad - \frac{e}{2}\varepsilon_{\imath\jmath\kappa}\left[S_\imath, S_\jmath\right]\alpha^\kappa_{rs} - ie\gamma_5\left[S_\imath, P_\jmath\right]\alpha^\imath_{rt}\beta^\jmath_{ts} - \frac{e}{2}\varepsilon_{\imath\jmath\kappa}\left[P_\imath, P_\jmath\right]\beta^\kappa_{rs}\Big)\epsilon_s, \qquad \text{(B.39)}
\end{aligned}
$$

which completes the verification that the supersymmetries of the 4-dimensional Lagrangian (4.1) are given by (4.4).

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