0.1 Unitary operator

The unitary operator $\hat{S}$ in (4.13) which is used to transform the Hamiltonian $\hat{H}_2$ in (4.17) gives the corresponding transformed time evolution operator

$$
\hat{U}_{\text{trans}}(t) = \exp \left[ -i \omega_k t \hat{k} \hat{k}^\dagger + i \frac{g^2 k \omega_m}{\zeta^2} t (\hat{k} \hat{k}^\dagger)^2 \right] \exp \left[ -i \zeta t \hat{a}^\dagger \hat{a} - i \beta t (\hat{a} \hat{a}^\dagger)^2 \right],
$$

(1)

where $\zeta = \omega_m + \beta$. The untransformed operator $\hat{U}(t)$ then becomes

$$
\hat{U}(t) = e^{-\hat{S}} \hat{U}_{\text{trans}}(t) e^\hat{S} = \exp \left[ -i \omega_k t \hat{k} \hat{k}^\dagger + i \frac{g^2 k \omega_m}{\zeta^2} t (\hat{k} \hat{k}^\dagger)^2 \right] \\
\exp(-\hat{S}) \exp \left[ -i \zeta t \hat{a}^\dagger \hat{a} - i \beta t (\hat{a} \hat{a}^\dagger)^2 \right] \exp(\hat{S}).
$$

(2)

Using the Baker-Campbell-Hausdorff expansion [79] together with making the rotating wave approximation, and also neglecting quadratic and higher order terms in $g_c/\zeta$, (2) simplifies to

$$
\hat{U}(t) = \exp \left\{ -i [\omega_k t \hat{k} \hat{k}^\dagger - \frac{g^2 k}{\zeta^2} t (\hat{k} \hat{k}^\dagger)^2] \left[ \omega_m t - \sin(\zeta t) (\hat{k} \hat{k}^\dagger)^2 + \beta t (\hat{a} \hat{a}^\dagger)^2 \right] \right\} \\
\times \exp \left[ \frac{g_k}{\zeta} \hat{k} \hat{k}^\dagger (\hat{a} \hat{a}^\dagger - \hat{a} \hat{a}^\dagger) - \frac{g_k}{\zeta} \hat{k} \hat{k}^\dagger (\hat{a} \hat{a}^\dagger e^{-i\zeta t} - \hat{a} \hat{a}^\dagger e^{i\zeta t}) \right] \exp \left[ -i \zeta t \hat{a} \hat{a} \right].
$$

(3)