APPENDIX A

Seismic Modelling Using Finite-Difference and 1D Convolution

* The contents of Section A.1 “Solving the elastic wave equation with finite-differences” are from the following document with minor modifications.

Bohlen T., De Nil D., Köhn D. and Jetschny S. 2012. SOFI3D – Seismic modeling with finite differences 3D - acoustic and viscoelastic version (user guide). Department of Physics, Geophysical Institute, Karlsruhe Institute of Technology.
A.1 Solving the elastic wave equation with finite-differences

Seismic wave propagation in linearly elastic and isotropic medium can be expressed as elastodynamic equations in terms of stress and displacement vectors. This system could be transformed into the following first-order velocity-stress hyperbolic system in terms of the particle velocities \( v \), the stresses \( \tau_{ij} \), the density \( \rho \), the Lame parameters \( \lambda \) and \( \mu \):

\[
\rho \frac{\partial v_x}{\partial t} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}
\]

\[
\rho \frac{\partial v_y}{\partial t} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}
\]

\[
\rho \frac{\partial \tau_{xx}}{\partial t} = (\lambda + 2\mu) \frac{\partial v_x}{\partial x} + \lambda \frac{\partial v_y}{\partial y}
\]

\[
\rho \frac{\partial \tau_{yy}}{\partial t} = (\lambda + 2\mu) \frac{\partial v_y}{\partial y} + \lambda \frac{\partial v_x}{\partial x}
\]

\[
\rho \frac{\partial \tau_{xy}}{\partial t} = \mu \left( \frac{\partial v_x}{\partial x} + \lambda \frac{\partial v_x}{\partial y} \right)
\]

Finite-difference (FD) implementation

To solve the above equations, the particle velocities \( v \), the stresses \( \tau_{ij} \), the density \( \rho \), the Lame parameters \( \lambda \) and \( \mu \) are calculated at discrete Cartesian coordinates \( x = i \times dx \), \( y = i \times dy \) and at discrete times \( t = i \times dt \) on a grid. \( dx \) and \( dy \) denote the spatial grid point distance in \( x \) and \( y \) direction and \( dt \) the difference between two succeeding time steps with \( i \in N[[1, NX] \), \( j \in N[[1, NY] \) and \( n \in N[[1, NT] \), where \( NX \), \( NY \) and \( NT \) denote the number of spatial grid points and time steps. Finally the partial derivatives are replaced by (finite)-difference operators. The derivative of a function \( y \) after a variable \( x \) can be approximated by a forward \( D^+ \) or a backward operator \( D^- \)

\[
D^+_x y = \frac{y[i+1]-y[i]}{dt}, \quad \text{forward operator}
\]

\[
D^-_x y = \frac{y[i]-y[i-1]}{dt}, \quad \text{backward operator}
\]
To calculate with a larger grid point distance the variables are arranged on a staggered grid (Virieux 1986) and (Levander 1988) (Figure A.1). Please note, that the vertical axis is denoted by $Y$, e.g. the indices of the stress components are labelled accordingly. To satisfy the stability of the Standard Staggered Grid (SSG) code the density $\rho$, respectively. The Lame parameter $\mu$ are arithmetically and harmonically averaged (Bohlen and Saenger 2006).

![Figure A.1 Grid geometry for a Standard Staggered Grid (SSG) (Bohlen and Saenger 2006).](image)

### Accuracy of FD-operators

In the previous section, the partial derivation was simply replaced by a finite difference quotient. In this section, a more methodical approach is used. First the first derivation of the variable $f$ at a grid point $i$ is calculated using the following Taylor expansion:

$$(2k - 1) \frac{\partial f}{\partial x_i} = \frac{1}{dh} \left( f_{i+(k-1/2)} - f_{i-(k-1/2)} \right) + \frac{1}{dh} \sum_{l=2}^{N} \frac{(k-dh/2)^{2l-1}}{(2l-1)!} \left( \frac{\partial^{(2l-1)}f}{\partial x_i^{(2l-1)}} \right)_i + O(dh)^{2N}$$

For an FD operator with length $2N$, $N$ equations with a weighting factor $\beta_k$ are added:

$$[\sum_{k=1}^{N} \beta_k (2k - 1)] \frac{\partial f}{\partial x_i} =$$

$$\frac{1}{dh} \sum_{k=1}^{N} \beta_k \left( f_{i+(k-1/2)} - f_{i-(k-1/2)} \right) + \frac{1}{dh} \sum_{k=1}^{N} \sum_{l=2}^{N} \beta_k \frac{(k-dh/2)^{2l-1}}{(2l-1)!} \left( \frac{\partial^{(2l-1)}f}{\partial x_i^{(2l-1)}} \right)_i + O(dh)^{2N}$$
For the case $N = 1$ we get the FD operator from the previous section with length $2N = 2$. The Taylor series expansion will be aborted after the first term $O(dh)^2$. This operator is called $2^\text{nd}$ order FD-operator which denotes the abortion error of the Taylor series and not the order of the desired approximated derivation. To understand this equation in more detail, the coefficients for a $4^\text{th}$ order FD-operator are calculated. The $4^\text{th}$ order FD operator has the length $2N = 4$, so $N = 2$. By calculating the sums in equation above we get:

\[
(\beta_1 + 3\beta_2) \frac{\partial f}{\partial x} = \frac{1}{dh} \left( \beta_k (f_{i-1/2} - f_{i+1/2}) + \beta_k (f_{i-3/2} - f_{i+3/2}) \right) + \frac{d^2h}{dh} \left( \beta_2 \frac{1}{8 \cdot 3!} + \right.
\]

\[
\beta_2 \frac{d^3f}{dx^3} = \frac{d^3f}{dx^3}.
\]

The weights $\beta_k$ are calculated in the following way. The coefficients in front of the derivation on the left-hand side of this equation should be equal to 1:

\[\beta_1 + 3\beta_2 = 1\]

The coefficients before the derivation $\frac{d^3f}{dx^3}$ on the right-hand side are vanishing:

\[\beta_1 + 27\beta_2 = 0\]

The weighting coefficients $\beta_k$ can be calculated by inverting the following matrix equation:

\[
\begin{bmatrix}
1 & 3 \\
1 & 27
\end{bmatrix}
\begin{bmatrix}
\beta_1 \\
\beta_2
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
0
\end{bmatrix}
\]

The resulting coefficients are $\beta_1 = 9/8$ and $\beta_2 = -1/24$, so the forward and backward $4^\text{th}$ order FD operators look like:

\[
\left. \frac{\partial f}{\partial x} \right|_{i+1/2} = \frac{1}{dh} \left[ \beta_1 (f_{i+1} - f_i) + \beta_2 (f_{i-2} - f_{i-1}) \right], \quad \text{forward operator}
\]

\[
\left. \frac{\partial f}{\partial x} \right|_{i-1/2} = \frac{1}{dh} \left[ \beta_1 (f_i - f_{i-1}) + \beta_2 (f_{i+1} - f_{i-2}) \right], \quad \text{backward operator}
\]
The coefficients $\beta_i$ in the FD operator are called Taylor coefficients. The accuracy of higher order FD operators can be improved significantly simply by slightly changes in the FD coefficients (Holberg 1987). These numerically optimized coefficients are called Holberg coefficients.

**Numerical artefacts and instabilities**

To avoid numerical artefacts and instabilities during a FD modelling run, a spatial and temporal sampling condition for the wave field has to be satisfied. These will be discussed in the following two sections.

**Grid dispersion**

The first question when building a FD model is: What is the maximum spatial grid point distance $dh$, so that the wave field is correctly sampled? To answer this question we take a look at this simple example: The particle velocity in x-direction is defined by a sine function:

$$v_x = \sin \left(2\pi \frac{x}{\lambda}\right) \quad (A-1)$$

where $\lambda$ denotes the wavelength. When calculating the derivation of this function analytically at $x = 0$ and setting $\lambda = 1m$ we get:

$$\left. \frac{dv_x}{dx} \right|_{x=0} = \frac{2\pi}{\lambda} \cos \left(2\pi \frac{x}{\lambda}\right)\left|_{x=0}\right. = 2\pi \quad (A-2)$$

In the next step, the derivation is approximated numerically by a 2nd order finite-difference operator:

$$\left. \frac{dv_x}{dx} \right|_{x=0} \approx \frac{v_x(x+\Delta x) - v_x(x)_{x=0}}{\Delta x} = \frac{\sin(2\pi \frac{\Delta x}{\lambda})}{\Delta x}$$

Using the Nyquist-criteria, it should be sufficient to sample the wave field with $\Delta x = \lambda/2$. In Table A.1, the numerical solutions of Equation A-1 and the analytical solution A-2 are compared for different sample intervals $\Delta x = \lambda/n$, where $n$ is the number of grid points per wavelength. For the case $n = 2$, which
corresponds to the Nyquist criteria, the numerical solution is \( \frac{dv_x}{dx} \bigg|_{x=0} = 0 \), which is not equal with the analytical solution \( 2\pi \). A refinement of the spatial sampling of the wave field results in an improvement of the finite difference solution. To avoid the occurrence of grid dispersion, the following criteria for the spatial grid spacing case \( dh \) has to be satisfied:

\[
dh \leq \frac{\lambda_{\text{min}}}{n} = \frac{V_{s,\text{min}}}{n f_{\text{max}}}
\]

Here \( \lambda_{\text{min}} \) denotes the minimum wavelength, \( V_{s,\text{min}} \) the minimum S-wave velocity in the model and \( f_{\text{max}} \) is the maximum frequency of the source signal. Depending on the accuracy of the used FD operator the parameter \( n \) is different. In table A.2, \( n \) is listed for different FD operator lengths and types (Taylor and Holberg operators). For short operators, \( n \) should be chosen relatively large, so the spatial grid spacing is small, while for longer FD operators \( n \) is smaller and the grid spacing can be larger.

| \( n \)   | \( \Delta x \,[m] \) | \( \frac{dv_x}{dx} \bigg|_{x=0} \) |
|----------|----------------|------------------|
| analytical | -      | \( 2\pi \approx 6.283 \) |
| 2        | \( \lambda/2 \) | 0                |
| 4        | \( \lambda/4 \) | 4.0              |
| 8        | \( \lambda/8 \) | 5.657            |
| 16       | \( \lambda/16 \) | 6.123            |

Table A.1 Comparison of the analytical solution of Equation A.1 with the numerical solution (Equation A.2) for different grid spacing \( \Delta x = \lambda/n \)

<table>
<thead>
<tr>
<th>FD-order</th>
<th>( n ) (Taylor)</th>
<th>( n ) (Holberg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2(^{\text{rd}})</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>4(^{\text{th}})</td>
<td>8</td>
<td>8.32</td>
</tr>
<tr>
<td>6(^{\text{th}})</td>
<td>7</td>
<td>4.77</td>
</tr>
<tr>
<td>8(^{\text{th}})</td>
<td>6</td>
<td>3.69</td>
</tr>
<tr>
<td>10(^{\text{th}})</td>
<td>5</td>
<td>3.19</td>
</tr>
<tr>
<td>12(^{\text{th}})</td>
<td>4</td>
<td>2.91</td>
</tr>
</tbody>
</table>

Table A.2 The number of grid points per minimum wavelength \( n \) for different orders (2\(^{\text{nd}}\)-12\(^{\text{th}}\)) and types (Taylor and Holberg) of FD operators.
The Courant Instability

In analogy to the spatial, the temporal discretization has to satisfy a sampling criterion to ensure the stability of the FD code. If a wave is crossing a discrete grid, then the time step $dt$ must be less than the time for the wave to travel between two adjacent grid points with grid spacing $dh$. For a 2D grid this means mathematically:

$$dt = \frac{dh}{n\sqrt{2}v_{p,max}}$$

where $v_{p,max}$ is the maximum P-wave velocity in the model. The factor $h$ again depends on the order of the FD operator. In Table A.3, $h$ is listed for different FD operator lengths and types (Taylor and Holberg operators). The above criterion is called Courant-Friedrichs-Lewy criterion.

<table>
<thead>
<tr>
<th>FD-order</th>
<th>n (Taylor)</th>
<th>n (Holberg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2\textsuperscript{nd}</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4\textsuperscript{th}</td>
<td>7/8</td>
<td>1.184614</td>
</tr>
<tr>
<td>6\textsuperscript{th}</td>
<td>149/120</td>
<td>1.283482</td>
</tr>
<tr>
<td>8\textsuperscript{th}</td>
<td>2161/1680</td>
<td>1.345927</td>
</tr>
<tr>
<td>10\textsuperscript{th}</td>
<td>53089/40320</td>
<td>1.387660</td>
</tr>
<tr>
<td>12\textsuperscript{th}</td>
<td>1187803/887040</td>
<td>1.417065</td>
</tr>
</tbody>
</table>

Table A.3 The factor $h$ in the Courant criterion for different orders FD-order (2\textsuperscript{nd}-12\textsuperscript{th}) and types (Taylor and Holberg) of FD operators.
A.2 Seismic modelling using 1D convolution

In this section, the implementation of the seismic modelling using 1D convolution is explained. The theory of convolutional theory is covered in Yilmaz (2000). In this model, earth’s reflectivity is convolved with the seismic wavelet to calculate the seismic trace:

\[ S = RC \otimes W + n \]

\( S \) is the seismic trace, \( RC \) the earth reflectivity series, \( W \) the seismic wavelet and \( n \) is the noise component. Figure A.2 shows the workflow for 1D convolution in practice. At each CMP location within a seismic bin, the vertical pseudo-logs are extracted. At each CMP, Zoeppritz equations are used to calculate the reflectivity depth series considering the angle of incidence associated with each CMP \( (RC_d(\theta_1), RC_d(\theta_2), \ldots, RC_d(\theta_n)) \). The mean reflectivity depth series \( (\overline{RC}_d) \) is assigned to the centre of the seismic bin. In practice the geometry of the CMPs within the bin is not known. Therefore, the reflectivity series are calculated using the extracted vertical pseudo-log at the centre of the bin (Inline/Cross-line location) and a range of angles of incidence. The mean reflectivity depth series \( (\overline{RC}_d) \) is converted to the reflectivity (zero-offset) tow-way-time series \( (\overline{RC}_{TWT}) \). Finally, the mean reflectivity time series and the wavelet are digitised using the same sampling interval and convolved to create the seismic trace. This algorithm is referred as one-dimensional because the zero-offset two-way-time is calculated using a vertical ray path from surface to the target. However, for accurate calculation of the two-way-times, the perturbations in ray trajectory due to the complexity in geometry and velocity heterogeneity of the subsurface should be taken into account. To include such complexities, 2D/3D ray tracing methods are implemented in more sophisticated convolution based seismic modelling algorithms.
Figure A.2 Seismic modelling based on 1D convolution at each Inline/Cross-line location (IL/XL). (a) The simplistic geometry of wave propagation through subsurface for two pairs of shot and receivers (S1, R1 and S2, R2). (b) The angle of incidence associated with all the CMPs within the seismic bin are taken into account for reflectivity calculations. (c) Reflectivity depth series for each angle of incidence is created by calculating the reflection coefficients at the interfaces of subsurface layers. The average of reflectivity series is calculated to create the angle stack response; (d) Mean reflectivity depth series is transformed to the mean reflection two-way-time series and digitised using a sampling interval. (e) The digitised mean reflectivity time series is coevolved with the digitised seismic wavelet to calculate the stacked seismic trace.
References


